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A STUDY ON NEW TYPE OF IDEAL IN TOPOLOGICAL SPACES

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Abstract: In this paper new ideals namely semi prime ideals are introduced in topological spaces and study some of their properties. Also local functions are introduced using semi prime ideals and characterize them. Also semi prime closure operator by using semi prime ideals was introduced discuss some of their properties.

Key words: semi-prime ideal; semi-prime local function; semi prime closure.

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1. INTRODUCTION

Ideals in a topological space (X,τ) is treated in the classic text by kuratowski. An ideal I on a topological space (X,τ) is a nonempty collection of subsets of X which satisfies (i) $A \in I$ and $B \subseteq A$ implies $B \in I$ (ii) $A \in I$ and $B \in I$ implies $A \cup B \in I$. A topological space together with an ideal I is called an ideal space and is denoted by $(X,\tau,I_{\wp})$. He also defined the local function for each subset of X with respect to an ideal I and τ. Given a topological space (X,τ) with an ideal I on X and if $\wp(X)$ is the set of all subsets of X, a set operator $(.)^*: \wp(X) \to \wp(X)$ is called a local function of A with respect to τ and I is defined as follows: For $A \subseteq X, A^*(I,\tau) =$

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$\{x \in U/U \cap A \notin I for\ every\ U \in r(x)\}$ where $r(x) = \{U \in \tau/x \in U\}$. A Kuratowski closure operator $(cl)^*(.)$ for a topology $\tau^*(I,\tau)$ called the $*$-topology finer than τ is defined by $(cl)^*(A) = A \cup A^*(I,\tau)$. We denote A^* for $A^*(I,\tau)$ and $(\tau)^*$ for $\tau^*(I,\tau)$. Further Vaidyanathaswamy extended the study of ideals and local functions. The properties of the topology generated by the ideal I and τ, called the star topology which is finer than τ, denoted by τ^* are studied by Vaidyanathaswamy, Hashimoto, Hayashi and Samuels. In 1990, Jankovic and Hamlet in addition to their findings, consolidated all the results. In 2021[7] we introduce prime ideals in topological space and studied some properties. In this research paper we introduced maximal ideal and compare with the already existing ideals.

Let (X,τ) be a topological space and $A \subset X$. A point $x \in X$ is called an accumulation point or limit point of A if every open set containing x contains a point of A other than x. The derived set of A is the collection of all limit points of A. It is denoted by D(A). Any point of A which is not a limit point is called an isolated point of A. A subset A of a space (X,τ) is said to be discrete if every point of A is not an accumulation point of A. $D(A) = \phi$ iff A is closed and discrete. Let A be a subset of a topological space X. Then $x \in cl(A)$ if and only if every open set containing x intersects A other than x. A subset A of a topological space (X,τ) is said to be nowhere dense in X if the interior of its closure is empty.

2. PRELIMINARIES

Definition2.1[7]:

An ideal $I_{\mathcal{P}}$ in a topological space (X,τ) is said to be a prime ideal if it satisfies the following condition: if $A \cap B \in I_{\mathcal{P}}$ then either $A \in I_{\mathcal{P}}$ or $B \in I_{\mathcal{P}}$.

The topological space with prime ideal is said to be a prime ideal topological spaces and denoted by $(X,\tau,I_{\mathcal{P}})$.

Definition2.2[7]:

A set operator $(.)^*_{\mathcal{P}}: \wp(X) \to \wp(X)$ is called a local function of A with respect to τ and $I_{\mathcal{P}}$ is defined as follows: For $A \subseteq X, A^*(I_{\mathcal{P}},\tau) = \{x \in U/U \cap A \notin I_{\mathcal{P}} for\ every\ U \in r(x)\}$ where $r(x) = \{U \in \tau/x \in U\}$

Definition 2.3[7]:

A kuratowski closure operator $cl^*_{\mathcal{P}}(.)$ for a topology $\tau^*_{\mathcal{P}}(I_{\mathcal{P}},\tau)$, called the $*_{\mathcal{P}}$-topology finer than τ is defined by $cl^*_{\mathcal{P}}(A) = A \cup A^*(I_{\mathcal{P}},\tau)$

We denote $A^*_{\mathcal{P}}$ for $A^*(I_{\mathcal{P}}, \tau)$ and $\tau^*_{\mathcal{P}}$ for $\tau^*(I_{\mathcal{P}}, \tau)$.

Theorem 2.4[8]:

Let X be a topological space and $X_1 = X - \{x\}, x \in X$, then $\wp(X_1)$ is a prime ideal in X.

Theorem 2.5[8]:

Let X be a topological space with n elements. Then $I_{\mathcal{P}}$ is a prime ideal in X iff it is $\wp(Y)$ where Y contains n-1 elements or n elements in X.

Theorem 2.6:[8]

Let X be a topological space with n elements then it has n+1 prime ideals.

Definition 2.7[7]:

Let $(X, \tau, \mathcal{P})$ be a prime ideal space and let A be the nonempty collection of the subsets of X. Then the radical of A is denoted by $Rad(A)$ or $\sqrt{A}$ and it is defined by $\sqrt{A} = \bigcap$\{P/P is a prime ideal containing A\}

Theorem 2.8[7]:

Radical of any prime ideal is itself.

3. SEMI-PRIME IDEAL IN TOPOLOGICAL SPACES

In this section we introduce semi-prime ideal and discuss of its basic properties.

Definition 3.1:

A semi-prime ideal $\mathcal{S}$ on a topological space (X, τ) is a nonempty collection of subsets of X which satisfies (i) $A \in \mathcal{S}$ and $B \subseteq A$ implies $B \in \mathcal{S}$ (ii) If $A \bigcap B \in \mathcal{S}$ then either $A \in \mathcal{S}$ or $B \in \mathcal{S}$. The space $(X, \tau, \mathcal{S})$ is said to be a semi-prime ideal space.

Example3.2:

Consider $X = \{a, b, c\}, \tau = \{\phi, \{a\}, \{b, c\}, X\}, \mathcal{S} = \{\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}\}$. Then $\mathcal{S}$ is a semi-prime ideal. The collection $\{\phi, \{a\}, \{b\}, \{c\}, \{a, c\}\}$ is not a semi prime ideal.

Remark 3.3:

Let X be a nonempty set. Then the collections $\wp(X)$ forms semi prime ideal and it is called a trivial semi prime ideal. A proper semi prime ideal is called non trivial semi prime ideal.

Note 3.4:

From definition of semi prime ideal, clearly $\phi \in \mathcal{S}$. Also if $X \in \mathcal{S}$, then $\mathcal{S} = \wp(X)$.

Theorem 3.5:

Every prime ideal is a semi-prime ideal.

Proof:

Let $\mathcal{S}$ be a prime ideal in a topological space X. Then it satisfies the condition of semi prime ideal. Hence $\mathcal{S}$ is a semi prime ideal.

Remark3.6:

The converse of the above theorem need not be true as shown in the following example.

Example3.7:

Consider $X = \{a, b, c\}, \tau = \{\phi, \{a\}, \{b, c\}, X\}, \mathcal{S} = \{\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}\}$. Then $\mathcal{S}$ is semi-prime ideal but not prime ideal.

Remark 3.8:

The concepts ideal and semi-prime ideal are independent to each other.

For, Consider $X = \{a, b, c\}, \tau = \{\phi, \{a\}, \{b, c\}, X\}, \mathcal{S} = \{\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}\}$. Then $\mathcal{S}$ is a semi-prime ideal but not ideal.

Consider $X = \{a, b, c\}, \tau = \{\phi, \{a\}, \{b, c\}, X\}, \mathcal{S} = \{\phi, \{a\}\}$. Then $\mathcal{S}$ is ideal but not semi-prime ideal.

Theorem3.9:

Union of two semi-prime ideal is also a semi-prime ideal.

Proof: Let $\mathcal{S}_1, \mathcal{S}_2$ be two semi-prime ideals in a topological space X.

i)Let $A \in \mathcal{S}_1 \cup \mathcal{S}_2$ and $B \subseteq A$. Since $A \in \mathcal{S}_1 \cup \mathcal{S}_2$, either $A \in \mathcal{S}_1$ or $A \in \mathcal{S}_2$. Since $\mathcal{S}_1, \mathcal{S}_2$ be two semi-prime ideals and $B \subseteq A$, either $B \in \mathcal{S}_1$ or $B \in \mathcal{S}_2$which implies $B \in \mathcal{S}_1 \cup \mathcal{S}_2$.

ii) Let $A \cap B \in \mathcal{S}_1 \cup \mathcal{S}_2$. Then $A \cap B \in \mathcal{S}_1$ or $A \cap B \in \mathcal{S}_2$. Since $\mathcal{S}_1$ and $\mathcal{S}_1$ are semi-prime ideals, $(A \in \mathcal{S}_1$ or $B \in \mathcal{S}_1)$ or $(A \in \mathcal{S}_2$ or $B \in \mathcal{S}_2)$. This gives $A \in \mathcal{S}_1$ or $A \in \mathcal{S}_2$ or $B \in \mathcal{S}_1$ or $B \in \mathcal{S}_2$ which implies $A \in \mathcal{S}_1 \cup \mathcal{S}_2$ or $B \in \mathcal{S}_1 \cup \mathcal{S}_2$. Hence $\mathcal{S}_1 \cup \mathcal{S}_2$ is semi-prime ideal.

Remark3.10:

Intersection of two semi-prime ideal need not be a semi-prime ideal as shown in the following example.

Example3.11:

Consider $X = \{a, b, c\},\ \tau = \{\phi, \{a\}, \{a, b\}, X\}$ with

$\mathcal{S}_1 = \{\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}\},\ \mathcal{S}_2 = \{\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}\}$.

Then $\mathcal{S}_1, \mathcal{S}_2$ are semi-prime ideal. But $\mathcal{S}_1 \cap \mathcal{S}_2 = \{\phi, \{a\}, \{b\}, \{c\}, \{a, b\}$ is not semi-prime ideal.

Note 3.12:

The set of all collection of semi prime ideals in a topological space (X,τ) forms a semi group under the union operation.

Note3.13:

The set of all closed and discrete subsets of a topological space X is not semi prime ideal.

For, consider $X=\{a,b,c\}, \tau=\{\phi,\{a\},X\}$. Then the only closed and discrete subset is ϕ. Clearly it is not a semi prime ideal.

Theorem 3.14:

$\mathcal{S}_n$, the set of all nowhere dense sets in (X,τ) is a semi prime ideal.

Proof:

Let $A\in\mathcal{S}_n$ and $B\subseteq A$. Then $int(cl(A))=\phi$. Since $B\subseteq A$, $int(cl(B))\subseteq int(cl(A))=\phi$. Hence $int(cl(B))=\phi$. Therefore $B\in\mathcal{S}_n$. Let $A,B\in\wp(X)$ with $A\cap B\in\mathcal{S}_n$. Then $int(cl(A\cap B))=\phi$. That is, there is no open set contained in $cl(A\cap B)$. This implies there is no open set contained in $cl(A)\cap cl(B)$. That is there is no open set G such that $G\not\subset cl(A)\cap cl(B)$. This implies either $G\not\subset cl(A)$ or $G\not\subset cl(B)$. Hence either $int(cl(A))=\phi$ or $int(cl(B))=\phi$ and hence either $A\in\mathcal{S}_n$ or $B\in\mathcal{S}_n$. Therefore $\mathcal{S}_n$ is a semi prime ideal.

4. ALGEBRAIC STRUCTURE OF SEMI PRIME IDEAL

Theorem4.1:

Let X be a topological space. Then the collection $\wp(X)-\{X,X_1\}$, where $X_1=X-x, x\in X$ is a semi prime ideal.

Proof:

Let $\mathcal{S}=\wp(X)-\{X,X_1\}$ and let $A\in\mathcal{S}$ with $B\subseteq A$. Then clearly $B\in\mathcal{S}$. Let $A,B\in\wp(X)$ with $A\cap B\in\mathcal{S}$. We have to prove that either $A\in\mathcal{S}$ or $B\in\mathcal{S}$. In either cases $A=B$, $A\subseteq B$ or $B\subseteq A$ the proof is obvious. Assume that A and B are unequal and one is not contained in another. Suppose $A,B\notin\mathcal{S}$. Then $A,B\in\{X,X_1\}$ which gives $A\cap B=X_1\notin\mathcal{S}$ gives a contradiction to our assumption. Hence either $A\in\mathcal{S}$ or $B\in\mathcal{S}$ and therefore $\mathcal{S}$ is a semi prime ideal.

Note4.2:

In a topological space X, $\wp(X)-\{X,X_1\}$, where $X_1=X-x, x\in X$ is not a prime ideal. Then $\mathcal{S}=\{\phi,\{a\},\{b\},\{c\},\{a,b\},\{b,c\}\}$ is not a prime ideal. For $\{a,b\}\{b,c\}\in\mathcal{S}$ but $\{a,b\}\cup\{b,c\}\notin\mathcal{S}$.

Theorem4.3:

Let X be a topological space. Then $\wp(X) - X$ is a semi prime ideal.

Proof:

Let $\mathcal{S} = \wp(X) - X$ and let $A \in \mathcal{S}$ with $B \subseteq A$. Then clearly $B \in \mathcal{S}$. Let $A, B \in \wp(X)$ with $A \cap B \in \mathcal{S}$. We have to prove that either $A \in \mathcal{S}$ or $B \in \mathcal{S}$. In either cases $A = B$, $A \subseteq B$ or $B \subseteq A$ the proof is obvious. On the other hand, Suppose $A, B \notin \mathcal{S}$. Then $A, B \in \{X\}$ which gives $A \cap B = X \notin \mathcal{S}$ gives a contradiction to our assumption. Hence either $A \in \mathcal{S}$ or $B \in \mathcal{S}$ and therefore $\mathcal{S}$ is a semi prime ideal.

Note4.4:

In a topological space X, $\wp(X) - X$ is not prime ideal. Then $\mathcal{S} = \{\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}\}$ is not a prime ideal. For $\{a, b\}\{b, c\} \in \mathcal{S}$ but $\{a, b\} \cup \{b, c\} \notin \mathcal{S}$.

Theorem4.5:

Let X be a topological space and let $X_1 = X - \{x\}, x \in X$. Then $\wp(X_1)$ is a semi prime ideal.

Proof:

Since $\wp(X_1)$ is a prime ideal, $\wp(X_1)$ is a semi prime ideal.

Note4.6:

The set of all collections of semi prime ideals in a topological space X is the union of two disjoint sets $\mathcal{P}$ and $\mathcal{S}$ where $\mathcal{P}$ is the collection of prime ideals and $\mathcal{S}$ is the collection of semi prime ideals which are not prime ideals.

We say the semi prime ideals which are not prime ideals is strictly semi prime ideals.

Theorem4.7:

A semi prime ideal of the topological space X is of the form either $\wp(X) - X$ or $\wp(X) - \{X, X_1\}$ or $\wp(X_1)$ where $X_1 = X - \{x\}, x \in X$.

Proof:

Let $\mathcal{S}$ be a semi prime ideal. then it may be prime ideal or strictly semi prime ideal.

If $\mathcal{S}$ is a prime ideal, then it is of the form $\wp(X_1)$, where $X_1 = X - x, x \in X$.

If $\mathcal{S}$ is a strictly semi prime ideal, then we have the following cases.

Case(i):

Suppose $\mathcal{S} = \wp(X) - A$, where A is any non empty subset of X and $A \neq X$. Then $X \in \mathcal{S}$ and $A \notin \mathcal{S}$. Hence $\mathcal{S}$ is not a semi prime ideal. similarly we can prove that $\wp(X) - \mathbb{C}$ is not a semi prime ideal, where $\mathbb{C}\wp(X) - X$.

Case(ii)

Suppose $\mathcal{S} = \wp(X) - \{X, A\}$, where A is any proper non empty subset of X and $A \neq X'$ where $X' = X - \{x\}, x \in X$. Then $X \notin \mathcal{S}$ and $A \notin \mathcal{S}$. But $A \subseteq X'$ for some X' and $X' \in \mathcal{S}$. Hence $\mathcal{S}$ is not a semi prime ideal. Similarly we can prove that $\wp(X) - \{X, \mathbb{C}\}$ is not a semi prime ideal, where $\mathbb{C} \subseteq \wp(X) - X'$.

Case(iii):

Suppose $\mathcal{S} = \wp(X) - \{X, X_1, X_2\}$, where A is any non empty subset of X and $X_1 \neq X_2$ such that $X_1 = X - \{x\}, X_2 = X - \{y\}, x, y \in X$. Here $X_1 \cap X_2 \in \mathcal{S}$, but $X_1 \notin \mathcal{S}$ and $X_2 \notin \mathcal{S}$ Hence $\mathcal{S}$ is not a semi prime ideal. Similarly we can prove that $\wp(X) - \{X, \mathbb{C}\}$ is not a semi prime ideal, where $\mathbb{C} \subseteq \mathbb{C}_{\boldsymbol{i}}, \mathbb{C}_{\boldsymbol{i}} = \{X - \{x_{\boldsymbol{i}}\}/x_{\boldsymbol{i}} \in X, i = 1,2, \ldots |X|\}$ with $2 \leq |\mathbb{C}| \leq |X|$.

In similar way we can prove that $\wp(X) - \{X, \mathbb{C}\}$ is not a semi prime ideal, where $\mathbb{C} \subseteq \mathbb{C}_{\boldsymbol{i}}, \mathbb{C}_{\boldsymbol{i}} = \{X - A/A \subseteq X\}$ with $2 \leq |\mathbb{C}| \leq |X|$.

From above discussions and 4.1,4.3 we conclude that there is is not a strictly semi prime ideal except of the form $\wp(X) - X$ or $\wp(X) - \{X, X_1\}$.

Theorem4.8:

Let X be topological space with n elements (n>2). Then their exists 2(n+1) semi prime ideals in X.

Proof:

The number of semi prime ideals in X is equal to the sum of the number of prime ideals in X and the number of strictly semi prime ideals in X. Clearly the number of prime ideals in X is n+1. The number of strictly semi prime ideals is equal to the number of elements in the format $\wp(X) - \{X, X_1\}$ or $\wp(X) - X$ which is equal to n+1 and hence the total number of semi prime ideals is n+1+n+1=2(n+1).

Theorem 4.9:

For every proper prime ideal $\mathcal{P}$, there exists a strictly semi prime ideal $\mathcal{S}$ such that $\mathcal{P} \subset \mathcal{S}$.

Proof:

Since $\mathcal{P}$ is a proper prime ideal, by 2.5 it is of the form $\wp(X_1)$, where $X_1 = X - \{x\}, x \in X$. Then clearly $\mathcal{P} \subset \wp(X) - X$. From 4.3 and 4.4 $\wp(X) - X$ is a strictly semi prime ideal which gives the proof.

Theorem 4.10:

Consider the topological space (X,τ). Then there is no proper prime ideal between the strictly semi prime ideal and the power set of X.

Proof:

Proof is obvious from above theorem.

Note 4.11:

Let $\mathcal{S}$ be a proper stricly semi prime ideal in a topological space X, then the only prime ideal containing itself is the power set of X.

Theorem 4.12:

The radical of any semi prime ideal in a topological space X is either itself or the power set of X.

Proof:

Let $\mathcal{S}$ be any semi prime ideal. We have to prove this in two cases.

Case (i): If $\mathcal{S}$ is prime ideal. Then radical of $\mathcal{S}$ is $\mathcal{S}$.

Case (ii): If $\mathcal{S}$ is not prime ideal, then the only prime ideal containing $\mathcal{S}$ is $\wp(X)$ and hence radical of $\mathcal{S}$ is $\wp(X)$.

5. LOCAL FUNCTIONS ON SEMI PRIME IDEALS

Definition 5.1:

Given a topological space (X,τ) with a semi-prime ideal $\mathcal{S}$ on X and if $\wp(X)$ is the set of all subsets of X, a set operator $(.)^*_{\mathcal{S}}: \wp(X) \to \wp(X)$ is called a semi prime local function of A with respect to τ and $\mathcal{S}$ is defined as follows: For $A \subseteq X, A^*_{\mathcal{S}}(\mathcal{S},\tau) = \{x \in U / U \cap A \notin \mathcal{S}\ for\ every\ U \in r(x)\}$ where $r(x) = \{U \in \tau / x \in U\}$.

Theorem 5.2:

$A^*_{\mathcal{S}}(\wp(X),\tau) = \phi$ for every $A \subset X$.

Proof:

$A^*_{\mathcal{S}}(\wp(X),\tau) = \{x \in U / U \cap A \notin \wp(X) for\ every\ U \in r(x)\} = \phi$

Theorem5.3:

Let (X,τ) be a topological space with $\mathcal{S}_1, \mathcal{S}_2$ be two semi-prime ideals on X and let A and B be two subsets on X. Then

i) $A \subseteq B \Rightarrow A^*_{\mathcal{S}} \subseteq B^*_{\mathcal{S}}$

ii) $\mathcal{S}_1 \subseteq \mathcal{S}_2 \Rightarrow A^*_{\mathcal{S}}(\mathcal{S}_2) \subseteq A^*_{\mathcal{S}}(\mathcal{S}_1)$

iii) $A^*_{\mathcal{S}} \cup B^*_{\mathcal{S}} \subseteq (A \cup B)^*_{\mathcal{S}}$.

iv) $(A \cap B)^*_{\mathcal{S}} = A^*_{\mathcal{S}} \cap B^*_{\mathcal{S}}$ for any semi-prime ideal $\mathcal{S}$.

v) $\phi^*_{\mathcal{S}} = \phi$

vi) $X^*_{\mathcal{S}} \subseteq X$

Proof:

i)Let $x \in A^*_{\mathcal{S}}$. Then $U \cap A \notin \mathcal{S}$ for every $U \in r(x)$. Since $A \subseteq B$ and $U \cap A \notin \mathcal{S}$, using semi-prime ideal condition we have $U \cap B \notin \mathcal{S}$ for every $U \in r(x)$. This gives $x \in B^*_{\mathcal{S}}$. Therefore $A^*_{\mathcal{S}} \subseteq B^*_{\mathcal{S}}$.

ii)Let $\mathcal{S}_1$ and $\mathcal{S}_2$ being semi-prime ideals on X such that $\mathcal{S}_1 \subseteq \mathcal{S}_2$. Let $x \in A^*_{\mathcal{S}}(\mathcal{S}_2)$. Then $U \cap A \notin \mathcal{S}_2$ for every $U \in r(x)$. Since $\mathcal{S}_1 \subseteq \mathcal{S}_2$, we have $U \cap A \notin \mathcal{S}_1$for every $U \in r(x)$. This gives $x \in A^*_{\mathcal{S}}(\mathcal{S}_1)$ and hence $A^*_{\mathcal{S}}(\mathcal{S}_2) \subseteq A^*_{\mathcal{S}}(\mathcal{S}_1)$.

iii)using (i) its obvious.

iv)using (i), obviously $(A \cap B)^*_{\mathcal{S}} \subseteq A^*_{\mathcal{S}} \cap B^*_{\mathcal{S}}$ ....(1)

Let $x \in A^*_{\mathcal{S}} \cap B^*_{\mathcal{S}}$. Then $x \in A^*_{\mathcal{S}}$ and $x \in B^*_{\mathcal{S}}$ which implies $(U \cap A) \notin \mathcal{S}$ for every $U \in r(x)$ and $(U \cap B) \notin \mathcal{S}$ for every $U \in r(x)$.This implies $(U \cap A) \cap (U \cap A) \notin \mathcal{S}$ for every $U \in r(x)$ and hence $U \cap (A \cap B) \notin \mathcal{S}$ for every $U \in r(x)$. This implies $x \in (A \cap B)^*_{\mathcal{S}}$. Therefore

$A^*_{\mathcal{S}} \cap B^*_{\mathcal{S}} \subseteq (A \cap B)^*_{\mathcal{S}}$ ....(2)

From (1) and (2) we have $(A \cap B)^*_{\mathcal{S}} = A^*_{\mathcal{S}} \cap B^*_{\mathcal{S}}$.

(v) and (vi) are obvious.

Example5.4:

In the above theorem equality does not holds in (iii) and (vi). For consider the semi prime ideal space $X = \{a, b, c\}, \tau = \{\phi, \{a\}, X\}, \mathcal{S} = \{\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}\}$. Let $A = \{a\}$ and $B = \{c\}$. Then $A^*_{\mathcal{S}} = \phi$ $B^*_{\mathcal{S}} = \phi, A^*_{\mathcal{S}} \cup B^*_{\mathcal{S}} = \phi$, $A \cup B = \{a, c\}$ and $(A \cup B)^*_{\mathcal{S}} = \{b, c\}$ which gives $A^*_{\mathcal{S}} \cup B^*_{\mathcal{S}} \neq (A \cup B)^*_{\mathcal{S}}$.

In the above space $X^*_{\mathcal{S}} = \{b, c\} \neq X$

Theorem5.5:

Let (X, τ) be a topological space with a semi-prime ideal $\mathcal{S}$ on X and let A and B be two subsets on X. Then

i) $A^*_{\mathcal{S}} = cl(A^*_{\mathcal{S}}) \subset cl(A)$

ii) $(A^*_{\mathcal{S}})^*_{\mathcal{S}} \subseteq A^*_{\mathcal{S}}$

iii) $U \in \tau \Rightarrow U \cap A^*_{\mathcal{S}} \subseteq (U \cap A)^*_{\mathcal{S}}$

iv) $U \in \tau \Rightarrow U \cap A^*_{\mathcal{S}} = U \cap (U \cap A)^*_{\mathcal{S}}$

v) $(A-B)^*_{\mathcal{S}} \subseteq A^*_{\mathcal{S}} - B^*_{\mathcal{S}}$

Proof:

(i)We know that $A^*_{\mathcal{S}} \subseteq cl(A^*_{\mathcal{S}})$. For other inclusion, let $x \in cl(A^*_{\mathcal{S}})$. Then $U \cap A^*_{\mathcal{S}} \neq \phi$ for every neighbourhood U of x. Let $y \in U \cap A^*_{\mathcal{S}}$. Then $y \in U$ and $y \in A^*_{\mathcal{S}}$ which implies $V \cap A \notin \mathcal{S}$ for every $V \in r(y)$. In particular U$\cap A \notin \mathcal{S}$. Therefore $U \cap A \notin \mathcal{S}$ for every $U \in r(x)$ and hence $x \in A^*_{\mathcal{S}}$ which gives $cl(A^*_{\mathcal{S}}) \subseteq A^*_{\mathcal{S}}$. Therefore $A^*_{\mathcal{S}} = cl(A^*_{\mathcal{S}})$.

Let $x \in A^*_{\mathcal{S}}$. Then $U \cap A \notin \mathcal{S}$ for every $U \in r(x)$. Since $\phi \in \mathcal{S}$, $U \cap A \neq \phi$ for every neighbourhood U of x. This gives $x \in cl(A)$ and hence $A^*_{\mathcal{S}} = cl(A^*_{\mathcal{S}}) \subset cl(A)$.

(ii) $(A^*_{\mathcal{S}})^*_{\mathcal{S}} \subseteq cl(A^*_{\mathcal{S}}) = A^*_{\mathcal{S}}$.

(iii)Let $x \in U \cap A^*_{\mathcal{S}}$. Then $x \in U$ and $x \in A^*_{\mathcal{S}}$. This implies $V \cap A \notin \mathcal{S}$ for every $V \in r(x)$. Since $x \in U$, $U \in r(x)$ and $U \cap V \in r(x)$ for every $V \in r(x)$. This gives $(U \cap V) \cap A \notin \mathcal{S}$ for every $V \in r(x)$. Hence $V \cap (U \cap A) \notin \mathcal{S}$ for every $V \in r(x)$. Therefore $x \in (U \cap A)^*_{\mathcal{S}}$ which gives $U \cap A^*_{\mathcal{S}} \subseteq (U \cap A)^*_{\mathcal{S}}$.

(iv) $U \cap A^*_{\mathcal{S}} = U \cap (U \cap A^*_{\mathcal{S}}) \subseteq U \cap (U \cap A)^*_{\mathcal{S}}$ (using (iii))

Let $x \in U \cap (U \cap A)^*_{\mathcal{S}}$. Then $x \in U$ and $x \in (U \cap A)^*_{\mathcal{S}}$. This implies $V \cap (U \cap A) \notin \mathcal{S}$ for every $V \in r(x)$. Since $x \in U$, $x \in U \cap V$ and $U \cap V \in r(x)$. Therefore $(U \cap V) \cap A \notin \mathcal{S}$ for every $V \in r(x)$. This gives $x \in A^*_{\mathcal{S}}$ and hence $x \in U \cap A^*_{\mathcal{S}}$. Therefore $U \cap (U \cap A)^*_{\mathcal{S}} \subseteq U \cap A^*_{\mathcal{S}}$ and hence $U \cap A^*_{\mathcal{S}} = U \cap (U \cap A)^*_{\mathcal{S}}$.

(v)Let $x \in (A-B)^*_{\mathcal{S}}$. Then $U \cap (A-B) \notin \mathcal{S}$ for every $U \in r(x)$. Which implies $(U \cap A) - (U \cap B) \notin \mathcal{S}$. Since $\mathcal{S}$ is a semi prime ideal and $(U \cap A) - (U \cap B) \subseteq U \cap A$, $U \cap A \notin \mathcal{S}$ for every $V \in r(x)$. Hence $x \in A^*_{\mathcal{S}}$. Suppose $x \in B^*_{\mathcal{S}}$. Then $U \cap B \notin \mathcal{S}$ for every $U \in r(x)$. Now $\big((U \cap A) - (U \cap B)\big) \cap (U \cap B) = \phi \in \mathcal{S}$. Since $\mathcal{S}$ is a semi prime ideal, either $(U \cap A) - (U \cap B) \in \mathcal{S}$ or $(U \cap B) \in \mathcal{S}$. This is a contradiction. Hence $x \notin B^*_{\mathcal{S}}$. Therefore $x \in A^*_{\mathcal{S}} - B^*_{\mathcal{S}}$ and hence $(A-B)^*_{\mathcal{S}} \subseteq A^*_{\mathcal{S}} - B^*_{\mathcal{S}}$

Example 5.6:

In the above theorem equality does not hold. For, consider the prime ideal space $X = \{a,b,c\}, \tau = \{\phi, \{a\}, \{b\}, \{a,b\}, X\}, \mathcal{S} = \{\phi, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}\}$. Let $A = \{b,c\}$. Then $A^*_{\mathcal{S}} = \{c\}$ and $(A^*_{\mathcal{S}})^*_{\mathcal{S}} = \phi$. Therefore $(A^*_{\mathcal{S}})^*_{\mathcal{S}} \neq A^*_{\mathcal{S}}$. In this space, let $A = \{b,c\}, B = \{b\}$. Then $A^*_{\mathcal{S}} = \{c\}, B^*_{\mathcal{S}} = \phi, A^*_{\mathcal{S}} - B^*_{\mathcal{S}} = \{c\}, A - B = \{c\}$ and $(A-B)^*_{\mathcal{S}} = \phi$. Therefore $(A-B)^*_{\mathcal{S}} \neq A^*_{\mathcal{S}} - B^*_{\mathcal{S}}$.

Consider the prime ideal space $X = \{a, b, c\}, \tau = \{\phi, \{a\}, \{a, c\}, X\}, \mathcal{S} = \{\phi, \{a\}, \{b\}, \{a, b\}\}$. Let $A = \{c\}$ and $U = \{a, c\}$. Then $A^*_{\mathcal{S}} = \{b, c\}, U \cap A^*_{\mathcal{S}} = \{c\}, U \cap A = \{c\}, (U \cap A)^*_{\mathcal{S}} = \{b, c\}$. Hence $U \cap A^*_{\mathcal{S}} \neq (U \cap A)^*_{\mathcal{S}}$. In this topology, let $A = \{b\}$. Then $A^*_{\mathcal{S}} = \phi, cl(A) = \{b\}$. Hence $A^*_{\mathcal{S}} \neq cl(A)$.

Theorem5.7:

Let $(X, \tau, \mathcal{S})$ be a ideal topological space and let A and B be two subsets on X. Then

i) $A^*_{\mathcal{S}} \cap (B^*_{\mathcal{S}} \cup C^*_{\mathcal{S}}) = (A^*_{\mathcal{S}} \cap B^*_{\mathcal{S}}) \cup (A^*_{\mathcal{S}} \cap C^*_{\mathcal{S}})$

ii) $A^*_{\mathcal{S}} \cup (B^*_{\mathcal{S}} \cap C^*_{\mathcal{S}}) = (A^*_{\mathcal{S}} \cup B^*_{\mathcal{S}}) \cap (A^*_{\mathcal{S}} \cup C^*_{\mathcal{S}})$

iii) $(A^*_{\mathcal{S}} \cap B^*_{\mathcal{S}}) \cup (A^*_{\mathcal{S}} \cap C^*_{\mathcal{S}}) \subseteq \left(A \cap (B \cup C)\right)^*_{\mathcal{S}}$

iv) $(A^*_{\mathcal{S}} \cup B^*_{\mathcal{S}}) \cap (A^*_{\mathcal{S}} \cup C^*_{\mathcal{S}}) \subseteq \left(A \cup (B \cap C)\right)^*_{\mathcal{S}}$

v) $(A \cup B)^{C^*}{}_{\mathcal{S}} = A^{C^*}{}_{\mathcal{S}} \cap B^{C^*}{}_{\mathcal{S}}$

vi) $A^{C^*}{}_{\mathcal{S}} \cup B^{C^*}{}_{\mathcal{S}} \subseteq (A \cap B)^{C^*}{}_{\mathcal{S}}$

Proof:

Proof of (i) and (ii) are obvious using set theory.

iii) $(A^*_{\mathcal{S}} \cap B^*_{\mathcal{S}}) \cup (A^*_{\mathcal{S}} \cap C^*_{\mathcal{S}}) = A^*_{\mathcal{S}} \cap (B^*_{\mathcal{S}} \cup C^*_{\mathcal{S}}) \subseteq A^*_{\mathcal{S}} \cap (B \cup C)^*_{\mathcal{S}} = \left(A \cap (B \cup C)\right)^*_{\mathcal{S}}$.

iv) $(A^*_{\mathcal{S}} \cup B^*_{\mathcal{S}}) \cap (A^*_{\mathcal{S}} \cup C^*_{\mathcal{S}}) = A^*_{\mathcal{S}} \cup (B^*_{\mathcal{S}} \cap C^*_{\mathcal{S}}) = A^*_{\mathcal{S}} \cup (B \cap C)^*_{\mathcal{S}} \subseteq \left(A \cup (B \cap C)\right)^*_{\mathcal{S}}$.

Proof of (v) and (vi) are obvious using demargan's law in set theory and theorem 5.3.

Theorem 5.8:

In a semi prime ideal space $(X, \tau, \mathcal{S})$, if $A \in \mathcal{S}$, then $A^*_{\mathcal{S}} = \phi$.

Proof:

$A^*_{\mathcal{S}} = \{x \in X / U \cap A \notin \mathcal{S} \; for \; every \; U \in r(x)\}$. Since $A \in \mathcal{S}$ and $\mathcal{S}$ is a semi prime ideal space $U \cap A \in \mathcal{S}$ for every $U \in r(x)$. Hence $A^*_{\mathcal{S}} = \phi$.

Theorem 5.9:

If $V \in \mathcal{S}$ and A is any non empty subset of a semi prime ideal space $(X, \tau, \mathcal{S})$, then $(V - A)^*_{\mathcal{S}} = \phi$.

Proof:

$(V - A)^*_{\mathcal{S}} \subseteq V^*_{\mathcal{S}} - A^*_{\mathcal{S}}$. Since $V \in \mathcal{S}$, $V^*_{\mathcal{S}} = \phi$. Hence $(V - A)^*_{\mathcal{S}} = \phi$.

Definition5.10:

Consider the semi prime ideal space $(X, \tau, \mathcal{S})$. Then the semi prime closure of any subset of X is denoted by $Cl^*_{\mathcal{S}}(A) = A \cup A^*_{\mathcal{S}}$. The set of all collections of semi prime closures denoted by $C^*_{\mathcal{S}}(X)$.

Note 5.11:

Clearly $A \subseteq Cl_{\mathcal{S}}^*(A)$ and $A_{\mathcal{S}}^* \subseteq Cl_{\mathcal{S}}^*(A)$

Theorem 5.12:

If $A \in \mathcal{S}$, then $Cl_{\mathcal{S}}^*(A) = A$.

Proof:

Since $A \in \mathcal{S}$, $A_{\mathcal{S}}^* = \phi$. Now $Cl_{\mathcal{S}}^*(A) = A \cup A_{\mathcal{S}}^* = A$.

Theorem 5.13:

Let $(X, \tau, \mathcal{S})$ be a semi prime ideal space. Then the following results holds.

i) $Cl_{\mathcal{S}}^*(\phi) = \phi$

ii) $Cl_{\mathcal{S}}^*(X) = X$

iii) If $A \subseteq B$, then $Cl_{\mathcal{S}}^*(A) \subseteq Cl_{\mathcal{S}}^*(B)$

iv) $Cl_{\mathcal{S}}^*(A) \cup Cl_{\mathcal{S}}^*(A) \subseteq Cl_{\mathcal{S}}^*(A \cup B)$

v) $Cl_{\mathcal{S}}^*(A \cap B) \subseteq Cl_{\mathcal{S}}^*(A) \cap Cl_{\mathcal{S}}^*(A)$

Proof:

Proof of (i) and (ii) are obvious.

(iii)Assume $A \subseteq B$. $Cl_{\mathcal{S}}^*(A) = A \cup A_{\mathcal{S}}^* \subseteq B \cup B_{\mathcal{S}}^* = Cl_{\mathcal{S}}^*(B)$

(iv)Since $A \subseteq A \cup B, B \subseteq A \cup B$, from (iii) $Cl_{\mathcal{S}}^*(A) \subseteq Cl_{\mathcal{S}}^*(A \cup B)$ and $Cl_{\mathcal{S}}^*(B) \subseteq Cl_{\mathcal{S}}^*(A \cup B)$ and hence $Cl_{\mathcal{S}}^*(A) \cup Cl_{\mathcal{S}}^*(B) \subseteq Cl_{\mathcal{S}}^*(A \cup B)$

(v) $A \cap B \subseteq A$ and $A \cap B \subseteq B$, from (iii) $Cl_{\mathcal{S}}^*(A \cap B) \subseteq Cl_{\mathcal{S}}^*(A) \cap Cl_{\mathcal{S}}^*(B)$

Note 5.14:

Equality does not holds in the above theorem. For consider the semi prime ideal space $X = \{a, b, c\}, \tau = \{\phi, \{a\}, X\}, \mathcal{S} = \{\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}\}$. Let $A = \{a, c\}$ and $B = \{b, c\}$. Here $Cl_{\mathcal{S}}^*(A) = X$, $Cl_{\mathcal{S}}^*(B) = \{b, c\}, Cl_{\mathcal{S}}^*(A \cap B) = \{c\}$ $Cl_{\mathcal{S}}^*(A) \cap Cl_{\mathcal{S}}^*(B) = \{b, c\}$. Then $Cl_{\mathcal{S}}^*(A \cap B) \neq Cl_{\mathcal{S}}^*(A) \cap Cl_{\mathcal{S}}^*(B)$.

Consider $A = \{a\}$ and $B = \{c\}$. Here $Cl_{\mathcal{S}}^*(A) = \{a\}$, $Cl_{\mathcal{S}}^*(B) = \{c\}, Cl_{\mathcal{S}}^*(A \cup B) = X$ $Cl_{\mathcal{S}}^*(A) \cup Cl_{\mathcal{S}}^*(B) = \{a, c\}$. Then $Cl_{\mathcal{S}}^*(A \cup B) \neq Cl_{\mathcal{S}}^*(A) \cup Cl_{\mathcal{S}}^*(B)$.

Note 5.15:

From above discussions $Cl_{\mathcal{S}}^*(.)$ is not a kuratowski operator.

6. CONCLUSION

In this paper, we have defined semi prime ideal in topological space and introduced semi prime closure using them. Also we proved that semi prime closure operator is not a kuratowski closure operator.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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