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MIXED VARIATIONAL LIKE INEQUALITIES INVOLVING PERTURBED OPERATOR

TIRTH RAM*, RAVDEEP KOUR

Department of Mathematics, University of Jammu, Jammu 180006, India

Abstract. In this paper, we study mixed variational like inequalities involving perturbed operator (in short, PMVLI) in real Hilbert spaces. Furthermore, we prove the existence of solution of PMVLI by using strongly monotonicity and Lipschitz continuity of perturbed operator, and auxiliary principle technique. Some special cases have also been discussed. Our results in this paper are new which can be considered as a significant extension and refinement of previously known results in the literature.

Keywords: strongly pseudomonotone; Lipschitz continuity; fixed points; contraction mapping.

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1. INTRODUCTION

Since its birth in the mid 1960's, the theory of variational inequalities introduced by Stampacchia [14] has become a very effective and powerful tool for studying a wide range of problems arising in pure and applied sciences, such as differential equations, mechanics, contact problems in elasticity, control problems, general equilibrium problems in economics and transportation, and optimization problems etc. The variational- like inequality was first introduced by Aubin and Ekeland [2], and later, this theory is extended in different directions by numerous authors

*Corresponding author
E-mail address: tir1ram2@yahoo.com
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by using novel and innovative techniques, see for example, Bloch and Noor [3], Noor[7], Noor et al. [10], Kim et al. [13], Parida and Sen [11], and the references therein. The existence of solution of perturbed mixed variational like inequalities cannot be studied by resolvent and projection techniques, as these perturbed mixed variational like inequalities involve the nonlinear terms and bifunction $\eta(.,.)$. This fact motivated us to use auxiliary principle technique, introduced and developed by Glowinski et al.[4] to deal with the existence theory for variational like inequalities.

In this paper, we introduce and study a new class of variational-like inequalities, which is known as mixed variational like inequalities involving perturbed operator. To study the existence of solution of PMVLI, we use the auxiliary principle technique. Some special cases of PMVLI have also been presented.

2. Preliminaries

Let us assume $\mathscr{H}$ to be a real Hilbert space. Denote $\langle .,. \rangle$ and $\|.\|$ to be the inner product and norm, respectively. Let K_η be a nonempty invex set in $\mathscr{H}$ and $\eta(.,.) : \mathscr{H} \times \mathscr{H} \to \mathbb{R}$ be a continuous bifunction. Let us recall the following definitions and concepts:

Definition 2.1. [5] Let K be a nonempty closed set in $\mathscr{H}$. Then K is said to be *convex set*, if

$$x + \lambda(y - x) \in K, \forall x, y \in K,\ \lambda \in [0,1].$$

Definition 2.2. [5]Let K be a nonempty closed set in $\mathscr{H}$. Then a function $F : K \to \mathbb{R}$ is said to *convex function*, if

$$F(x + \lambda(y - x)) \leq (1 - \lambda)F(x) + \lambda F(y),\ \forall x, y \in K,\ \lambda \in [0,1].$$

Definition 2.3. [12] A function $F : K \to \mathbb{R}$ is said to be *strongly convex function*, if there exists a constant $\mu > 0$, such that

$$F(x + \lambda(y - x)) \leq (1 - \lambda)F(x) + \lambda F(y) - \mu\lambda(1 - \lambda)\|y - x\|^2,\ \forall x, y \in K,\ \lambda \in [0,1].$$

Definition 2.4. [15] A set K_η is said to be an invex set, if there exists a bifunction $\eta(.,.)$, such that

$$x + \lambda\eta(y, x) \in K_\eta,\ \forall\ x, y \in K_\eta, \lambda \in [0,1].$$

Definition 2.5. [15] A function $F : K_\eta \to \mathbb{R}$ is said to be a preinvex function, if there exists a bifunction $\eta(.,.)$, such that

$$F(x+\lambda\eta(y,x)) \leq (1-\lambda)F(x)+\lambda F(y),\ \forall\, x,y \in K_\eta, \lambda \in [0,1].$$

Definition 2.6. [9] A function $F : K_\eta \to \mathbb{R}$ is said to be a strongly preinvex function w.r.to a bifunction $\eta(.,.)$, if there exists a constant $\mu > 0$, such that

$$F(x+\lambda\eta(y,x)) \leq (1-\lambda)F(x)+\lambda F(y) - \mu\lambda(1-\lambda)\|\eta(y,x)\|^2,\ \forall\, x,y \in K_\eta, \lambda \in [0,1].$$

Remark 2.7. We note that, if $\eta(y,x) = y-x$, the invex set K_η reduces to the convex set K, preinvex functions and strongly preinvex functions reduce to convex functions and strongly convex functions, respectively.

Now consider the functional $I[y]$, defined as

$$I(y) = F(y)+\varphi(y), \tag{2.1}$$

where F is a differentiable preinvex function and φ is a non-differentiable strongly preinvex function. The minimum $x \in K_\eta$ of the functional $I(y)$, defined by (2.1), can be characterized by a class of variational- like inequalities.

$$\langle F'(x), \eta(y,x)\rangle + \varphi(y) - \varphi(x) + \mu\|\eta(y,x)\|^2 \geq 0,\ \forall\, y \in K_\eta. \tag{2.2}$$

The inequality (2.2) is called perturbed mixed variational-like inequality.

Given an operator $G : \mathscr{H} \to \mathbb{R}$ and a functional $\varphi : \mathscr{H} \to \mathbb{R}$, we consider the following mixed variational like inequalities involving perturbed operator (for short, PMVLI):

Find $x \in \mathscr{H}$ and constant $\mu > 0$, such that

$$\langle G_\varepsilon x, \eta(y,x)\rangle + \varphi(y) - \varphi(x) + \mu\|\eta(y,x)\|^2 \geq 0,\ \forall\, y \in \mathscr{H}, \tag{2.3}$$

where $G_\varepsilon = (G+\varepsilon I)$ is perturbed operator.

If $G_{\varepsilon}x = F'(x)$, then (PMVLI) (2.3) includes inequality (2.2) as a special case.

Now we need the following assumption, lemma and some important definitions for the establishment of main result:

Assumption 2.1: The bifunction $\eta(.,.) : \mathscr{H} \times \mathscr{H}$ satisfies the following condition

$$\eta(x,y) = \eta(x,z) + \eta(z,y),\ \forall x,y,z \in \mathscr{H}.$$

We assume that $\eta(y,x)$ is skew symmetric, i.e., $\eta(y,x) = -\eta(x,y),\ \forall x,y \in \mathscr{H}$. This assumption played an important part in the study of existence of a solution of variational like inequalities, see for example [6].

Definition 2.8. [7] An operator $G : \mathscr{H} \to \mathscr{H}$ is said to be:

(i) strongly $\eta-$ monotone, if there exists a constant $\alpha > 0$ such that

$$\langle G(x) - G(y), \eta(x,y) \rangle \geq \alpha \|\eta(x,y)\|,\ \forall x,y \in \mathscr{H},$$

(ii) $\eta-$ monotone, if

$$\langle G(x) - G(y), \eta(x,y) \rangle \geq 0,\ \forall x,y \in \mathscr{H},$$

(iii) partially relaxed strongly $\eta-$ monotone, if there exists a constant $\alpha > 0$ such that

$$\langle G(x) - G(y), \eta(z,y) \rangle \geq -\alpha \|\eta(x,y)\|^2,\ \forall x,y,z \in \mathscr{H}.$$

(iv) Lipschitz continuous, if there exists a constant $\beta > 0$ such that

$$\|G(x) - G(y)\| \leq \beta \|y - x\|,\ \forall x,y \in \mathscr{H}.$$

We note that for $z = x$, partially relaxed strongly monotonicity reduces to monotonicity.

Definition 2.9. [9] The bifunction $\eta(.,.) : \mathscr{H} \times \mathscr{H} \to \mathscr{H}$ is said to be:

(i) strongly monotone, if there exists a constant $\sigma > 0$ such that

$$\langle \eta(y,x), y - x \rangle \leq \sigma \|y - x\|^2,\ \forall x,y \in \mathscr{H},$$

(ii) Lipschitz continuous, if there exists a constant $\delta > 0$ such that

$$\|\eta(y,x)\| \leq \delta\|y-x\|, \ \forall x,y \in \mathscr{H}.$$

From (i) and (ii), we have seen that $\sigma \leq \delta$. If $\eta(y,x) = G(y) - G(x)$, then the above Definition 2.9 reduces to the strongly monotonicity and Lipschitz continuity of the nonlinear operator G.

Lemma 2.10. [1] *Suppose $\mathscr{H}$ is a real Hilbert space. If $G : \mathscr{H} \to \mathscr{H}$ be strongly monotone operator with constant $\alpha > 0$ and Lipschitz continuous with constant β, then the perturbed operator $G_\varepsilon = (G+\varepsilon I)$ is also strongly monotone with constant $\alpha+\varepsilon$ and Lipschitz continuous with constant $\beta+\varepsilon$, where I is identity operator.*

3. Main Result

In this section, we first prove the uniqueness of solution of perturbed mixed variational like inequalities (2.3). Also, we prove the existence of solution of (2.3) by using the auxiliary principle technique of Glowinski et al.[4].

Theorem 3.1. *Suppose the operator $G_\varepsilon : \mathscr{H} \to \mathscr{H}$, where $G_\varepsilon = G+\varepsilon I$, be strongly monotone and Lipschitz continuous with constants $\alpha+\varepsilon > 0$ and $\beta+\varepsilon > 0$, respectively. Suppose the bi-function $\eta(.,.) : \mathscr{H} \times \mathscr{H} \to \mathscr{H}$ be strongly monotone and Lipschitz continuous with constants $\sigma > 0, \delta > 0$, respectively. If Assumption 2.1 holds and there is a constant $\rho > 0$, such that*

$$0 < \rho < 2\frac{\alpha+\varepsilon-(\gamma+2\mu\delta^2)}{(\beta+\varepsilon)^2-(\gamma+2\mu\delta^2)^2}, \ \rho < \frac{1}{\gamma+2\mu\delta^2}, \ \alpha+\varepsilon > 2\mu\delta^2, \tag{3.1}$$

where

$$\gamma = (\beta+\varepsilon)\sqrt{1-2\sigma+\delta^2}. \tag{3.2}$$

Then, there exists a unique solution of PMVLI (2.3).

Proof. Uniqueness: Suppose $x_1 \neq x_2 \in \mathscr{H}$, be two solutions of problem (2.3). Then, we have

$$\langle G_\varepsilon x_1, \eta(y,x_1)\rangle + \varphi(y) - \varphi(x_1) \geq -\mu\|\eta(y,x_1)\|^2, \ \forall y \in \mathscr{H}. \tag{3.3}$$

and

$$\langle G_\varepsilon x_2, \eta(y,x_2)\rangle + \varphi(y) - \varphi(x_2) \geq -\mu\|\eta(y,x_2)\|^2,\ \forall y \in \mathscr{H}. \tag{3.4}$$

Replacing y by x_2 in (3.3) and y by x_1 in (3.4), adding the resultants and using the Assumption 2.1, we have

$$\langle G_\varepsilon x_1 - G_\varepsilon x_2, \eta(x_1,x_2)\rangle \leq 2\mu\|\eta(x_1,x_2)\|^2, \tag{3.5}$$

which can be written as

$$\langle G_\varepsilon x_1 - G_\varepsilon x_2, x_1 - x_2\rangle \leq \langle G_\varepsilon x_1 - G_\varepsilon x_2, x_1 - x_2 - \eta(x_1,x_2)\rangle + 2\mu\|\eta(x_1,x_2)\|^2.$$

Since G_ε is a strongly monotone and Lipschitz continuous with constants $\alpha+\varepsilon > 0$ and $\beta+\varepsilon > 0$, we have

$$\begin{aligned}
(\alpha+\varepsilon)\|x_1-x_2\|^2 &\leq \langle G_\varepsilon x_1 - G_\varepsilon x_2, x_1 - x_2\rangle \\
&\leq \|G_\varepsilon x_1 - G_\varepsilon x_2\|\|x_1 - x_2 - \eta(x_1,x_2)\| + 2\mu\|\eta(x_1,x_2)\|^2 \\
&\leq (\beta+\varepsilon)\|x_1-x_2\|\|x_1 - x_2 - \eta(x_1,x_2)\| + 2\mu\|\eta(x_1,x_2)\|^2 \\
&\leq (\beta+\varepsilon)\|x_1-x_2\|\|x_1 - x_2 - \eta(x_1,x_2)\| + 2\mu\delta^2\|x_1-x_2)\|^2. \qquad (3.6)
\end{aligned}$$

Since $\eta(.,.)$ is strongly monotone and Lipschitz continuous with constants $\sigma > 0$ and $\delta > 0$, we have

$$\begin{aligned}
\|x_1 - x_2 - \eta(x_1,x_2)\|^2 &\leq \|x_1-x_2\|^2 - 2\langle x_1 - x_2, \eta(x_1,x_2)\rangle + \|\eta(x_1,x_2)\|^2 \\
&\leq \left(1 - 2\sigma + \delta^2\right)\|x_1-x_2\|^2, \qquad (3.7)
\end{aligned}$$

From (3.2), (3.6) and (3.7), we have

$$\begin{aligned}
(\alpha+\varepsilon)\|x_1-x_2\|^2 &\leq \left((\beta+\varepsilon)\sqrt{1-2\sigma+\delta^2} + 2\mu\delta^2\right)\|x_1-x_2\|^2 \\
&= \left(\gamma + 2\mu\delta^2\right)\|x_1-x_2\|^2. \qquad (3.8)
\end{aligned}$$

It follows that

$$\left(\alpha+\varepsilon-\gamma-2\mu\delta^2\right)\|x_1-x_2\|^2 \leq 0.$$

Thus, for $\alpha+\varepsilon>\gamma+2\mu\delta^2$, we have

$$\|x_1-x_2\|^2\leq 0.$$

Which implies that $x_1=x_2$ is the uniqueness of the solution of (2.3).

Existence of solution: In order to prove the existence of solution of problem (2.3), we use auxiliary principle technique as developed by Noor [8], which is mainly due to Glowinski et al.[3].

For a given $x\in\mathscr{H}$ satisfying (2.3), we consider the problem of finding $z\in\mathscr{H}$, such that

$$\langle\rho G_\varepsilon x,\eta(y,z)\rangle+\langle z-x,y-z\rangle+\rho\varphi(y)-\rho\varphi(z)+\rho\mu\|\eta(y,z)\|^2\geq 0,\ y\in\mathscr{H}, \tag{3.9}$$

where ρ is a constant.

The relation (3.9) defines a mapping $z=z(x)$ between the problem (2.3) and (3.9). In order to prove the existence of solution of problem (2.3), it is sufficient to show that the connecting mapping is a contraction mapping.

For this, suppose $z_1\neq z_2$(corresponding to $x_1\neq x_2\in\mathscr{H}$).

Let us take x_1,x_2 be two solutions of (3.9). Then, we have

$$\langle\rho G_\varepsilon x_1,\eta(y,z_1)\rangle+\langle z_1-x_1,y-z_1\rangle+\rho\varphi(y)-\rho\varphi(z_1)+\rho\mu\|\eta(y,z_1)\|^2\geq 0,\ y\in\mathscr{H}. \tag{3.10}$$

and

$$\langle\rho G_\varepsilon x_2,\eta(y,z_2)\rangle+\langle z_2-x_2,y-z_2\rangle+\rho\varphi(y)-\rho\varphi(z_2)+\rho\mu\|\eta(y,z_2)\|^2\geq 0,\ y\in\mathscr{H}. \tag{3.11}$$

Taking $y=z_2$ in (3.10) and $y=z_1$ in (3.11) and adding the resultants and using the Assumption 2.1, we have

$$\begin{aligned}\langle z_1-z_2,z_1-z_2\rangle &\leq \langle x_1-x_2,z_1-z_2\rangle-\rho\langle G_\varepsilon x_1-G_\varepsilon x_2,\eta(z_1,z_2)\rangle+2\rho\mu\|\eta(z_1,z_2)\|^2\\ &= \langle x_1-x_2-\rho(G_\varepsilon x_1-G_\varepsilon x_2),z_1-z_2\rangle\\ &+ \rho\langle G_\varepsilon x_1-G_\varepsilon x_2,z_1-z_2-\eta(z_1,z_2)\rangle+2\rho\mu\|\eta(z_1,z_2)\|^2.\end{aligned}$$

Therefore, we have

$$\begin{aligned}\|z_1-z_2\|^2 &\le \|x_1-x_2-\rho(G_\varepsilon x_1-G_\varepsilon x_2)\|\|z_1-z_2\| \\ &+ \rho\|G_\varepsilon x_1-G_\varepsilon x_2\|\|z_1-z_2-\eta(z_1,z_2)\|+2\rho\mu\delta^2\|z_1-z_2\|^2. \end{aligned} \tag{3.12}$$

By the strongly monotonicity and Lipschitz continuity of G_ε from Lemma 2.10 with constants $\alpha+\varepsilon>0$ and $\beta+\varepsilon>0$, we have

$$\begin{aligned}\|x_1-x_2-\rho(G_\varepsilon x_1-G_\varepsilon x_2)\|^2 &\le \|x_1-x_2\|^2-2\rho\langle G_\varepsilon x_1-G_\varepsilon x_2, x_1-x_2\rangle \\ &+ \rho^2\|G_\varepsilon x_1-G_\varepsilon x_2\|^2 \\ &\le \left(1-2\rho(\alpha+\varepsilon)+\rho^2(\beta+\varepsilon)^2\right)\|x_1-x_2\|^2. \end{aligned} \tag{3.13}$$

From (3.2), (3.7), (3.12) and (3.13) and using the Lipschitz continuity of G_ε, and in view of Lemma 2.10, we have

$$\begin{aligned}\|z_1-z_2\| &\le \{\sqrt{1-2\rho(\alpha+\varepsilon)+\rho^2(\beta+\varepsilon)^2}+\rho(\beta+\varepsilon)\sqrt{1-2\sigma+\delta^2}\}\|x_1-x_2\| \\ &+ 2\rho\mu\delta^2\|z_1-z_2\| \\ &\le (t(\rho)+\rho\gamma)\|x_1-x_2\|+2\rho\mu\delta^2\|z_1-z_2\|, \end{aligned} \tag{3.14}$$

where

$$t(\rho)=\sqrt{1-2\rho(\alpha+\varepsilon)+\rho^2(\beta+\varepsilon)^2}.$$

On simplification (3.14), we have

$$\|z_1-z_2\|\le\{\frac{t(\rho)+\rho\gamma}{1-2\rho\mu\delta^2}\}\|x_1-x_2\|=\theta\|x_1-x_2\|,$$

where

$$\theta=\frac{t(\rho)+\rho\gamma}{1-2\rho\mu\delta^2}.$$

From (3.1), we have $\theta<1$, therefore the mapping z defined by (3.9) is a contraction mapping and hence it has a fixed point $z(x)=x\in\mathscr{H}$ which satisfy (PMVLI)(2.3). ∎

Some special cases of problem (2.3)

(i) If $G_\varepsilon=G$, then Theorem 3.1 reduces to the following theorem of Bloch and Noor[3]:

Theorem 3.2. *Suppose the operator $G : \mathscr{H} \to \mathscr{H}$ be strongly monotone and Lipschitz continuous with constants $\alpha > 0, \beta > 0$ and the bifunction $\eta : \mathscr{H} \to \mathscr{H}$ be strongly monotone and Lipschitz continuous with constants $\sigma > 0, \delta > 0$, respectively. If Assumption 2.1 holds and there exists a constant $\rho > 0$ such that*

$$0 < \rho < 2\frac{\alpha - (\gamma + 2\mu\delta^2)}{\beta^2 - (\gamma + 2\mu\delta^2)^2},\ \rho < \frac{1}{\gamma + 2\mu\delta^2},\ \alpha > 2\mu\delta^2,$$

where

$$\gamma = \beta\sqrt{1 - 2\sigma + \delta^2}.$$

Then, there exists a unique solution $x \in \mathscr{H}$ such that

$$\langle Gx, \eta(y,x)\rangle + \varphi(y) - \varphi(x) + \mu\|\eta(y,x)\|^2 \geq 0,\ \forall\, y \in \mathscr{H}.$$

(ii) If $G_\varepsilon = G$, and $\eta(y,x) = y - x$, then Theorem 3.1 reduces to the following theorem of Noor [6]:

Theorem 3.3. *Suppose the operator $G : \mathscr{H} \to \mathscr{H}$ be strongly monotone and Lipschitz continuous with constants $\alpha > 0, \beta > 0$, respectively. If there is a constant $\rho > 0$ such that*

$$0 < \rho < 2\frac{\alpha - 2\mu}{\beta^2},\ \rho < \frac{1}{4\mu},\ \mu < \frac{\alpha}{2}.$$

Then, there exists a unique solution $x \in \mathscr{H}$ such that

$$\langle Gx, y - x\rangle + \varphi(y) - \varphi(x) + \mu\|y - x\|^2 \geq 0,\ \forall\, y \in \mathscr{H}.$$

4. Conclusion

In this work, a new class of mixed variational like inequalities involving perturbed operator (PMVLI) has been studied. The existence of unique solutions of PMVLI has been proved by using auxiliary principle technique which is quite interesting. Some special cases of Theorem 3.1 have also been discussed under some suitable conditions. Further research is needed to develop the scheme of finding the approximate solution of such problems which will motivate the researchers working in this direction.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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