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STABILITY ANALYSIS FOR GENERALIZED NONLINEAR MIXED ORDERED IMPLICIT QUASI-VARIATIONAL INCLUSION PROBLEM INVOLVING $\oplus$ OPERATION

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Abstract. In this article, we introduce a generalized nonlinear mixed ordered implicit quasi-variational inclusion problem involving $\oplus$ operation in the setting of real ordered positive Hilbert spaces. We propose a perturbed three-step iterative algorithm for solving generalized nonlinear mixed ordered implicit quasi-variational inclusion problem involving $\oplus$ operator. By using a new resolvent operator method with XOR technique, we prove the existence of solution of generalized nonlinear mixed ordered implicit quasi-variational inclusion problem involving $\oplus$ operator and discuss the convergence analysis as well as stability analysis of the proposed algorithm. The iterative algorithm and results presented in this paper generalize, improve and significantly refine many previously known results of this area. Moreover, we construct a numerical example and a convergence graph in support of our main result by using MATLAB programming.

Keywords: convergence; quasi-variational inclusion; stability; algorithm; XOR operation.

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1. Introduction

A number of solutions of nonlinear equations were introduced and studied by Amann [1] in 1972. In recent past, the fixed point theory and their applications have been intensively studied in real ordered Banach spaces. Therefore, it is very important and natural for generalized nonlinear ordered variational inequalities (ordered equations) to be studied and discussed. In 2008, Li [8] introduced the generalized nonlinear ordered variational inequalities and proposed an algorithm to approximate the solution for a class of generalized nonlinear ordered variational inequalities (ordered equations) in real ordered Banach spaces.

In 2009, Li [9] introduced and studied a new class of general nonlinear ordered variational inequalities (ordered equations), and established an existence theorem in real ordered Banach spaces by using the B-restricted-accretive method. By using different kind of mappings such as RME set-valued mapping, ordered (α,λ)-NODM set-valued mapping, (γ_G,λ)-weak-GRD set-valued mapping and ordered (α_A,λ)-ANODM set-valued mappings with strong comparison mapping and their respective resolvent operators, Li et al. [9–13] studied different classes of nonlinear inclusion problems and obtained their solutions in real ordered Hilbert spaces. Very recently, Ahmad et al. [2–4] considered some classes of ordered variational inclusions involving XOR operator in different settings.

Recently, three-step forward-backward splitting methods have been developed by Glowinski et al. [7] and Noor [19–21] for solving various classes of variational inequalities by using the Lagrangian multiplier and the auxiliary principle techniques. Thus, one can conclude that three-step iterative algorithms play an important and significant part in solving various problems, which aries in pure and applied sciences. Glowinski et al. [7] shown that the three-step schemes gave better numerical results than the two-step and one-step approximation iterations. It has been proved that three step iterative algorithms are natural generalization of the splitting methods for solving partial differential equations. For applications of splitting and decomposition methods, see [7, 16–20, 22] and the references therein.

Motivated and inspired by on going research in this direction, we introduce a new class of generalized nonlinear mixed ordered implicit quasi-variational inclusion problem involving $\oplus$

operation in real ordered Hilbert spaces. Using the concept of XOR operation, we propose a perturbed three-step iterative algorithm which is more powerful than the previous iterative algorithms considered by Li et al. [8–14]. Furthermore, we prove the existence of solution of generalized nonlinear mixed ordered implicit quasi-variational inclusion problem involving $\oplus$ operation and analyze the convergence criteria of the iterative sequences of the proposed algorithm. Finally, we discuss stability analysis. We also construct a numerical example and a convergence graph is given by using MATLAB programming.

2. Preliminaries

Throughout this paper, we suppose that $\mathscr{H}_p$ is a real ordered positive Hilbert space endowed with a norm $\|\cdot\|$ and an inner product $\langle\cdot,\cdot\rangle$, d is the metric induced by the norm $\|\cdot\|$ and $2^{\mathscr{H}_p}$ is the family of all nonempty subsets of $\mathscr{H}_p$.

For the presentation of the results, let us demonstrate some known definitions and results.

Definition 2.1 ([6,23]). *A nonempty subset C of $\mathscr{H}_p$ is called*

(i) *a normal cone if there exists a constant $N > 0$ such that for $0 \leq p \leq q$, we have $||p|| \leq N||q||$, for any $p,q \in \mathscr{H}_p$;*

(ii) *for any $p,q \in \mathscr{H}_p$, $p \leq q$ if and only if $q - p \in C$;*

(iii) *p and q are said to be comparative to each other if and only if, we have either $p \leq q$ or $q \leq p$ and is denoted by $p \propto q$.*

Definition 2.2 ([23]). *For arbitrary elements $p,q \in \mathscr{H}_p$, $lub\{p,q\}$ and $glb\{p,q\}$ mean least upper bound and greatest upper bound of the set $\{p,q\}$. Suppose $lub\{p,q\}$ and $glb\{p,q\}$ exist, some binary operations are defined as follows:*

(i) $p \vee q = lub\{p,q\}$;

(ii) $p \wedge q = glb\{p,q\}$;

(iii) $p \oplus q = (p-q) \vee (q-p)$;

(iv) $p \odot q = (p-q) \wedge (q-p)$.

The operations $\vee, \wedge$, $\oplus$ and $\odot$ are called OR, AND, XOR and XNOR operations, respectively.

Lemma 2.1 ([6]). *If $p \propto q$, then $lub\{p,q\}$ and $glb\{p,q\}$ exist, $p-q \propto q-p$ and $0 \leq (p-q) \vee (q-p)$.*

Lemma 2.2 ([6]). *For any natural number n, $p \propto q_n$ and $q_n \to q^*$ as $n \to \infty$, then $p \propto q^*$.*

Proposition 2.1 ([8, 11, 14]). *Let $\oplus$ be an XOR operation and $\odot$ be an XNOR operation. Then the following relations hold:*

- (i) $p \odot p = 0$, $p \odot q = q \odot p = -(p \oplus q) = -(q \oplus p)$;
- (ii) *if* $p \propto 0$, *then* $-p \oplus 0 \leq p \leq p \oplus 0$;
- (iii) $(\lambda p) \oplus (\lambda q) = |\lambda|(p \oplus q)$;
- (iv) $0 \leq p \oplus q$, *if* $p \propto q$;
- (v) *if* $p \propto q$, *then* $p \oplus q = 0$ *if and only if* $p = q$;
- (vi) $(p+q) \odot (u+v) \geq (p \odot u) + (q \odot v)$;
- (vii) $(p+q) \odot (u+v) \geq (p \odot v) + (q \odot u)$;
- $(viii)$ *if* p, q *and* w *are comparative to each other, then* $(p \oplus q) \leq p \oplus w + w \oplus q$;
- (ix) *if* $p \propto q$, *then* $((p \oplus 0) \oplus (q \oplus 0)) \leq (p \oplus q) \oplus 0 = p \oplus q$;
- (x) $\alpha p \oplus \beta p = |\alpha - \beta| p = (\alpha \oplus \beta) p$, *if* $p \propto 0$, *for all* $p, q, u, v, w \in \mathscr{H}_p$ *and* $\alpha, \beta, \lambda \in \mathbb{R}$.

Proposition 2.2 ([6]). *Let C be a normal cone in $\mathscr{H}_p$ with normal constant N, then for each $p, q \in \mathscr{H}_p$, the following relations hold:*

- (i) $\|0 \oplus 0\| = \|0\| = 0$;
- (ii) $\|p \vee q\| \leq \|p\| \vee \|q\| \leq \|p\| + \|q\|$;
- (iii) $\|p \oplus q\| \leq \|p - q\| \leq N\|p \oplus q\|$;
- (iv) *if* $p \propto q$, *then* $\|p \oplus q\| = \|p - q\|$.

Definition 2.3 ([11, 14]). *Let $A : \mathscr{H}_p \to \mathscr{H}_p$ be a single-valued mapping. Then*

- (i) *A is said to be strongly comparison mapping, if A is a comparison mapping and $A(p) \propto A(q)$ if and only if $p \propto q$, for all $p, q \in \mathscr{H}_p$;*
- (ii) *A is said to be γ-ordered non-extended mapping, if there exists $\gamma > 0$ such that*

$$\gamma(p \oplus q) \leq A(p) \oplus A(q), \text{ for all } p, q \in \mathscr{H}_p.$$

Definition 2.4 ([11]). *A mapping $A : \mathscr{H}_p \to \mathscr{H}_p$ is said to be β-ordered compression mapping, if A is a comparison mapping and*

$$A(p) \oplus A(q) \leq \beta(p \oplus q), \ for \ 0 < \beta < 1.$$

Definition 2.5. *A mapping $F : \mathscr{H}_p \times \mathscr{H}_p \to \mathscr{H}_p$ is said to be (π, ν)-ordered Lipschitz continuous, if $p \propto q$, $u \propto v$, then $N(p,u) \propto N(q,v)$ and there exist constants $\pi, \nu > 0$ such that*

$$F(p,u) \oplus F(q,v) \leq \pi(p \oplus q) + \nu(u \oplus v), \ for \ all \ p,q,u,v \in \mathscr{H}_p.$$

Definition 2.6 ([10–12]). *Let $A : \mathscr{H}_p \to \mathscr{H}_p$ be a strong comparison and γ-ordered non-extended mapping and $M : \mathscr{H}_p \to 2^{\mathscr{H}_p}$ be a set-valued mapping. Then*

- (i) *M is said to be a comparison mapping, if for any $v_p \in M(p)$, $p \propto v_p$, and if $p \propto q$, then for any $v_p \in M(p)$ and $v_q \in M(q)$, $v_p \propto v_q$, for all $p,q \in \mathscr{H}_p$;*
- (ii) *a weak comparison mapping M is said to be α_A-weak-non-ordinary difference mapping with respect to A, if for each $p,q \in \mathscr{H}_p$, there exist $\alpha_A > 0$ and $v_p \in M(A(p))$ and $v_q \in M(A(q))$ such that*

$$(v_p \oplus v_q) \oplus \alpha_A(A(p) \oplus A(q)) = 0.$$

Now, we introduce some new definitions of XOR-ordered different weak compression mapping, XOR-weak-ANODD set-valued mapping and a resolvent operator associated with XOR-weak-ANODD set-valued mapping.

Definition 2.7. *A weak compression mapping $M : \mathscr{H}_p \to 2^{\mathscr{H}_p}$ is said to be λ-XOR-ordered different weak compression mapping with respect to A, if for each $p,q \in \mathscr{H}_p$, there exists a constant $\lambda > 0$ and $v_p \in M(A(p)), v_q \in M(A(q))$ such that*

$$\lambda(v_p \oplus v_q) \geq p \oplus q,$$

holds.

Definition 2.8. *Let $A : \mathscr{H}_p \to \mathscr{H}_p$ be a strongly comparison and γ-ordered non-extended mapping. Then, a weak comparison mapping $M : \mathscr{H}_p \to 2^{\mathscr{H}_p}$ is said to be (α_A, λ)-XOR-weak-ANODD set-valued mapping if M is a α_A-weak-non-ordinary difference mapping with respect to A and λ-XOR-ordered different weak compression mapping with respect to A, and $[A \oplus \lambda M](\mathscr{H}_p) = \mathscr{H}_p$, for $\lambda, \beta, \alpha > 0$.*

Definition 2.9. *Let $A : \mathscr{H}_p \to \mathscr{H}_p$ be a strongly comparison and γ-ordered non-extended mapping. Let $M : \mathscr{H}_p \to 2^{\mathscr{H}_p}$ be a (α_A, λ)-XOR-weak-ANODD set-valued mapping. The resolvent operator $\mathscr{J}^A_{\lambda,M} : \mathscr{H}_p \to \mathscr{H}_p$ associated with A and M is defined by*

$$\mathscr{J}^A_{\lambda,M}(p) = [A \oplus \lambda M]^{-1}(p), \forall p \in \mathscr{H}_p, \tag{2.1}$$

where $\lambda > 0$ is a constant.

Definition 2.10 ([24]). *Let $S, T : \mathscr{H}_p \to \mathscr{H}_p$ be a single-valued mapping, $p_0 \in \mathscr{H}_p$ and let*

$$p_{n+1} = S(T, p_n)$$

defines an iterative sequence which yields a sequence of points $\{p_n\}$ in $\mathscr{H}_p$. Suppose that $F(T) = \{p \in \mathscr{H}_p : Tp = p\} \neq \emptyset$ and $\{p_n\}$ converges to a fixed point p^* of T. Let $\{u_n\} \subset \mathscr{H}_p$ and

$$\vartheta_n = \|u_{n+1} - S(T, u_n)\|.$$

If $\lim\limits_{n\to\infty} \vartheta_n = 0$, which implies that $u_n \to p^*$, then the iterative sequence $\{p_n\}$ is said to be T-stable or stable with respect to T.

Lemma 2.3 ([25]). *Let $\{\chi_n\}$ be a nonnegative real sequence and $\{\zeta_n\}$ be a real sequence in $[0,1]$ such that $\sum\limits_{n=0}^{\infty} \zeta_n = \infty$. If there exists a positive integer m such that*

$$\chi_n \leq (1 - \zeta_n)\chi_n + \zeta_n \eta_n,\ \forall n \geq m, \tag{2.2}$$

where $\eta_n \geq 0$, for all $n \geq 0$ and $\eta_n \to 0 (n \to 0)$, then $\lim\limits_{n\to\infty} \chi_n = 0$.

Now, we show that the resolvent operator defined by (2.1) is single-valued, a comparison mapping as well as continuous.

Proposition 2.3 ([5]). *Let $A : \mathscr{H}_p \to \mathscr{H}_p$ be a strongly comparison, γ-ordered non-extended mapping and $M : \mathscr{H}_p \to 2^{\mathscr{H}_p}$ be a α_A-weak-non-ordinary difference set-valued mapping with respect to A with $\lambda\alpha_A \neq 1$. Then the resolvent operator $\mathscr{J}^A_{\lambda,M} : \mathscr{H}_p \to \mathscr{H}_p$ is a single-valued, for all $\alpha, \lambda > 0$.*

Proposition 2.4 ([5]). *Let $M : \mathscr{H}_p \to 2^{\mathscr{H}_p}$ be a (α_A, λ)-XOR-weak-ANODD set-valued mapping with respect to $\mathscr{J}^A_{\lambda,M}$. Let $A : \mathscr{H}_p \to \mathscr{H}_p$ be a strongly comparison mapping with respect to $\mathscr{J}^A_{\lambda,M}$. Then, the resolvent operator $\mathscr{J}^A_{\lambda,M} : \mathscr{H}_p \to \mathscr{H}_p$ is a comparison mapping.*

Proposition 2.5. *Let $M : \mathscr{H}_p \to 2^{\mathscr{H}_p}$ be a (α_A, λ)-XOR-weak-ANODD set-valued mapping with respect to $\mathscr{J}^A_{\lambda,M}$. Let $A : \mathscr{H}_p \to \mathscr{H}_p$ be a comparison and γ-ordered non-extended mapping with respect to $\mathscr{J}^A_{\lambda,M}$, for $\mu \geq 1$ and $\lambda\alpha_A > \mu$. Then the following condition holds:*

$$\mathscr{J}^A_{\lambda,M}(p) \oplus \mathscr{J}^A_{\lambda,M}(q) \leq \frac{\mu}{(\lambda\alpha_A \oplus \mu)}(p \oplus q),\ for\ all\ p, q \in \mathscr{H}_p.$$

Proof. Let $p, q \in \mathscr{H}_p$, $u_p = \mathscr{J}^A_{\lambda,M}(p)$, $u_q = \mathscr{J}^A_{\lambda,M}(q)$, and let

$$v_{p^*} = \frac{1}{\lambda}\big(p \oplus A(u_p))\big) \in M(u_p) \text{ and } v_{q^*} = \frac{1}{\lambda}\big(q \oplus A(u_q))\big) \in M(u_q).$$

As M be an (α_A, λ)-XOR-weak-ANODD set-valued mapping with respect to $\mathscr{J}^A_{\lambda,M}$. It follows that M is also an α_A-weak-non-ordinary difference mapping with respect to $\mathscr{J}^A_{\lambda,M}$, we have

$$(v_{p^*} \oplus v_{q^*}) \oplus \alpha_A(A(u_p) \oplus A(u_q)) = 0, \tag{2.3}$$

and

$$\begin{aligned} v_{p^*} \oplus v_{q^*} &= \frac{1}{\lambda}[(p \oplus A(u_p)) \oplus (q \oplus A(u_q))] \\ &= \frac{1}{\lambda}[(p \oplus q) \oplus (A(u_p) \oplus A(u_q))] \\ &\leq \frac{\mu}{\lambda}[(p \oplus q) \oplus (A(u_p) \oplus A(u_q))], \text{ for } \mu \geq 1. \end{aligned}$$

From (2.3), we have

$$\begin{aligned} \alpha_A(A(u_p) \oplus A(u_q)) &= v_{p^*} \oplus v_{q^*} \\ &\leq \frac{\mu}{\lambda}[(p \oplus q) \oplus (A(u_p) \oplus A(u_q))], \end{aligned}$$

i.e.,

$$\frac{\lambda\alpha_A}{\mu}(A(u_p)\oplus A(u_q)) \leq \quad [(p\oplus q)\oplus(A(u_p)\oplus A(u_q))].$$

Now,

$$\begin{aligned}\Big[\frac{\lambda\alpha_A}{\mu}(A(u_p)\oplus A(u_q))\Big]\oplus\big(A(u_p)\oplus A(u_q)\big) &\leq (p\oplus q)\oplus 0 = p\oplus q\\ \Big(\frac{\lambda\alpha_A}{\mu}\oplus 1\Big)\big(A(u_p)\oplus A(u_q)\big) &\leq (p\oplus q).\end{aligned}$$

It follows that

$$A(u_p)\oplus A(u_q) \leq \Big(\frac{\mu}{(\lambda\alpha_A\oplus\mu)}\Big)(p\oplus q). \tag{2.4}$$

Since A is γ-ordered non-extended mapping with respect to $\mathscr{J}^A_{\lambda,M}$ and using (2.4), we have

$$u_p\oplus u_q \leq \Big(\frac{\mu}{\gamma(\lambda\alpha_A\oplus\mu)}\Big)(p\oplus q) \tag{2.5}$$

and consequently, we have

$$\mathscr{J}^A_{\lambda,M}(p)\oplus\mathscr{J}^A_{\lambda,M}(q) \leq \frac{\mu}{\gamma(\lambda\alpha_A\oplus\mu)}(p\oplus q),\ \forall\ p,q\in\mathscr{H}_p.$$

□

Proposition 2.6. *Let $M:\mathscr{H}_p\to 2^{\mathscr{H}_p}$ be a (α_A,λ)-XOR-weak-ANODD set-valued mapping with respect to $\mathscr{J}^A_{\lambda,M}$. Let $A:\mathscr{H}_p\to\mathscr{H}_p$ be a comparison and γ-ordered non-extended mapping with respect to $\mathscr{J}^A_{\lambda,M}$, for $\mu\geq 1$ and $\lambda\alpha_A>\mu$. Then the resolvent operator $\mathscr{J}^A_{\lambda,M}$ is continuous.*

Proof. Let M be an (α_A,λ)-XOR-weak-ANODD set-valued mapping with respect to $\mathscr{J}^A_{\lambda,M}$. If $\mu\geq 1$ and $\lambda\alpha_A>\mu$, then

$$\mathscr{J}^A_{\lambda,M}(p)\oplus\mathscr{J}^A_{\lambda,M}(q) \leq \frac{\mu}{\gamma(\lambda\alpha_A\oplus\mu)}(p\oplus q),$$

holds. Let the sequence $\{p_n\}\subseteq\mathscr{H}_p$ and $q\in\mathscr{H}_p$, then we have

$$0\leq\mathscr{J}^A_{\lambda,M}(p_n)\oplus\mathscr{J}^A_{\lambda,M}(q) \leq \frac{\mu}{\gamma(\lambda\alpha_A\oplus\mu)}(p_n\oplus q),$$

By using (i) of the Definition 2.1, we have

$$\Big\|\mathscr{J}^A_{\lambda,M}(p_n)\oplus\mathscr{J}^A_{\lambda,M}(q)\Big\| \leq \frac{\mu}{\gamma(\lambda\alpha_A\oplus\mu)}\|p_n\oplus q\|.$$

From Proposition 2.2 and Proposition 2.4, we have

$$\left\| \mathscr{J}^{A}_{\lambda,M}(p_n) - \mathscr{J}^{A}_{\lambda,M}(q) \right\| \leq \frac{\mu}{\gamma(\lambda\alpha_A \oplus \mu)} \|p_n - q\|.$$

If $p_n \to q$, then $\mathscr{J}^{A}_{\lambda,M}(p_n) \to \mathscr{J}^{A}_{\lambda,M}(q)$. Therefore, the resolvent operator $\mathscr{J}^{A}_{\lambda,M}$ is continuous. This completes the proof. □

3. Formulation of the Problem and Existence Results of Solutions

Let $\mathscr{H}_p$ be a real ordered Hilbert space and C be a normal cone with normal constant N. Let $P, g, h, Q : \mathscr{H}_p \to \mathscr{H}_p$ and $F : \mathscr{H}_p \times \mathscr{H}_p \to \mathscr{H}_p$ be the single-valued mappings. Let $M : \mathscr{H}_p \to 2^{\mathscr{H}_p}$ be a set-valued mapping such that $\tau M = \{\tau u | u \in M(\mathscr{H}_p)\}$, for some $\tau > 0$. We consider the following problem:

For some $\tau > 0$ and any $\xi \in \mathbb{R}$, find $p \in \mathscr{H}_p$ such that

$$0 \in P(p) \oplus \tau M(p) - \xi Q(h(p) \oplus F(p, g(p))). \tag{3.1}$$

We call this problem as generalized nonlinear mixed ordered implicit quasi-variational inclusion problem involving $\oplus$ operation (in short, GNMOIQVIP).

Some special cases of problem (3.1) are as follows:

(i) If $\tau, \xi = 1$, $g, F = 0$, $h = I$ (identity mapping) and $Q(p) = \omega$, $(\omega \in \mathscr{H}_p)$, then problem (3.1) reduces to the problem of finding $p \in \mathscr{H}_p$ such that

$$\omega \in P(p) \oplus M(p). \tag{3.2}$$

Problem (3.2) was considered and studied by Li et al. [11].

(ii) If $P, h = 0$ and $Q = I$ then problem (3.1) reduces to the problem of finding $p \in \mathscr{H}_p$ such that

$$0 \in \tau M(p) - \xi F(p, g(p)). \tag{3.3}$$

Problem (3.3) was considered and studied by Li et al. [14].

(iii) If $\xi = -1$, $h = 0$, $Q = I$ (identity mapping) and $F(p, g(p)) = f(p) - \omega$, $(\omega \in \mathscr{H}_p)$, then problem (3.1) reduces to the problem of finding $p \in \mathscr{H}_p$ such that

$$\omega \in f(p) + \tau M(p). \tag{3.4}$$

Problem (3.4) was considered and studied by Li et al. [15].

(iv) If $P,g,F=0$, $h=I$ (identity mapping) and $Q=1$ (constant mapping), then problem (3.1) reduces to the problem of finding $p\in\mathscr{H}_p$ such that

$$\xi\in\tau M(p). \tag{3.5}$$

Problem (3.5) was considered and studied by Li et al. [13].

(v) If $\xi=0,\tau=1$ and $P,g,h,F,Q=0$, then problem (3.1) reduces to the problem of finding $p\in\mathscr{H}_p$ such that

$$0\in M(p). \tag{3.6}$$

Problem (3.6) was considered and studied by Li [10, 12].

(vi) If $\xi=-1$, $P,M=0$, and $Q=I$ (identity mapping), then problem (3.1) reduces to the problem of finding $p\in\mathscr{H}_p$ such that

$$0\leq h(p)\oplus F(p,g(p)). \tag{3.7}$$

Problem (3.7) was considered and studied by Li [9].

(vii) If $h=0$ and $F(p,g(p))=F(g(p))$, then problem (3.7) reduces to the problem of finding $p\in\mathscr{H}_p$ such that

$$0\leq F(g(p)). \tag{3.8}$$

Problem (3.8) was considered and studied by Li [8].

Hence, we see that our problem is much more general that the previous problems existing in the literature.

The following Lemma is a fixed point formulation of GMOQVIP (3.1).

Lemma 3.1. *The GNMOIQVIP* (3.1) *admits a solution* $p\in\mathscr{H}_p$ *if and only if it satisfies the following equation:*

$$p=\mathscr{J}^{A}_{\lambda,M}[A(p)\oplus\frac{\lambda}{\tau}(P(p)\oplus\xi Q(h(p)\oplus F(p,g(p))))], \tag{3.9}$$

where $\lambda>0$ *is constant.*

Proof. The proof directly follows from the definition of the resolvent operator $\mathscr{J}^{A}_{\lambda,M}$. $\square$

Now, we prove the existence of solution for GNMOIQVIP (3.1).

Theorem 3.1. *Let $\mathscr{H}_p$ be a real ordered Hilbert space and C be a normal cone with normal constant N. Let $A,P,g,h,Q:\mathscr{H}_p\to\mathscr{H}_p$ and $F:\mathscr{H}_p\times\mathscr{H}_p\to\mathscr{H}_p$ be the single-valued mappings such that A is δ_A-ordered compression and γ-ordered non-extended mapping, P is comparison, δ_p-ordered compression mapping, g is comparison, δ_g-ordered compression mapping, h is comparison and δ_h-ordered compression mapping, Q is comparison and δ_Q-ordered compression mapping with respect to $h\oplus F$ and F is comparison and (π,ν)-ordered Lipschitz continuous mapping with respect to g, respectively. Let $M:\mathscr{H}_p\to 2^{\mathscr{H}_p}$ be an (α_A,λ)-XOR-weak-ANODD set-valued mapping. In addition, if $A,P,g,h,Q,F,M,h\oplus F$ and $[A(.)\oplus\frac{\lambda}{\tau}(P(.)\oplus(\xi Q(.,g(.))))]$ are compared to each other, and for all $\tau,\lambda>0$, the following conditions are satisfied:*

$$\begin{cases} N|\mu\big(\delta_A\tau\oplus\lambda(\delta_P\oplus|\xi|\delta_Q(\delta_h\oplus(\pi+\nu\delta_g)))\big)|<|\gamma\tau(\lambda\alpha_A\oplus\mu)|,\\ \lambda\alpha_A>\mu \text{ and } \mu\geq 1, \end{cases}\tag{3.10}$$

then, GNMOIQVIP (3.1) admits a unique solution $p^*\in\mathscr{H}_p$, which is a fixed point of $\mathscr{J}^{A}_{\lambda,M}[A(p^*)\oplus\frac{\lambda}{\tau}(P(p^*)\oplus\xi Q(h(p^*)\oplus F(p^*,g(p^*))))]$.

Proof. For $i=1,2$, let us define $R(p_i)=P(p_i)\oplus\xi Q(h(p_i)\oplus F(p_i,g(p_i)))$, for all $p_i\in\mathscr{H}_p$. Using Proposition 2.1 and 2.5, we obtain

$$\begin{aligned} 0 &\leq \mathscr{J}^{A}_{\lambda,M}[A(.)\oplus\frac{\lambda}{\tau}R(.)](p_1)\oplus\mathscr{J}^{A}_{\lambda,M}[A(.)\oplus\frac{\lambda}{\tau}R(.)](p_2)\\ &\leq \frac{\mu}{\gamma(\lambda\alpha_A\oplus\mu)}\big[(A(.)\oplus\frac{\lambda}{\tau}R(.))(p_1)\oplus(A(.)\oplus\frac{\lambda}{\tau}R(.))(p_2)\big]\\ &\leq \frac{\mu}{\gamma(\lambda\alpha_A\oplus\mu)}\big[(A(p_1)\oplus A(p_2))\oplus\frac{\lambda}{\tau}(R(p_1)\oplus R(p_2))\big]\\ &\leq \frac{\mu}{\gamma(\lambda\alpha_A\oplus\mu)}\big[\delta_A(p_1\oplus p_2)\oplus\frac{\lambda}{\tau}(R(p_1)\oplus R(p_2))\big]. \end{aligned}\tag{3.11}$$

Since P is δ_p-ordered compression mapping, g is δ_g-ordered compression mapping, h is δ_h-ordered compression mapping, Q is δ_Q-ordered compression mapping with respect to $h\oplus F$

and F is (π,ν)-ordered Lipschitz continuous mapping with respect to g, we have

$$\begin{aligned}
& R(p_1)\oplus R(p_2)\\
= \;& [(P(p_1)\oplus\xi Q(h(p_1)\oplus F(p_1,g(p_1))))\oplus(P(p_2)\oplus\xi Q(h(p_2)\oplus F(p_2,g(p_2))))]\\
= \;& [(P(p_1)\oplus P(p_2))\oplus(\xi Q(h(p_1)\oplus F(p_1,g(p_1)))\oplus\xi Q(h(p_2)\oplus F(p_2,g(p_2))))]\\
\leq \;& (\delta_P(p_1\oplus p_2))\oplus(\xi Q(h(p_1)\oplus F(p_1,g(p_1)))\oplus\xi Q(h(p_2)\oplus F(p_2,g(p_2))))\\
\leq \;& (\delta_P(p_1\oplus p_2))\oplus|\xi|(Q(h(p_1)\oplus F(p_1,g(p_1)))\oplus Q(h(p_2)\oplus F(p_2,g(p_2))))\\
\leq \;& (\delta_P(p_1\oplus p_2))\oplus|\xi|\delta_Q(h(p_1)\oplus F(p_1,g(p_1))\oplus h(p_2)\oplus F(p_2,g(p_2)))\\
\leq \;& (\delta_P(p_1\oplus p_2))\oplus|\xi|\delta_Q(h(p_1)\oplus h(p_2)\oplus(F(p_1,g(p_1))\oplus F(p_2,g(p_2)))\\
\leq \;& (\delta_P(p_1\oplus p_2))\oplus|\xi|\delta_Q(\delta_h(p_1\oplus p_2)\oplus(\pi(p_1\oplus p_2)+\nu(g(p_1)\oplus g(p_2)))\\
\leq \;& (\delta_P(p_1\oplus p_2))\oplus|\xi|\delta_Q(\delta_h(p_1\oplus p_2)\oplus(\pi(p_1\oplus p_2)+\nu\delta_g(p_1\oplus p_2))\\
= \;& (\delta_P(p_1\oplus p_2))\oplus|\xi|\delta_Q(\delta_h\oplus(\pi+\nu\delta_g))(p_1\oplus p_2)\\
= \;& (\delta_P\oplus|\xi|\delta_Q(\delta_h\oplus(\pi+\nu\delta_g)))(p_1\oplus p_2). \qquad (3.12)
\end{aligned}$$

Using (3.12), (3.11) becomes

$$\begin{aligned}
0 \;\leq\;& \mathscr{J}^A_{\lambda,M}[A(.)\oplus\frac{\lambda}{\tau}R(.)](p_1)\oplus\mathscr{J}^A_{\lambda,M}[A(.)\oplus\frac{\lambda}{\tau}R(.)](p_2)\\
\leq\;& \frac{\mu}{\gamma(\lambda\alpha_A\oplus\mu)}\big[\delta_A(p_1\oplus p_2)\oplus\frac{\lambda}{\tau}(\delta_P\oplus|\xi|\delta_Q(\delta_h\oplus(\pi+\nu\delta_g)))(p_1\oplus p_2)\big]\\
\leq\;& \frac{\mu\big(\delta_A\tau\oplus\lambda(\delta_P\oplus|\xi|\delta_Q(\delta_h\oplus(\pi+\nu\delta_g)))\big)}{\gamma\tau(\lambda\alpha_A\oplus\mu)}(p_1\oplus p_2),
\end{aligned}$$

which implies that

$$0\leq\mathscr{J}^A_{\lambda,M}[A(.)\oplus\frac{\lambda}{\tau}R(.)](p_1)\oplus\mathscr{J}^A_{\lambda,M}[A(.)\oplus\frac{\lambda}{\tau}R(.)](p_2)\leq\psi(p_1\oplus p_2), \tag{3.13}$$

$$\text{where } \psi=\left[\frac{\mu\big(\delta_A\tau\oplus\lambda(\delta_P\oplus|\xi|\delta_Q(\delta_h\oplus(\pi+\nu\delta_g)))\big)}{\gamma\tau(\lambda\alpha_A\oplus\mu)}\right].$$

Using the definition of normal cone, we conclude that

$$\left\|\mathscr{J}^A_{\lambda,M}[A(.)\oplus\frac{\lambda}{\tau}R(.)](p_1)\oplus\mathscr{J}^A_{\lambda,M}[A(.)\oplus\frac{\lambda}{\tau}R(.)](p_2)\right\|\leq|\psi|N\|p_1\oplus p_2\|. \tag{3.14}$$

Using the conditions (3.10), we can see that $|\psi| < \frac{1}{N}$. It follows from (3.14) that $\mathscr{J}^A_{\lambda,M}[A(.) \oplus \frac{\lambda}{\tau}R(.)]$ is contraction operator. Hence, there exists a unique $p^* \in \mathscr{H}_p$ such that

$$p^* = \mathscr{J}^A_{\lambda,M}[A(p^*) \oplus \frac{\lambda}{\tau}(P(p^*) \oplus \xi Q(h(p^*) \oplus F(p^*, g(p^*))))].$$

From Lemma 3.1, p^* is a unique solution of GNMOIQVIP (3.1). □

4. Convergence and Stability Analysis

In this section, we suggest the following perturbed three-step iterative algorithm based on Lemma 3.1 for finding the approximate solution of GNMOIQVIP (3.1). We also discuss the convergence and stability analysis of the proposed algorithm.

Iterative Algorithm 4.1. *Let $A, P, g, h, Q : \mathscr{H}_p \to \mathscr{H}_p$ and $F : \mathscr{H}_p \times \mathscr{H}_p \to \mathscr{H}_p$ be the single-valued mappings. Let $M : \mathscr{H}_p \to 2^{\mathscr{H}_p}$ be a set-valued mapping. Given any initial point $p_0 \in \mathscr{H}_p$, assume that $p_1 \propto p_0$. We define the sequence $\{p_n\}$ and let $p_{n+1} \propto p_n$ such that*

$$(4.1)\qquad \begin{cases} p_{n+1} = (1-a_n)p_n + a_n\Big(\mathscr{J}^A_{\lambda,M}[A(q_n) \oplus \frac{\lambda}{\tau}(P(q_n) \oplus \xi Q(h(q_n) \\ \oplus F(q_n, g(q_n))))]\Big) + a_n\alpha_n, \\ q_n = (1-b_n)p_n + b_n\Big(\mathscr{J}^A_{\lambda,M}[A(z_n) \oplus \frac{\lambda}{\tau}(P(z_n) \oplus \xi Q(h(z_n) \\ \oplus F(z_n, g(z_n))))]\Big) + b_n\beta_n, \\ z_n = (1-c_n)p_n + c_n\Big(\mathscr{J}^A_{\lambda,M}[A(p_n) \oplus \frac{\lambda}{\tau}(P(p_n) \oplus \xi Q(h(p_n) \\ \oplus F(p_n, g(p_n))))]\Big) + c_n\delta_n. \end{cases}$$

Let $\{u_n\}$ be any sequence in $\mathscr{H}_p$ and define $\{\vartheta_n\}$ by

$$(4.2)\qquad \begin{cases} \vartheta_n = \Big\|u_{n+1} - \Big[(1-a_n)u_n + a_n\Big(\mathscr{J}^A_{\lambda,M}[A(t_n) \oplus \frac{\lambda}{\tau}(P(t_n) \oplus \xi Q(h(t_n) \\ \oplus F(t_n, g(t_n))))]\Big) + a_n\alpha_n\Big]\Big\|, \\ t_n = (1-b_n)u_n + b_n\Big(\mathscr{J}^A_{\lambda,M}[A(s_n) \oplus \frac{\lambda}{\tau}(P(s_n) \oplus \xi Q(h(s_n) \\ \oplus F(s_n, g(s_n))))]\Big) + b_n\beta_n, \\ s_n = (1-c_n)u_n + c_n\Big(\mathscr{J}^A_{\lambda,M}[A(u_n) \oplus \frac{\lambda}{\tau}(P(u_n) \oplus \xi Q(h(u_n) \\ \oplus F(u_n, g(u_n))))]\Big) + c_n\delta_n, \end{cases}$$

where $0 \leq a_n, b_n, c_n \leq 1$, $\sum\limits_{n=0}^{\infty} a_n = \infty, \forall\, n \geq 0$. *Here* $\{\alpha_n\}$, $\{\beta_n\}$ *and* $\{\delta_n\}$ *are three sequences in* $\mathscr{H}_p$ *introduced to take into account the possible inexact computation provided that* $\alpha_n \oplus 0 = \alpha_n$, $\beta_n \oplus 0 = \beta_n$ *and* $\delta_n \oplus 0 = \delta_n, \forall\, n \geq 0$.

Remark 4.1. *If* $c_n = 0, \forall\, n \geq 0$, *then Algorithm* 4.1 *becomes Ishikawa type iterative algorithm. On taking* $b_n, c_n = 0, \forall\, n \geq 0$, *Algorithm* 4.1 *becomes Mann type iterative algorithm. Also, we remark that for suitable choices of operators involved in Algorithm* 4.1, *we can easily obtain many more algorithms studied by several authors for solving ordered variational inclusion problems, see e.g. [2, 3, 9–12, 14].*

Theorem 4.1. *Let* $\mathscr{H}_p, C, A, P, h, g, F$ *and* M *be the same as in Theorem 3.1 such that all the conditions of Theorem* 3.1 *are satisfied. In addition, assume that the following conditions are satisfied:*

$$\begin{cases} |\mu\big(\delta_A\tau \oplus \lambda(\delta_P \oplus |\xi|\delta_Q(\delta_h \oplus (\pi + \nu\delta_g)))\big)| < |\gamma\tau(\lambda\alpha_A \oplus \mu)|\min\{\frac{1}{N}, 1\}, \\ \lambda\alpha_A > \mu \text{ and } \mu \geq 1. \end{cases} \tag{4.3}$$

If $\lim\limits_{n\to\infty} \|\alpha_n \vee (-\alpha_n)\| = \lim\limits_{n\to\infty} \|\beta_n \vee (-\beta_n)\| = \lim\limits_{n\to\infty} \|\delta_n \vee (-\delta_n)\| = 0$, *then*

(I) *the sequence* $\{p_n\}$ *generated by Algorithm 4.1 converges strongly to the unique solution* p^* *of GNMOIQVIP (3.1).*

(II) *moreover, if* $0 < \kappa \leq a_n$, *then* $\lim\limits_{n\to\infty} u_n = p^*$ *if and only if* $\lim\limits_{n\to\infty} \vartheta_n = 0$, *where* ϑ_n *is defined in (4.2) i.e., the sequence* $\{p_n\}$ *generated by (4.1) is stable with respect to* $\mathscr{J}^A_{\lambda,M}$.

Proof. (**I**). First, we prove the sequence $\{p_n\}$ strongly converges to the unique solution p^* of GNMOQVIP (3.1). By the conditions (4.3), we can assume that the conditions (3.10) hold. Then, by Theorem 3.1, let us suppose that p^* be an unique solution of GNMOQVIP (3.1). Then, we have

$$\begin{aligned} p^* &= (1-a_n)p^* + a_n\Big(\mathscr{J}^A_{\lambda,M}[A(p^*) \oplus \frac{\lambda}{\tau}(P(p^*) \oplus \xi Q(h(p^*) \oplus F(p^*, g(p^*))))]\Big) \\ &= (1-b_n)p^* + b_n\Big(\mathscr{J}^A_{\lambda,M}[A(p^*) \oplus \frac{\lambda}{\tau}(P(p^*) \oplus \xi Q(h(p^*) \oplus F(p^*, g(p^*))))]\Big) \\ &= (1-c_n)p^* + c_n\Big(\mathscr{J}^A_{\lambda,M}[A(p^*) \oplus \frac{\lambda}{\tau}(P(p^*) \oplus \xi Q(h(p^*) \oplus F(p^*, g(p^*))))]\Big). \end{aligned} \tag{4.4}$$

Using Algorithm 4.1, (4.4), Proposition 2.1 and Proposition 2.5, it follows that

$$\begin{aligned}
0 \leq\ & p_{n+1} \oplus p^* = \Big[(1-a_n)p_n + a_n\Big(\mathscr{J}^A_{\lambda,M}[A(q_n) \oplus \frac{\lambda}{\tau}(P(q_n) \\
& \oplus \xi Q(h(q_n) \oplus F(q_n, g(q_n))))]\Big) + a_n\alpha_n\Big] \oplus \Big[(1-a_n)p^* + a_n\Big(\mathscr{J}^A_{\lambda,M}[A(p^*) \\
& \oplus \frac{\lambda}{\tau}(P(p^*) \oplus \xi Q(h(p^*) \oplus F(p^*, g(p^*))))]\Big)\Big] \\
\leq\ & (1-a_n)(p_n \oplus p^*) + a_n(\alpha_n \oplus 0) \\
& + a_n\Big(\Big(\mathscr{J}^A_{\lambda,M}[A(q_n) \oplus \frac{\lambda}{\tau}(P(q_n) \oplus \xi Q(h(q_n) \oplus F(q_n, g(q_n))))]\Big) \\
& \oplus \Big(\mathscr{J}^A_{\lambda,M}[A(p^*) \oplus \frac{\lambda}{\tau}(P(p^*) \oplus \xi Q(h(p^*) \oplus F(p^*, g(p^*))))]\Big)\Big) \\
\leq\ & (1-a_n)(p_n \oplus p^*) + \psi a_n(q_n \oplus p^*) + a_n(\alpha_n \oplus 0), \qquad (4.5)
\end{aligned}$$

$$\text{where } \psi = \left[\frac{\mu\big(\delta_A\tau \oplus \lambda(\delta_P \oplus |\xi|\delta_Q(\delta_h \oplus (\pi + \nu\delta_g)))\big)}{\gamma\tau(\lambda\alpha_A \oplus \mu)}\right].$$

Using the same argument as for (4.5), we calculate

$$\begin{aligned}
0 \leq\ & q_n \oplus p^* = \Big[(1-b_n)p_n + b_n\Big(\mathscr{J}^A_{\lambda,M}[A(z_n) \oplus \frac{\lambda}{\tau}(P(z_n) \\
& \oplus \xi Q(h(z_n) \oplus F(z_n, g(z_n))))]\Big) + b_n\beta_n\Big] \oplus \Big[(1-b_n)p^* + b_n\Big(\mathscr{J}^A_{\lambda,M}[A(p^*) \\
& \oplus \frac{\lambda}{\tau}(P(p^*) \oplus \xi Q(h(p^*) \oplus F(p^*, g(p^*))))]\Big)\Big] \\
\leq\ & (1-b_n)(p_n \oplus p^*) + b_n(\beta_n \oplus 0) \\
& + b_n\Big(\Big(\mathscr{J}^A_{\lambda,M}[A(z_n) \oplus \frac{\lambda}{\tau}(P(z_n) \oplus \xi Q(h(z_n) \oplus F(z_n, g(z_n))))]\Big) \\
& \oplus \Big(\mathscr{J}^A_{\lambda,M}[A(p^*) \oplus \frac{\lambda}{\tau}(P(p^*) \oplus \xi Q(h(p^*) \oplus F(p^*, g(p^*))))]\Big)\Big) \\
\leq\ & (1-b_n)(p_n \oplus p^*) + \psi b_n(z_n \oplus p^*) + b_n(\beta_n \oplus 0), \qquad (4.6)
\end{aligned}$$

and

$$\begin{aligned}
0 \leq\ & z_n \oplus p^* = \Big[(1-c_n)p_n + c_n\Big(\mathscr{J}^A_{\lambda,M}[A(p_n) \oplus \frac{\lambda}{\tau}(P(p_n) \\
& \oplus \xi Q(h(p_n) \oplus F(p_n, g(p_n))))]\Big) + c_n\delta_n\Big] \oplus \Big[(1-c_n)p^* + c_n\Big(\mathscr{J}^A_{\lambda,M}[A(p^*) \\
& \oplus \frac{\lambda}{\tau}(P(p^*) \oplus \xi Q(h(p^*) \oplus F(p^*, g(p^*))))]\Big)\Big]
\end{aligned}$$

$$
\begin{aligned}
&\leq (1-c_n)(p_n \oplus p^*) + c_n(\delta_n \oplus 0) \\
&\quad + c_n\Big(\Big(\mathscr{J}^{A}_{\lambda,M}[A(p_n) \oplus \frac{\lambda}{\tau}(P(p_n) \oplus \xi Q(h(p_n) \oplus F(p_n, g(p_n))))]\Big) \\
&\quad \oplus \Big(\mathscr{J}^{A}_{\lambda,M}[A(p^*) \oplus \frac{\lambda}{\tau}(P(p^*) \oplus \xi Q(h(p^*) \oplus F(p^*, g(p^*))))]\Big)\Big) \\
&\leq (1-c_n)(p_n \oplus p^*) + \psi c_n(p_n \oplus p^*) + c_n(\delta_n \oplus 0) \\
&\leq [1-c_n(1-\psi)](p_n \oplus p^*) + c_n(\delta_n \oplus 0) \\
&\leq (p_n \oplus p^*) + c_n(\delta_n \oplus 0), \text{ since } (1-c_n(1-\psi)) \leq 1. \qquad (4.7)
\end{aligned}
$$

Using (4.7), (4.6) becomes

$$
\begin{aligned}
0 &\leq q_n \oplus p^* \\
&\leq (1-b_n)(p_n \oplus p^*) + \psi b_n((p_n \oplus p^*) + c_n(\delta_n \oplus 0)) + b_n(\beta_n \oplus 0) \\
&\leq (1-b_n(1-\psi))(p_n \oplus p^*) + \psi c_n(\delta_n \oplus 0) + b_n(\beta_n \oplus 0) \\
&\leq (p_n \oplus p^*) + \psi c_n(\delta_n \oplus 0) + b_n(\beta_n \oplus 0), \text{ since } (1-b_n(1-\psi)) \leq 1. \qquad (4.8)
\end{aligned}
$$

Combining (4.5), (4.7) and (4.8) becomes

$$
\begin{aligned}
0 &\leq p_{n+1} \oplus p^* \\
&\leq (1-a_n)(p_n \oplus p^*) + \psi a_n\big[(p_n \oplus p^*) + \psi c_n(\delta_n \oplus 0) \\
&\quad + b_n(\beta_n \oplus 0)\big] + a_n(\alpha_n \oplus 0) \\
&\leq (1-a_n(1-\psi))(p_n \oplus p^*) + \psi a_n\big[\psi c_n(\delta_n \oplus 0) + b_n(\beta_n \oplus 0)\big] \\
&\quad + a_n(\alpha_n \oplus 0) \\
&\leq (1-a_n(1-\psi))(p_n \oplus p^*) + \big[\psi^2 a_n c_n(\delta_n \oplus 0) + \psi a_n b_n(\beta_n \oplus 0) \\
&\quad + a_n(\alpha_n \oplus 0)\big].
\end{aligned}
$$

Using definition of normal cone and Proposition 2.2, we have

$$
\begin{aligned}
\|p_{n+1} - p^*\| &\leq (1 - N a_n(1-\psi))\|p_n - p^*\| \\
&+ N a_n(1-\psi)\left(\frac{\psi^2\|\delta_n \vee (-\delta_n)\| + \psi\|\beta_n \vee (-\beta_n)\| + \|\alpha_n \vee (-\alpha_n)\|}{(1-\psi)}\right).
\end{aligned}
$$

(4.9)

On setting $\eta_n = \frac{\psi^2\|\delta_n\vee(-\delta_n)\|+\psi\|\beta_n\vee(-\beta_n)\|+\|\alpha_n\vee(-\alpha_n)\|}{(1-\psi)}$; $\chi_n = \|p_n - p^*\|$; $\zeta_n = Na_n(1-\psi)$ (4.9) can be written as

$$\chi_n \leq (1-\zeta_n)\chi_n + \zeta_n\eta_n. \tag{4.10}$$

From Lemma 2.3 and using the hypothesis $\lim_{n\to\infty}\|\alpha_n\vee(-\alpha_n)\| = \lim_{n\to\infty}\|\beta_n\vee(-\beta_n)\| = \lim_{n\to\infty}\|\delta_n\vee(-\delta_n)\| = 0$, we can deduce that $\chi_n \to 0$, as $n\to\infty$, and so $\{p_n\}$ converges strongly to a unique solution p^* of GNMOIQVIP (3.1).

(II). Let $H(p^*) = \mathscr{J}^A_{\lambda,M}[A(p^*)\oplus\frac{\lambda}{\tau}(P(p^*)\oplus\xi Q(h(p^*)\oplus F(p^*,g(p^*))))]$. Using Algorithm 4.1 and Proposition 2.1, we obtain

$$\begin{aligned}
0 &\leq u_{n+1}\oplus p^* \\
&\leq u_{n+1}\oplus\big((1-a_n)u_n + a_nH(t_n) + a_n\alpha_n\big) \\
&\quad +\big((1-a_n)u_n + a_nH(t_n) + a_n\alpha_n\big)\oplus\big((1-a_n)p^* + a_nH(p^*)\big) \\
&\leq \big[u_{n+1}\oplus\big((1-a_n)u_n + a_nH(t_n) + a_n\alpha_n\big)\big] \\
&\quad +(1-a_n)(u_n\oplus p^*) + a_n(H(t_n)\oplus H(p^*)) + a_n(\alpha_n\oplus 0) \\
&\leq \big[u_{n+1}\oplus\big((1-a_n)u_n + a_nH(t_n) + a_n\alpha_n\big)\big] \\
&\quad +(1-a_n)(u_n\oplus p^*) + a_n\psi(t_n\oplus p^*) + a_n(\alpha_n\oplus 0),
\end{aligned} \tag{4.11}$$

$$\text{where } \psi = \left[\frac{\mu\big(\delta_A\tau\oplus\lambda(\delta_P\oplus|\xi|\delta_Q(\delta_h\oplus(\pi+\nu\delta_g)))\big)}{\gamma\tau(\lambda\alpha_A\oplus\mu)}\right].$$

From (4.11), we have

$$\begin{aligned}
0 &\leq t_n\oplus p^* \\
&\leq \big((1-b_n)u_n + b_nH(s_n) + b_n\beta_n\big)\oplus\big((1-b_n)p^* + b_nH(p^*)\big) \\
&\leq (1-b_n)(u_n\oplus p^*) + b_n(H(s_n)\oplus H(p^*)) + b_n(\beta_n\oplus 0) \\
&\leq (1-b_n)(u_n\oplus p^*) + b_n\psi(s_n\oplus p^*) + b_n(\beta_n\oplus 0).
\end{aligned} \tag{4.12}$$

Similarly, from (4.12), we get

$$\begin{aligned}
0 &\leq s_n \oplus p^* \\
&\leq \big((1-c_n)u_n + c_n H(u_n) + c_n\delta_n\big) \oplus \big((1-c_n)p^* + b_n H(p^*)\big) \\
&\leq (1-c_n)(u_n \oplus p^*) + c_n(H(u_n) \oplus H(p^*)) + c_n(\delta_n \oplus 0) \\
&\leq (1-c_n)(u_n \oplus p^*) + c_n\psi(u_n \oplus p^*) + c_n(\delta_n \oplus 0) \\
&\leq (1-c_n(1-\psi))(u_n \oplus p^*) + c_n(\delta_n \oplus 0) \\
&\leq (u_n \oplus p^*) + c_n(\delta_n \oplus 0), \text{ since } (1-c_n(1-\psi)) \leq 1. \qquad (4.13)
\end{aligned}$$

Using (4.13), (4.12) becomes

$$\begin{aligned}
0 &\leq t_n \oplus p^* \\
&\leq (1-b_n)(t_n \oplus p^*) + \psi b_n((t_n \oplus p^*) + c_n(\delta_n \oplus 0)) + b_n(\beta_n \oplus 0) \\
&\leq (1-b_n(1-\psi))(t_n \oplus p^*) + \psi c_n(\delta_n \oplus 0) + b_n(\beta_n \oplus 0) \\
&\leq (t_n \oplus p^*) + \psi c_n(\delta_n \oplus 0) + b_n(\beta_n \oplus 0), \text{ since } (1-b_n(1-\psi)) \leq 1. \qquad (4.14)
\end{aligned}$$

By using (4.14) and (4.12), (4.11) becomes as

$$\begin{aligned}
0 &\leq u_{n+1} \oplus p^* \\
&\leq \big[u_{n+1} \oplus \big((1-a_n)u_n + a_n H(t_n) + a_n\alpha_n\big)\big] + (1-a_n)(u_n \oplus p^*) \\
&\quad + \psi a_n\big[(u_n \oplus p^*) + \psi c_n(\delta_n \oplus 0) + b_n(\beta_n \oplus 0)\big] + a_n(\alpha_n \oplus 0)
\end{aligned}$$

$$\begin{aligned}
&\leq \big[u_{n+1} \oplus \big((1-a_n)u_n + a_n H(t_n) + a_n\alpha_n\big)\big] + (1-a_n(1-\psi))(u_n \oplus p^*) \\
&\quad + \psi a_n\big[\psi c_n(\delta_n \oplus 0) + b_n(\beta_n \oplus 0)\big] + a_n(\alpha_n \oplus 0) \\
&\leq \big[u_{n+1} \oplus \big((1-a_n)u_n + a_n H(t_n) + a_n\alpha_n\big)\big] + (1-a_n(1-\psi))(u_n \oplus p^*) \\
&\quad + \big[\psi^2 a_n c_n(\delta_n \oplus 0) + \psi a_n b_n(\beta_n \oplus 0) + a_n(\alpha_n \oplus 0)\big].
\end{aligned}$$

Using definition of normal cone and Proposition 2.2, we have

$$\begin{aligned}
\|u_{n+1}-p^*\| &\leq N\|u_{n+1}-\big((1-a_n)u_n+a_nH(t_n)+a_n\alpha_n\big)\| \\
&+ (1-Na_n(1-\psi))\|u_n-p^*\| \\
&+ Na_n(1-\psi)\left(\frac{\psi^2\|\delta_n\vee(-\delta_n)\|+\psi\|\beta_n\vee(-\beta_n)\|+\|\alpha_n\vee(-\alpha_n)\|}{(1-\psi)}\right) \\
&\leq N\vartheta_n+(1-a_n(1-\psi))\|u_n-p^*\| \\
&+ Na_n(1-\psi)\left(\frac{\psi^2\|\delta_n\vee(-\delta_n)\|+\psi\|\beta_n\vee(-\beta_n)\|+\|\alpha_n\vee(-\alpha_n)\|}{(1-\psi)}\right). \qquad (4.15)
\end{aligned}$$

Since $0<\kappa\leq a_n$, (4.15) becomes

$$\begin{aligned}
\|u_{n+1}-p^*\| &\leq (1-Na_n(1-\psi))\|u_n-p^*\|+a_nN(1-\psi)\Bigg(\frac{\vartheta_n}{\kappa(1-\psi)} \\
&+ \frac{\psi^2\|\delta_n\vee(-\delta_n)\|+\psi\|\beta_n\vee(-\beta_n)\|+\|\alpha_n\vee(-\alpha_n)\|}{(1-\psi)}\Bigg). \qquad (4.16)
\end{aligned}$$

Assume that $\lim\limits_{n\to\infty}\vartheta_n=0$, hence $\lim\limits_{n\to\infty}u_n=p^*$, where $\lim\limits_{n\to\infty}\|\alpha_n\vee(-\alpha_n)\|=\lim\limits_{n\to\infty}\|\beta_n\vee(-\beta_n)\|=\lim\limits_{n\to\infty}\|\delta_n\vee(-\delta_n)\|=0$.

Conversely, suppose that $\lim\limits_{n\to\infty}u_n=p^*$. From (4.4) and $\lim\limits_{n\to\infty}\|\alpha_n\vee(-\alpha_n)\|=\lim\limits_{n\to\infty}\|\beta_n\vee(-\beta_n)\|=\lim\limits_{n\to\infty}\|\delta_n\vee(-\delta_n)\|=0$, we have

$$\begin{aligned}
0 &\leq u_{n+1}\oplus\big[(1-a_n)u_n+a_nH(t_n)+a_n\alpha_n\big] \\
&\leq u_{n+1}\oplus p^*+\big[((1-a_n)u_n+a_nH(t_n)+a_n\alpha_n)\oplus p^*\big] \\
&\leq u_{n+1}\oplus p^*+\big[((1-a_n)u_n+a_nH(t_n)+a_n\alpha_n)\oplus((1-a_n)p^*+a_nH(t_n)p^*)\big]
\end{aligned}$$

$$\begin{aligned}
&\leq u_{n+1}\oplus p^*+(1-a_n)(u_n\oplus p^*)+a_n(H(t_n)\oplus H(p^*))+a_n(\alpha_n\oplus 0) \\
&\leq u_{n+1}\oplus p^*+(1-a_n)(u_n\oplus p^*)+a_n\psi(t_n\oplus p^*)+a_n(\alpha_n\oplus 0) \\
&\leq (u_{n+1}\oplus p^*)+(1-a_n(1-\psi))(u_n\oplus p^*)+a_n\psi\big[\psi^2(\delta_n\oplus 0) \\
&+\psi(\beta_n\oplus 0)+(\alpha_n\oplus 0)\big]. \qquad (4.17)
\end{aligned}$$

Again applying the definition of normal cone and Proposition 2.2, it follows that

$$
\begin{aligned}
\vartheta_n &= \|u_{n+1} - \big[(1-a_n)u_n + a_n H(t_n) + a_n\alpha_n\big]\| \\
&\leq N\|u_{n+1} - p^*\| + N(1-a_n(1-\psi))\|u_n - p^*\| \\
&\quad + a_n N\psi\big[\psi^2\|\delta_n \vee (-\delta_n)\| + \psi\|\beta_n \vee (-\beta_n)\| + \|\alpha_n \vee (-\alpha_n)\|\big], \qquad (4.18)
\end{aligned}
$$

which implies that

$$
\lim_{n\to\infty} \vartheta_n = 0. \qquad (4.19)
$$

Hence, the iterative sequence $\{p_n\}$ generated by (4.2) is stable with respect to $\mathscr{J}^A_{\lambda,M}$. This completes the proof. □

If we take $\tau, \xi = 1$, $g, F = 0$, $h = I$ (identity mapping) and $Q(p) = \omega$, $(\omega \in \mathscr{H}_p)$ in GN-MOIQVIP (3.1), we suggest and analyze another class of perturbed three-step iterative scheme to get the following convergence analysis and stability of the problem (3.2).

We can obtain the following corollaries for Theorem 4.1.

Corollary 4.1. *Let $\mathscr{H}_p$ be a real ordered Hilbert space and C be a normal cone with normal constant N. Let $A, P : \mathscr{H}_p \to \mathscr{H}_p$ be the single-valued mappings such that A is δ_A-ordered compression and γ-ordered non-extended mapping and P is comparison, δ_p-ordered compression mapping, respectively. Let $M : \mathscr{H}_p \to 2^{\mathscr{H}_p}$ be an (α_A, λ)-XOR-weak-ANODD set-valued mapping. In addition, if A, P, M and $[A(.) \oplus \lambda(P(.) \oplus \omega)]$ are compared to each other, and for all $\lambda > 0$, the following conditions are satisfied:*

$$
\begin{cases}
|\mu(\delta_A \oplus \lambda\delta_P| < |\gamma(\lambda\alpha_A \oplus \mu)| \min\{\frac{1}{N}, 1\}, \\
\lambda\alpha_A > \mu \text{ and } \mu \geq 1.
\end{cases} \qquad (4.20)
$$

For a given $p_0 \in \mathscr{H}_p$, let the sequences $\{p_n\}$, $\{q_n\}$ and $\{z_n\}$ are defined by the following schemes:

$$
\begin{cases}
p_{n+1} = (1-a_n)p_n + a_n\Big(\mathscr{J}^A_{\lambda,M}[A(q_n) \oplus \lambda(\omega \oplus P(q_n))]\Big) + a_n\alpha_n, \\
q_n = (1-b_n)p_n + b_n\Big(\mathscr{J}^A_{\lambda,M}[A(z_n) \oplus \lambda(\omega \oplus P(z_n))]\Big) + b_n\beta_n, \\
z_n = (1-c_n)p_n + c_n\Big(\mathscr{J}^A_{\lambda,M}[A(p_n) \oplus \lambda(\omega \oplus P(p_n))]\Big) + c_n\delta_n.
\end{cases} \qquad (4.21)
$$

Let $\{u_n\}$ be any sequence in $\mathscr{H}_p$ and define $\{\vartheta_n\}$ by

$$
\begin{cases}
\vartheta_n = \left\| u_{n+1} - \left[(1-a_n)u_n + a_n \Big(\mathscr{J}^A_{\lambda,M}[A(t_n) \oplus \lambda(\omega \oplus P(t_n))] \Big) + a_n \alpha_n \right] \right\|, \\
t_n = (1-b_n)u_n + b_n \Big(\mathscr{J}^A_{\lambda,M}[A(s_n) \oplus \lambda(\omega \oplus P(s_n))] \Big) + b_n \beta_n, \\
s_n = (1-c_n)u_n + c_n \Big(\mathscr{J}^A_{\lambda,M}[A(u_n) \oplus \lambda(\omega \oplus P(u_n))] \Big) + c_n \delta_n,
\end{cases} \tag{4.22}
$$

where $0 \leq a_n, b_n, c_n \leq 1$, $\sum\limits_{n=0}^{\infty} a_n = \infty, \forall\, n \geq 0$. Here $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\delta_n\}$ are three sequences in $\mathscr{H}_p$ introduced to take into account the possible inexact computations provided that $\alpha_n \oplus 0 = \alpha_n$, $\beta_n \oplus 0 = \beta_n$ and $\delta_n \oplus 0 = \delta_n, \forall\, n \geq 0$. If $\lim\limits_{n\to\infty} \|\alpha_n \vee (-\alpha_n)\| = \lim\limits_{n\to\infty} \|\beta_n \vee (-\beta_n)\| = \lim\limits_{n\to\infty} \|\delta_n \vee (-\delta_n)\| = 0$, then

- (I) *the sequence $\{p_n\}$ converges strongly to the unique solution p^* of the problem (3.2).*
- (II) *moreover, if $0 < \kappa \leq a_n$, then $\lim\limits_{n\to\infty} u_n = p^*$ if and only if $\lim\limits_{n\to\infty} \vartheta_n = 0$, where ϑ_n is defined (4.22) i.e., the sequence $\{p_n\}$ generated by (4.21) is stable with respect to $\mathscr{J}^A_{\lambda,M}$.*

On taking $h, P = 0$ and $Q = I$ (identity mapping) in GNMOIQVIP (3.1), we suggest and analyze another class of perturbed three-step iterative scheme to get the following convergence analysis and stability of the problem (3.3).

Corollary 4.2. *Let $\mathscr{H}_p$ be a real ordered Hilbert space and C be a normal cone with normal constant N. Let $A, g : \mathscr{H}_p \to \mathscr{H}_p$ and $F : \mathscr{H}_p \times \mathscr{H}_p \to \mathscr{H}_p$ be the single-valued mappings such that A is δ_A-ordered compression and γ-ordered non-extended mapping, g is comparison, δ_g-ordered compression mapping and F is comparison and (π, ν)-ordered Lipschitz continuous mapping with respect to g, respectively. Let $M : \mathscr{H}_p \to 2^{\mathscr{H}_p}$ be an (α_A, λ)-XOR-weak-ANODD set-valued mapping. In addition, if A, g, F, M and $[A(.) \oplus \frac{\lambda}{\tau}(\xi F(., g(.)))]$ are compared to each other, and for all $\tau, \lambda > 0$, the following conditions are satisfied:*

$$
\begin{cases}
|\mu(\delta_A \tau \oplus \lambda \xi(\pi + \nu \delta_g)| < |\tau \gamma(\lambda \alpha_A \oplus \mu)| \min\{\frac{1}{N}, 1\}, \\
\lambda \alpha_A > \mu \text{ and } \mu \geq 1.
\end{cases} \tag{4.23}
$$

For a given $p_0 \in \mathscr{H}_p$, let the sequences $\{p_n\}$, $\{q_n\}$ and $\{z_n\}$ are defined by the following schemes:

$$(4.24)\qquad \begin{cases} p_{n+1} = (1-a_n)p_n + a_n\Big(\mathscr{J}^A_{\lambda,M}[A(q_n)\oplus\frac{\lambda}{\tau}(\xi F(q_n,g(q_n)))]\Big) + a_n\alpha_n, \\ q_n = (1-b_n)p_n + b_n\Big(\mathscr{J}^A_{\lambda,M}[A(z_n)\oplus\frac{\lambda}{\tau}(\xi F(z_n,g(z_n)))]\Big) + b_n\beta_n, \\ z_n = (1-c_n)p_n + c_n\Big(\mathscr{J}^A_{\lambda,M}[A(p_n)\oplus\frac{\lambda}{\tau}(\xi F(p_n,g(p_n)))]\Big) + c_n\delta_n. \end{cases}$$

Let $\{u_n\}$ be any sequence in $\mathscr{H}_p$ and define $\{\vartheta_n\}$ by

$$(4.25)\qquad \begin{cases} \vartheta_n = \Big\|u_{n+1} - \Big[(1-a_n)u_n + a_n\Big(\mathscr{J}^A_{\lambda,M}[A(t_n)\oplus\frac{\lambda}{\tau}(\xi F(t_n,g(t_n)))]\Big) + a_n\alpha_n\Big]\Big\|, \\ t_n = (1-b_n)u_n + b_n\Big(\mathscr{J}^A_{\lambda,M}[A(s_n)\oplus\frac{\lambda}{\tau}(\xi F(s_n,g(s_n)))]\Big) + b_n\beta_n, \\ s_n = (1-c_n)u_n + c_n\Big(\mathscr{J}^A_{\lambda,M}[A(u_n)\oplus\frac{\lambda}{\tau}(\xi F(u_n,g(u_n)))]\Big) + c_n\delta_n, \end{cases}$$

where $0 \le a_n, b_n, c_n \le 1$, $\sum_{n=0}^{\infty} a_n = \infty, \forall\, n \ge 0$. Here $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\delta_n\}$ are three sequences in $\mathscr{H}_p$ introduced to take into account the possible inexact computations provided that $\alpha_n \oplus 0 = \alpha_n$, $\beta_n \oplus 0 = \beta_n$ and $\delta_n \oplus 0 = \delta_n, \forall\, n \ge 0$. If $\lim_{n\to\infty}\|\alpha_n \vee (-\alpha_n)\| = \lim_{n\to\infty}\|\beta_n \vee (-\beta_n)\| = \lim_{n\to\infty}\|\delta_n \vee (-\delta_n)\| = 0$, then

- (I) *the sequence $\{p_n\}$ converges strongly to the unique solution p^* of the problem (3.3).*
- (II) *moreover, if $0 < \kappa \le a_n$, then $\lim_{n\to\infty} u_n = p^*$ if and only if $\lim_{n\to\infty}\vartheta_n = 0$, where ϑ_n is defined (4.25) i.e., the sequence $\{p_n\}$ generated by (4.24) is stable with respect to $\mathscr{J}^A_{\lambda,M}$.*

By setting $\xi = -1$, $h, P = 0$, $Q = I$ (identity mapping)and $F(p,g(p)) = f(p) - \omega$, $(\omega \in \mathscr{H}_p)$ in GNMOIQVIP (3.1), we suggest and analyze another class of perturbed three-step iterative scheme to get the following convergence analysis and stability of the problem (3.4).

Corollary 4.3. *Let $\mathscr{H}_p$ be a real ordered Hilbert space and C be a normal cone with normal constant N. Let $A, f : \mathscr{H}_p \to \mathscr{H}_p$ be the single-valued mappings such that A is δ_A-ordered compression and γ-ordered non-extended mapping, f is comparison, δ_f-ordered compression mapping, respectively. Let $M : \mathscr{H}_p \to 2^{\mathscr{H}_p}$ be an (α_A, λ)-XOR-weak-ANODD set-valued mapping. In addition, if A, f, M and $[A(.)\oplus\frac{\lambda}{\tau}(\omega - f(.))]$ are compared to each other, and for all*

$\tau,\lambda > 0$, *the following conditions are satisfied:*

$$\text{(4.26)} \qquad \begin{cases} |\mu(\tau\delta_A \oplus \lambda\delta_f)| < |\tau\gamma(\lambda\alpha_A \oplus \mu)| \min\{\frac{1}{N}, 1\}, \\ \lambda\alpha_A > \mu \text{ and } \mu \geq 1. \end{cases}$$

For a given $p_0 \in \mathscr{H}_p$, *let the sequences* $\{p_n\}$, $\{q_n\}$ *and* $\{z_n\}$ *defined by the following schemes:*

$$\text{(4.27)} \qquad \begin{cases} p_{n+1} = (1-a_n)p_n + a_n\Big(\mathscr{J}^A_{\lambda,M}[A(q_n) \oplus \frac{\lambda}{\tau}(\omega - f(q_n))]\Big) + a_n\alpha_n, \\ q_n = (1-b_n)p_n + b_n\Big(\mathscr{J}^A_{\lambda,M}[A(z_n) + \frac{\lambda}{\tau}(\omega - f(z_n))]\Big) + b_n\beta_n, \\ z_n = (1-c_n)p_n + c_n\Big(\mathscr{J}^A_{\lambda,M}[A(p_n) + \frac{\lambda}{\tau}(\omega - f(p_n))]\Big) + c_n\delta_n. \end{cases}$$

Let $\{u_n\}$ *be any sequence in* $\mathscr{H}_p$ *and define* $\{\vartheta_n\}$ *by*

$$\text{(4.28)} \qquad \begin{cases} \vartheta_n = \Big\|u_{n+1} - \Big[(1-a_n)u_n + a_n\Big(\mathscr{J}^A_{\lambda,M}[A(t_n) + \frac{\lambda}{\tau}(\omega - f(t_n))]\Big) + a_n\alpha_n\Big]\Big\|, \\ t_n = (1-b_n)u_n + b_n\Big(\mathscr{J}^A_{\lambda,M}[A(s_n) + \frac{\lambda}{\tau}(\omega - f(s_n))]\Big) + b_n\beta_n, \\ s_n = (1-c_n)u_n + c_n\Big(\mathscr{J}^A_{\lambda,M}[A(u_n) + \frac{\lambda}{\tau}(\omega - f(u_n))]\Big) + c_n\delta_n, \end{cases}$$

where $0 \leq a_n, b_n, c_n \leq 1$, $\sum\limits_{n=0}^{\infty} a_n = \infty, \forall\, n \geq 0$. *Here* $\{\alpha_n\}$, $\{\beta_n\}$ *and* $\{\delta_n\}$ *are three sequences in* $\mathscr{H}_p$ *introduced to take into account the possible inexact computations provided that* $\alpha_n \oplus 0 = \alpha_n$, $\beta_n \oplus 0 = \beta_n$ *and* $\delta_n \oplus 0 = \delta_n, \forall\, n \geq 0$. *If* $\lim\limits_{n\to\infty} \|\alpha_n \vee (-\alpha_n)\| = \lim\limits_{n\to\infty} \|\beta_n \vee (-\beta_n)\| = \lim\limits_{n\to\infty} \|\delta_n \vee (-\delta_n)\| = 0$, *then*

(I) *the sequence* $\{p_n\}$ *converges strongly to the unique solution* p^* *of the problem (3.4).*

(II) *moreover, if* $0 < \kappa \leq a_n$, *then* $\lim\limits_{n\to\infty} u_n = p^*$ *if and only if* $\lim\limits_{n\to\infty} \vartheta_n = 0$, *where* ϑ_n *is defined (4.28). That is, the sequence* $\{p_n\}$ *generated by (4.27) is stable with respect to* $\mathscr{J}^A_{\lambda,M}$.

If we take $P, g, F = 0, h = I$ (identity mapping) and $Q(p) = 1$ (constant mapping) in GN-MOIQVIP (3.1), we suggest and analyze another class of perturbed three-step iterative schemes get the following convergence analysis and stability of the problem (3.5).

Corollary 4.4. *Let* $\mathscr{H}_p$ *be a real ordered Hilbert space and* C *be a normal cone with normal constant* N. *Let* $A : \mathscr{H}_p \to \mathscr{H}_p$ *be the single-valued mapping such that* A *is comparison,* δ_A-*ordered compression and* γ-*ordered non-extended mapping, respectively. Let* $M : \mathscr{H}_p \to 2^{\mathscr{H}_p}$ *be* *an (α_A,λ)-XOR-weak-ANODD set-valued mapping. In addition, if A,M and $[A(.)\oplus\frac{\lambda}{\tau}\xi]$ are compared to each other, and for all $\tau,\lambda>0$, the following conditions are satisfied:*

$$\begin{cases}|\mu\delta_A|<|\tau\gamma(\lambda\alpha_A\oplus\mu)|\min\{\frac{1}{N},1\},\\ \lambda\alpha_A>\mu \text{ and } \mu\geq 1.\end{cases}\tag{4.29}$$

For a given $p_0\in\mathscr{H}_p$, let the sequences $\{p_n\}$, $\{q_n\}$ and $\{z_n\}$ defined by the following schemes:

$$\begin{cases}p_{n+1}=(1-a_n)p_n+a_n\Big(\mathscr{J}^A_{\lambda,M}[A(q_n)\oplus\frac{\lambda}{\tau}\xi]\Big)+a_n\alpha_n,\\ q_n=(1-b_n)p_n+b_n\Big(\mathscr{J}^A_{\lambda,M}[A(z_n)\oplus\frac{\lambda}{\tau}\xi]\Big)+b_n\beta_n,\\ z_n=(1-c_n)p_n+c_n\Big(\mathscr{J}^A_{\lambda,M}[A(p_n)\oplus\frac{\lambda}{\tau}\xi]\Big)+c_n\delta_n.\end{cases}\tag{4.30}$$

Let $\{u_n\}$ be any sequence in $\mathscr{H}_p$ and define $\{\vartheta_n\}$ by

$$\begin{cases}\vartheta_n=\Big\|u_{n+1}-\Big[(1-a_n)u_n+a_n\Big(\mathscr{J}^A_{\lambda,M}[A(t_n)\oplus\frac{\lambda}{\tau}\xi]\Big)+a_n\alpha_n\Big]\Big\|,\\ t_n=(1-b_n)u_n+b_n\Big(\mathscr{J}^A_{\lambda,M}[A(s_n)\oplus\frac{\lambda}{\tau}\xi]\Big)+b_n\beta_n,\\ s_n=(1-c_n)u_n+c_n\Big(\mathscr{J}^A_{\lambda,M}[A(u_n)\oplus\frac{\lambda}{\tau}\xi]\Big)+c_n\delta_n,\end{cases}\tag{4.31}$$

where $0\leq a_n,b_n,c_n\leq 1$, $\sum\limits_{n=0}^{\infty}a_n=\infty,\forall\ n\geq 0$. Here $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\delta_n\}$ are three sequences in $\mathscr{H}_p$ introduced to take into account the possible inexact computations provided that $\alpha_n\oplus 0=\alpha_n$, $\beta_n\oplus 0=\beta_n$ and $\delta_n\oplus 0=\delta_n,\forall\ n\geq 0$. If $\lim\limits_{n\to\infty}\|\alpha_n\vee(-\alpha_n)\|=\lim\limits_{n\to\infty}\|\beta_n\vee(-\beta_n)\|=\lim\limits_{n\to\infty}\|\delta_n\vee(-\delta_n)\|=0$, then

- (I) *the sequence $\{p_n\}$ converges strongly to the unique solution p^* of the problem (3.5).*
- (II) *moreover, if $0<\kappa\leq a_n$, then $\lim\limits_{n\to\infty}u_n=p^*$ if and only if $\lim\limits_{n\to\infty}\vartheta_n=0$, where ϑ_n is defined (4.31). That is, the sequence $\{p_n\}$ generated by (4.30) is stable with respect to $\mathscr{J}^A_{\lambda,M}$.*

If we take $\xi=0,\tau=1$ and $P,g,h,F,Q=0$ in GNMOIQVIP (3.1), we suggest and analyze another class of perturbed three-step iterative schemes get the following convergence analysis and stability of the problem (3.6).

Corollary 4.5. *Let $\mathscr{H}_p$ be a real ordered Hilbert space and C be a normal cone with normal constant N. Let $A:\mathscr{H}_p\to\mathscr{H}_p$ be the single-valued mapping such that A is comparison, δ_A-ordered compression and γ-ordered non-extended mapping, respectively. Let $M:\mathscr{H}_p\to 2^{\mathscr{H}_p}$ be* *an (α_A,λ)-XOR-weak-ANODD set-valued mapping. In addition, if M and A are compared to each other, and for all $\lambda > 0$, the following conditions are satisfied:*

$$\begin{cases} |\mu\delta_A| < |\gamma(\lambda\alpha_A \oplus \mu)|\min\{\frac{1}{N},1\}, \\ \lambda\alpha_A > \mu \text{ and } \mu \geq 1. \end{cases} \tag{4.32}$$

For a given $p_0 \in \mathscr{H}_p$, let the sequences $\{p_n\}$, $\{q_n\}$ and $\{z_n\}$ defined by the following schemes:

$$\begin{cases} p_{n+1} = (1-a_n)p_n + a_n\Big(\mathscr{J}^A_{\lambda,M}[A(q_n)]\Big) + a_n\alpha_n, \\ q_n = (1-b_n)p_n + b_n\Big(\mathscr{J}^A_{\lambda,M}[A(z_n)]\Big) + b_n\beta_n, \\ z_n = (1-c_n)p_n + c_n\Big(\mathscr{J}^A_{\lambda,M}[A(p_n)]\Big) + c_n\delta_n. \end{cases} \tag{4.33}$$

Let $\{u_n\}$ be any sequence in $\mathscr{H}_p$ and define $\{\vartheta_n\}$ by

$$\begin{cases} \vartheta_n = \Big\|u_{n+1} - \Big[(1-a_n)u_n + a_n\Big(\mathscr{J}^A_{\lambda,M}[A(t_n)]\Big) + a_n\alpha_n\Big]\Big\|, \\ t_n = (1-b_n)u_n + b_n\Big(\mathscr{J}^A_{\lambda,M}[A(s_n)]\Big) + b_n\beta_n, \\ s_n = (1-c_n)u_n + c_n\Big(\mathscr{J}^A_{\lambda,M}[A(u_n)]\Big) + c_n\delta_n, \end{cases} \tag{4.34}$$

where $0 \leq a_n, b_n, c_n \leq 1$, $\sum\limits_{n=0}^{\infty} a_n = \infty, \forall\, n \geq 0$. Here $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\delta_n\}$ are three sequences in $\mathscr{H}_p$ introduced to take into account the possible inexact computations provided that $\alpha_n \oplus 0 = \alpha_n$, $\beta_n \oplus 0 = \beta_n$ and $\delta_n \oplus 0 = \delta_n, \forall\, n \geq 0$. If $\lim\limits_{n\to\infty}\|\alpha_n \vee (-\alpha_n)\| = \lim\limits_{n\to\infty}\|\beta_n \vee (-\beta_n)\| = \lim\limits_{n\to\infty}\|\delta_n \vee (-\delta_n)\| = 0$, then

- (I) *the sequence $\{p_n\}$ converges strongly to the unique solution p^* of the problem (3.6).*
- (II) *moreover, if $0 < \kappa \leq a_n$, then $\lim\limits_{n\to\infty} u_n = p^*$ if and only if $\lim\limits_{n\to\infty} \vartheta_n = 0$, where ϑ_n is defined (4.34). That is, the sequence $\{p_n\}$ generated by (4.33) is stable with respect to $\mathscr{J}^A_{\lambda,M}$.*

5. Numerical Example

In this section, we provide a numerical example to illustrate Algorithm 4.1 and justify our main result.

Example 5.1. *Let $\mathscr{H}_p = [0,\infty)$ with the usual inner product and norm, and let $C = [0,1]$ be a normal cone with normal constant $N = 1$. Let $A, P, g, h, Q : \mathscr{H}_p \to \mathscr{H}_p$ and $F : \mathscr{H}_p \times \mathscr{H}_p \to \mathscr{H}_p$*

be the mappings defined by

$$A(p) = \Big(\frac{p}{2} \oplus \frac{5}{6}\Big),\ P(p) = \Big(\frac{p}{3} \oplus \frac{5}{6}\Big),\ g(p) = \frac{p}{5}\ h(p) = \Big(\frac{p}{9} \oplus 4\Big),$$

$$Q(p) = \frac{6p}{17} \oplus \frac{11}{5} \text{ and } F(p, g(p)) = \Big(p + 5g(p)\Big) \oplus 4,\ \forall p \in \mathscr{H}_p.$$

For each $p, q \in \mathscr{H}_p,\ p \propto q$*. Then, it is easy to check that A is* $\frac{3}{4}$*-ordered compression and* $\frac{1}{4}$*-ordered non-extending mapping, P is* $\frac{2}{3}$*-ordered compression mapping, Q is* $\frac{3}{7}$*-ordered compression mapping, g is* $\frac{1}{3}$*-ordered compression mapping and h is* $\frac{1}{7}$*-ordered compression mapping, respectively.*

For $p, q, u, v \in \mathscr{H}_p, p \propto u, q \propto v$*, we calculate*

$$\begin{aligned}
F(p, g(u)) \oplus F(q, g(v)) &= ((p + 5g(u)) \oplus 4) \oplus ((q + 5g(v)) \oplus 4)\\
&= (((p + 5g(u))) \oplus (q + 5g(v))) \oplus (4 \oplus 4)\\
&= (((p + 5g(u))) \oplus (q + 5g(v))) \oplus 0\\
&= ((p + 5g(u))) \oplus (q + 5g(v))\\
&\leq (p \oplus q) + (5g(u) \oplus 5g(v))\\
&= (p \oplus q) + 5(g(u) \oplus g(v))\\
&= (p \oplus q) + 5\Big(\frac{u}{5} \oplus \frac{v}{5}\Big)\\
&= (p \oplus q) + (u \oplus v),
\end{aligned}$$

i.e.,

$$F(p, g(u)) \oplus F(q, g(v)) \leq (p \oplus q) + (u \oplus v),\ \forall p, q, u, v \in \mathscr{H}_p.$$

Hence, F is $(1, 1)$*-ordered Lipschitz continuous mapping with respect to g.*

Suppose that $M : \mathscr{H}_p \to CB(\mathscr{H}_p)$ *is a set-valued mapping defined by*

$$M(p) = \{4p \oplus \frac{5}{6}\}, \forall p \in \mathscr{H}_p.$$

It can be easily verified that M is a comparison mapping and $(4, 1)$*-XOR-weak-ANODD set-valued mapping. Thus, M is a* 4*-weak-non-ordinary difference mapping with respect to A and* 1*-XOR-ordered different weak comparison mapping. Also, it is clear that for* $\lambda = 1$*,* $[A \oplus \lambda M](\mathscr{H}_p) = \mathscr{H}_p$*. Hence, M is an* $(4, 1)$*-XOR-weak-ANODD set-valued mapping.*

The resolvent operator defined by (2.1) *associated with A and M is given by*

$$\mathscr{J}^{A}_{\lambda,M}(p) = \frac{2p}{7}, \forall p \in \mathscr{H}_p, \tag{5.1}$$

It is easy to examine that the resolvent operator defined above is comparison and single-valued mapping.

In particular for $\mu = 1$, *we obtain*

$$\begin{aligned} \mathscr{J}^{A}_{\lambda,M}(p) \oplus \mathscr{J}^{A}_{\lambda,M}(q) &= \frac{2p}{7} \oplus \frac{2p}{7} \\ &\leq \frac{4}{3}(p \oplus q), \end{aligned}$$

i.e.,

$$\mathscr{J}^{A}_{\lambda,M}(p) \oplus \mathscr{J}^{A}_{\lambda,M}(q) \leq \frac{4}{3}(p \oplus q), \forall p, q \in \mathscr{H}_p.$$

That is, the resolvent operator $\mathscr{J}^{A}_{\lambda,M}$ *is* $\frac{4}{3}$*-ordered Lipschitz continuous.*

If we take $\tau = 1$ *and* $\xi = 1$, *we calculate*

$$\begin{aligned} \mathscr{J}^{A}_{\lambda,M}[A(p) &\oplus \frac{\lambda}{\tau}(P(p) \oplus \xi Q(h(p) \oplus F(p, g(p))))] \\ &= \mathscr{J}^{A}_{\lambda,M}[A(p) \oplus (P(p) \oplus Q(h(p) \oplus F(p, g(p))))] \\ &= \frac{2[A(p) \oplus (P(p) \oplus Q(h(p) \oplus F(p, g(p))))]}{7} \\ &= \frac{2\Big[\Big(\frac{p}{2} \oplus 1\Big) \oplus \Big(\Big(\frac{p}{3} \oplus \frac{5}{6}\Big) \oplus Q(h(p) \oplus F(p, g(p)))\Big)\Big]}{7} \\ &= \frac{2}{7}\Big(\frac{p}{2} \oplus \frac{5}{6} \oplus \Big(\frac{p}{3} \oplus \frac{5}{6} \oplus Q((\frac{p}{9} \oplus 4) \oplus (2p \oplus 4))\Big)\Big) \\ &= \frac{2}{7}\Big(\frac{p}{2} \oplus \frac{5}{6} \oplus \Big(\frac{p}{3} \oplus \frac{5}{6} \oplus Q((\frac{p}{9} \oplus 2p))\Big)\Big) \\ &= \frac{2}{7}\Big(\frac{p}{2} \oplus \frac{5}{6} \oplus \Big(\frac{p}{3} \oplus \frac{5}{6} \oplus Q\Big(\frac{17p}{9}\Big)\Big)\Big) \\ &= \frac{2}{7}\Big[\frac{p}{2} \oplus \frac{5}{6} \oplus \Big(\Big(\frac{p}{3} \oplus \frac{5}{6}\Big) \oplus \Big(\frac{6}{17}\frac{17p}{9}\Big) \oplus \frac{11}{5}\Big)\Big] \end{aligned}$$

$$\begin{aligned} &= \frac{2}{7}\Big[\frac{p}{2}\oplus\frac{5}{6}\oplus\frac{5}{6}\oplus\frac{p}{3}\oplus\frac{2p}{3}\oplus\frac{11}{5}\Big] \\ &= \frac{p}{21}\oplus\frac{22}{35}. \end{aligned}$$

Clearly, $\frac{3}{5}$ *is a fixed point of* $\mathscr{J}^{A}_{\lambda,M}[A(p)+(P(p)\oplus Q(h(p)\oplus F(p,g(p))))]$.

Let $a_n=\frac{1}{n+1}, b_n=\frac{3n}{3n+1}, c_n=\frac{2n+1}{2n+3}, \alpha_n=\frac{2}{5n}, \beta_n=\frac{1}{4n+1}$ *and* $\gamma_n=\frac{2n+3}{3n^2+5}$*. It is easy to show that the sequences* $\{a_n\},\{b_n\},\{c_n\},\{\alpha_n\},\{\beta_n\}$ *and* $\{\gamma_n\}$ *satisfying the conditions* $0\le a_n,b_n,c_n\le 1$, $\sum_{n=0}^{\infty} a_n=\infty$, $\alpha_n\oplus 0=\alpha_n, \beta_n\oplus 0=\beta_n$ *and* $\gamma_n\oplus 0=\gamma_n$.

Now, we can estimate the sequences $\{p_n\},\{q_n\}$ *and* $\{z_n\}$ *by the following schemes:*

$$\begin{aligned} p_{n+1} &= \frac{n}{n+1}p_n+\frac{1}{n+1}\Big(\frac{q_n}{21}\oplus\frac{4}{7}\Big)+\frac{2}{(n+1)(5n+1)}, \\ q_n &= \frac{1}{3n+1}p_n+\frac{3n}{3n+1}\Big(\frac{z_n}{21}\oplus\frac{4}{7}\Big)+\frac{3n}{(3n+1)(4n+1)}, \\ z_n &= \frac{2}{2n+3}p_n+\frac{2n+1}{2n+3}\Big(\frac{q_n}{21}\oplus\frac{4}{7}\Big)+\frac{2n+1}{3n^2+5}. \end{aligned}$$

It is also verified that condition (4.3) is satisfied. Thus, all the assumptions of Theorem 4.1 are fulfilled. Hence, the sequence $\{p_n\}$ *converges strongly to the unique solution* $p^*=\frac{3}{5}$ *of the problem* (3.1).

All codes are written in MATLAB Version 7.13*, we have the following different initial values* $p_0=4,6,8$ *which shows that the sequence* $\{p_n\}$ *converge to* $p^*=\frac{3}{5}$.

No. of Iteration	*For $p_0 = 4$* p_n	*For $p_0 = 6$* p_n	*For $p_0 = 8$* p_n
n=1	*4*	*6*	*8*
n=2	*6.287078*	*8.673421*	*11.2979*
n=3	*5.170505*	*7.302701*	*10.346027*
n=4	*3.653934*	*4.103810*	*6.1038*
n=5	*2.350428*	*2.587599*	*2.599085*
n=6	*1.872496*	*1.948989*	*2.190598*
n=7	*1.511503*	*1.659801*	*1.890222*
n=8	*1.101440*	*1.300220*	*1.120331*
n=9	*0.989578*	*0.998203*	*0.999360*
n=10	*0.903126*	*0.908721*	*0.907726*
n=15	*0.648478*	*0.641205*	*0.649038*
n=20	*0.600941*	*0.600831*	*0.600810*
n=25	*0.600043*	*0.600051*	*0.600046*
n=30	*0.600005*	*0.600007*	*0.600005*
n=35	*0.600002*	*0.600003*	*0.600001*

TABLE 1. The values of p_n with initial values $p_0 = 4$, $p_0 = 6$ and $p_0 = 8$

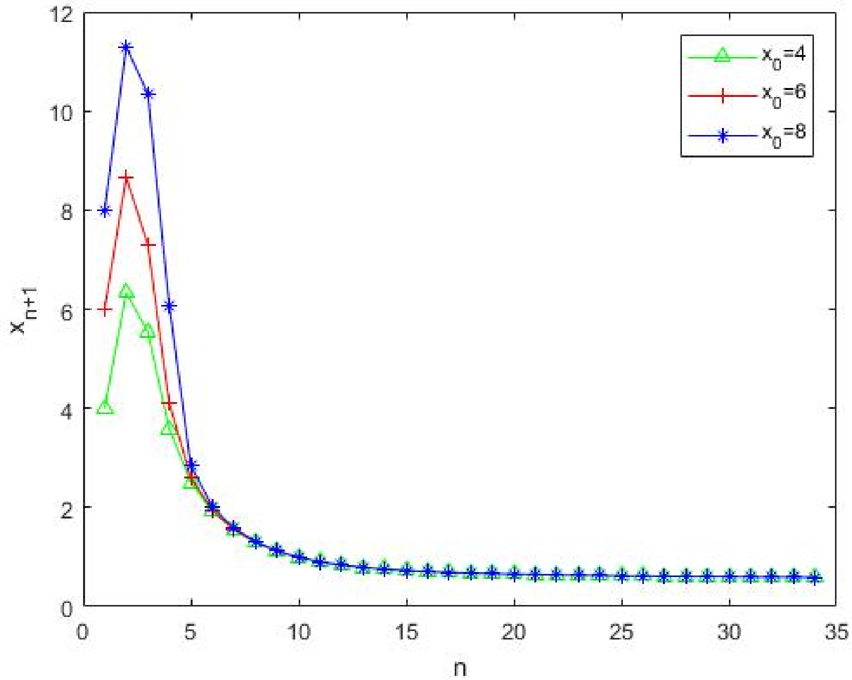

FIGURE 1. The convergence of p_n with initial values $p_0 = 4$, $p_0 = 6$ and $p_0 = 8$

6. CONCLUSION

In this article, we study a generalized mixed ordered quasi-variational inclusion problem based on XOR operator in a real ordered Hilbert spaces and prove the existence of solution. We have constructed a perturbed three-step iterative algorithm for this class of generalized mixed ordered quasi-variational inclusion problem which is more general than the Mann-type, Ishikawa-type iterative algorithms with errors, and many other iterative schemes studies by several author's, see e.g., [2–4, 10–14, 16, 17]. We prove the convergence analysis of our proposed algorithm which assumes that the suggested algorithm converges in norm to a unique solution of our considered problem and also show that the convergence is stable with some consequences. Finally, we give a numerical example in support of our main result. Our results extend and generalize most of the results of different authors existing in the literature.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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