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WOVEN K-G-FRAMES IN HILBERT C^*-MODULES

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Abstract. The aim of this paper is to introduce woven K-g-frames in Hilbert C^*-modules, to characterize them in term of atomic system for K, and to discuss the erasures and perturbations of weaving of K-g-frames in Hilbert C^*-modules.

Keywords: K-g-frames; woven K-g-frames; C^*-algebra; Hilbert C^*-modules.

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1. Introduction

As a generalization of bases in Hilbert spaces, frames were first introduced in 1952 by Duffin and Schaefer [2] in the study of nonharmonic fourier series. Frames possess many nice properties which make them very useful in wavelet analysis, irregular sampling theory, signal processing and many other fields.

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The notion of weaving was recently proposed by Bemrose et al. [1] to simulate a question in distributed signal processing and wireless sensor networks.

$K-g-$frames, which are more general than ordinary $g-$frames, naturally have become one of the most active fields in frame theory in recent years. $K-g-$frames share many properties with $g-$frames, but they have their own properties, like the inversibility of frame operator of $K-g-$frames for more see [4, 6, 8, 9, 10, 11, 12].

Hilbert C^*-modules are generalization of Hilbert spaces in that they allow the inner product to take values in a C^*-algebra rather than the field of complex numbers. There are many differences between Hilbert C^*-modules and Hilbert spaces. For example, we know that any closed subspace in a Hilbert space has an orthogonal complement, but it is not true for Hilbert C^*-modules. And the Riesz representation theorem of continuous functionals in Hilbert C^*-modules is invalid in general.

In this paper, we introduce the weaving of K-g-frames in Hilbert C^*-modules, we will characterize them in term of atomic system for K, and we will discuss the erasures and perturbations of weaving of K-g-frames in Hilbert C^*-modules.

A frame in a separable Hilbert space H is a sequence $\{x_i\}_{i\in I}$ for which there exist positive constants $A,B>0$ such that:

$$A\|x\|^2 \leq \sum_{i\in I} |\langle x,x_i\rangle|^2 \leq B\|x\|^2,$$

for all $x\in H$. The constants A,B are respectively called lower and upper bounds. If $A=B$, it is called a tight frame and it is said to be a normalized tight or Parseval frame if $A=B=1$. The collection $\{x_i\}_{i\in I}\subset H$ is called Bessel if the above second inequality holds. In this case, B is called the Bessel bound.

2. Background Material

Let I and J be finite or countable index sets and let $\mathbb{N}$ be the set of natural numbers. Throughout this paper, we assume that $\mathscr{U}$ and $\mathscr{V}$ are finitely or countably generated Hilbert A-modules, where A is a complex C^*-algebra with the norm $\|.\|_{\mathscr{A}}$, and $\{\mathscr{V}_i : i\in I\}$ is a sequence of closed Hilbert submodules of $\mathscr{V}$. $End_A^*(\mathscr{U},\mathscr{V}_i)$ is the collection of all adjointable $\mathscr{A}$-linear maps from $\mathscr{U}$ to $\mathscr{V}_i$ and $End_A^*(\mathscr{U})$ is abbreviated for $End_A^*(\mathscr{U},\mathscr{U})$.

In this section, we recall the definitions of $g-$frames, $K-g-$frames in Hilbert C^*-modules and some lemmas which are needed later.

Definition 2.1. [5] A sequence $\{\Lambda_i \in End_A^*(\mathscr{U},\mathscr{V}_i), i \in I\}$ is called a g-frame or a generalized frame in U with respect to $\{\mathscr{V}_i : i \in I\}$ if there exist constants $C;D > 0$ such that for every $f \in \mathscr{U}$,

$$C\langle f,f\rangle \leq \sum_{i\in I}\langle \Lambda_i f,\Lambda_i f\rangle \leq D\langle f,f\rangle$$

Definition 2.2. [13] Let $K \in End_A^*(\mathscr{U})$, a sequence $\{\Lambda_i \in End_A^*(\mathscr{U},\mathscr{V}_i), i \in I\}$ is called a K-g-frame if there exist constants $C;D > 0$ such that for every $f \in \mathscr{U}$,

$$C\langle K^*f,K^*f\rangle \leq \sum_{i\in I}\langle \Lambda_i f,\Lambda_i f\rangle \leq D\langle f,f\rangle$$

Lemma 2.3. [3] *Let U, V and W be Hilbert A-moduls, let $S \in End_A^*(W,V)$ and $T \in End_A^*(U,V)$ with $\overline{\mathscr{R}}(T^*)$ orthogonally complemented. The following statements are equivalent.*

(i) *$SS^* \leq \lambda TT^*$ for some $\lambda > 0$*
(ii) *There exists $\mu > 0$ such that $\|S^*z\| \leq \mu\|T^*z\|, \quad \forall z \in V$*
(iii) *There exists a $D \in End_A^*(W,V)$ such that $S = TD$, i.e :$TX = S$ has a solution.*
(iv) *$\mathscr{R}(S) \subset \mathscr{R}(T)$*

Lemma 2.4. [7] *Let $\mathscr{U}$ and $\mathscr{V}$ be Hilbert A-modules over a C^*-algebra A, and let $T : \mathscr{U} \longrightarrow \mathscr{V}$ be a linear map. Then the following conditions are equivalent:*

1. *The operator T is bounded and A-linear.*
2. *There exists $k \geq 0$ such that $\langle Tx,Tx\rangle \leq k\langle x,x\rangle$ for all $x \in \mathscr{U}$.*

One of the advantages of this equivalent definition of K-g-frames is that it is much easier to compare the norms of two elements than to compare two elements in C^*-algebras.

Theorem 2.5. *Let $K \in End_A^*(\mathscr{U})$, a sequence $\{\Lambda_i \in End_A^*(\mathscr{U},\mathscr{V}_i), i \in I\}$ is a K-g-frame if and only if there exists $0 < C;D < \infty$ such that;*

$$C\|\langle K^*f,K^*f\rangle\| \leq \|\sum_{i\in I}\langle \Lambda_i f,\Lambda_i f\rangle\| \leq D\|\langle f,f\rangle\|$$

for every $f \in \mathscr{U}$.

Proof. ($\Longrightarrow$) immediate.

($\Longleftarrow$) Assume that there exist constants $0 < C; D < \infty$ such that for all $f \in \mathscr{U}$

$$C\|\langle K^*f, K^*f\rangle\| \leq \|\sum_{i\in I}\langle \Lambda_i f, \Lambda_i f\rangle\| \leq D\|\langle f, f\rangle\|$$

Let S the frame operator of the bessel g-sequence $\{\Lambda_i\}_{i\in I}$.

S is a bounded positive self-adjoint operator, hence S has a unique positive square root, denoted by $S^{\frac{1}{2}}$. then

$$\sqrt{C}\|K^*f\| \leq \|S^{\frac{1}{2}}f\| \leq \sqrt{D}\|f\|.$$

By lemma 2.4, we obtain

$$\langle S^{\frac{1}{2}}f, S^{\frac{1}{2}}f\rangle = \langle Sf, f\rangle \leq B\langle f, f\rangle$$

From $(i) \Leftrightarrow (ii)$ in lemma (2.3) there exist some $\lambda > 0$ such that:

$$KK^* \leq \lambda S^{\frac{1}{2}}(S^{\frac{1}{2}})^*.$$

Then

$$\frac{1}{\lambda}\langle K^*f, K^*f\rangle \leq \langle Sf, f\rangle = \sum_{i\in I}\langle \Lambda_i f, \Lambda_i f\rangle, \quad \forall f \in \mathscr{U}.$$

□

Lemma 2.6. *Let H be an Hilbert A-module, let $T, P, Q \in End_A^*(H)$ with $\overline{\mathscr{R}}(P^*)$ and $\overline{\mathscr{R}}(Q^*)$ are orthogonally complemented. The following statements are equivalent:*

(i) $\mathscr{R}(T) \subset \mathscr{R}(P) + \mathscr{R}(Q)$

(ii) $TT^* \leq \lambda(PP^* + QQ^*)$ *for some* $\lambda > 0$

(iii) *There exists $X, Y \in End_A^*(H)$ such that $T = PX + QY$.*

3. Woven K-g-Frames

Definition 3.1. Two K-g-frames $\Lambda = \{\Lambda_i\}_{i\in I}$ and $\Gamma = \{\Gamma_i\}_{i\in I}$ for $\mathscr{U}$ are said to be woven K-g-frames if there exist universal positive constants A and B such that for any partition σ of I, the family $\{\Lambda_i\}_{i\in\sigma} \cup \{\Gamma_i\}_{i\in\sigma^c}$ is a K-g-frame for $\mathscr{U}$ with lower and upper K-g-frame bounds A and B, respectively, that is

$$A\langle K^*f, \langle K^*f\rangle \leq \sum_{i\in\sigma}\langle \Lambda_i f, \Lambda_i f\rangle + \sum_{i\in\sigma^c}\langle \Gamma_i f, \Gamma_i f\rangle \leq B\langle f, f\rangle, \quad \forall f \in \mathscr{U}.$$

Definition 3.2. A family of K-g-frames $\{\Lambda_j = \{\Lambda_{ij}\}_{i\in I}, \quad j \in [m]\}$ for $\mathscr{U}$ are said to be woven K-g-frames if there exist universal positive constants A and B such that for any partition $\{\sigma_j\}_{j\in[m]}$ of I, the family $\{\Lambda_{ij}\}_{i\in\sigma_j, j\in[m]}$ is a K-g-frame for $\mathscr{U}$ with lower and upper K-g-frame bounds A and B, respectively, that is

$$A\langle K^*f, \langle K^*f\rangle \leq \sum_{j\in[m]}\sum_{i\in\sigma_j}\langle \Lambda_{ij}f, \Lambda_{ij}f\rangle \leq B\langle f, f\rangle, \quad \forall f \in \mathscr{U}.$$

Suppose that $\{\Lambda_i\}_{i\in I}$ is a K-g-Bessel sequence for $\mathscr{U}$, then the synthesis operator of $\{\Lambda_i\}_{i\in I}$ is defined by $T_\Lambda : \bigoplus_{i\in I}\mathscr{V}_i \longrightarrow \mathscr{U}$,

$$T_\Lambda(\{f_i\}_{i\in I}) = \sum_{i\in I}\Lambda_i^* f_i, \quad \forall\{f_i\}_{i\in I} \in \bigoplus_{i\in I}\mathscr{V}_i.$$

Its adjoint operator, which is called the analysis operator $T_\Lambda^* : \mathscr{U} \longrightarrow \bigoplus_{i\in I}\mathscr{V}_i$,

$$T_\Lambda^*(f) = \{\Lambda_i f\}_{i\in I}, \quad \forall\, f \in \mathscr{U}.$$

And the K-g-frame operator $S_\Lambda : \mathscr{U} \longrightarrow \mathscr{U}$,

$$S_\Lambda f = T_\Lambda T_\Lambda^* f = \sum_{i\in I}\Lambda_i^*\Lambda_i f, \quad \forall\, f \in \mathscr{U}.$$

For any partition $\{\sigma_j\}_{j\in[m]}$ of I, we define these operators,

$$T_\Lambda^{\sigma_j}(\{f_i\}_{i\in I}) = \sum_{i\in\sigma_j}\Lambda_i^* f_i, \quad \forall\{f_i\}_{i\in I} \in \bigoplus_{i\in I}\mathscr{V}_i, \quad j \in [m],$$

$$(T_\Lambda^{\sigma_j})^*(f) = \{\Lambda_i f\}_{i\in\sigma_j}, \quad \forall\, f \in \mathscr{U}, \quad j \in [m],$$

$$S_\Lambda^{\sigma_j} f = T_\Lambda T_\Lambda^* f = \sum_{i\in\sigma_j}\Lambda_i^*\Lambda_i f, \quad \forall\, f \in \mathscr{U}.$$

Theorem 3.3. *Let $K \in End_A^*(\mathscr{U})$, $\Lambda = \{\Lambda_i\}_{i\in I}$ and $\Gamma = \{\Gamma_i\}_{i\in I}$ be two K-g-frames for $\mathscr{U}$ with respect to $\{\mathscr{V}_i : i \in I\}$. Then for every partition σ of I, Λ and Γ are woven K-g-frames for $\mathscr{U}$ with universal lower and upper K-g-frame bounds A and B, respectively, if and only if*

$$A\|\langle K^*f, \langle K^*f\rangle\| \leq \|\sum_{i\in\sigma}\langle \Lambda_i f, \Lambda_i f\rangle + \sum_{i\in\sigma^c}\langle \Gamma_i f, \Gamma_i f\rangle\| \leq B\|\langle f, f\rangle\|, \quad \forall f \in \mathscr{U}.$$

Proof. It follows from Theorem (2.4) □

Proposition 3.4. *Let $K \in L(\mathscr{U})$ and $\{\Lambda_{ij}\}_{i\in\sigma_j, j\in[m]}$ be a family of woven K-g-frames for $\mathscr{U}$. Then the frame operator S is self adjoint, positive, bounded on $\mathscr{U}$, and $KK^* \leq \lambda S$ for some $\lambda > 0$.*

Proof. For every $f \in \mathscr{U}$

$$Sf = \sum_{j\in[m]} \sum_{i\in\sigma_j} \Lambda_{ij}^* \Lambda_{ij} f$$

then

$$\langle Sf, f\rangle = \langle \sum_{j\in[m]} \sum_{i\in\sigma_j} \Lambda_{ij}^* \Lambda_{ij} f, f\rangle \quad = \sum_{j\in[m]} \sum_{i\in\sigma_j} \langle \Lambda_{ij} f, \Lambda_{ij} f\rangle$$

then

$$A\langle K^* f, K^* f\rangle \leq \langle Sf, f\rangle \leq B\langle f, f\rangle$$

hence

$$AKK^* \leq S \leq BI.$$

So, the frame operator S is bounded and positive.

Therefore, $S^* = (TT^*)^* = TT^* = S$ then S is self adjoint. □

Proposition 3.5. *Suppose for every $j \in [m]$; $\{\Lambda_j = \{\Lambda_{ij}\}_{i\in I}\}$ is a g-Bessel sequence for $\mathscr{U}$ with bound B_j. Then every weaving $\{\Lambda_{ij}\}_{i\in\sigma_j, j\in[m]}$ is a g-Bessel sequence with bound $\sum_{j\in[m]} B_j$.*

Proof.

$$\begin{aligned}\sum_{j\in[m]} \sum_{i\in\sigma_j} \langle \Lambda_{ij} f, \Lambda_{ij} f\rangle &\leq \sum_{j=1}^{m} \sum_{i\in\sigma} \langle \Lambda_{ij} f, \Lambda_{ij} f\rangle \\ &= \sum_{j=1}^{m} B_j \langle f, f\rangle.\end{aligned}$$

□

The following theorem gives a characterization for weaving K-g-frames in term of atomic system for K

Definition 3.6. Let $K \in End_A^*(\mathscr{U})$, then the family $\{\Lambda_i \in End_A^*(\mathscr{U}, \mathscr{V}_i), i \in I\}$ is called an atomic system for K, if the following conditions are satisfied

(i) The family $\{\Lambda_i \in End_A^*(\mathscr{U}, \mathscr{V}_i), i \in I\}$ is a g-Bessel sequence,

(ii) For every $f \in \mathscr{U}$, there exists $f_i \in \bigoplus_{i\in I} \mathscr{V}_i$ such that $\|\{f_i\}_{i\in I}\| \leq C\|f\|$ for some $C > 0$ and $Kf = \sum_{i\in I} \Lambda_i^* f_i$.

Theorem 3.7. *Let $K \in End_A^*(\mathscr{U})$, the families $\{\Lambda_i\}_{i\in I}$ and $\{\Gamma_i\}_{i\in I}$ be two K-g-frames for $\mathscr{U}$. The the following statements are equivalent*

(i) *The families $\{\Lambda_i\}_{i\in I}$ and $\{\Gamma_i\}_{i\in I}$ are woven K-g-frames.*

(ii) *The family $\{\Lambda_i\}_{i\in\sigma} \cup \{\Gamma_i\}_{i\in\sigma^c}$ is an atomic system for K, where σ is any subset of I.*

Proof. $i) \Longrightarrow ii)$. Suppose that the families $\{\Lambda_i\}_{i\in I}$ and $\{\Gamma_i\}_{i\in I}$ are woven K-g-frames with bounds A and B .

For every partition $[\sigma, \sigma^c]$ of I, we have

$$A\langle K^* f, \langle K^* f\rangle \leq \sum_{i\in\sigma} \langle \Lambda_i f, \Lambda_i f\rangle + \sum_{i\in\sigma^c} \langle \Gamma_i f, \Gamma_i f\rangle \leq B\langle f, f\rangle, \quad \forall f \in \mathscr{U}.$$

Then the family $\{\Lambda_i\}_{i\in\sigma} \cup \{\Gamma_i\}_{i\in\sigma^c}$ is g-Bessel sequence with bound B.

On the other hand

$$A\langle K^* f, \langle K^* f\rangle \leq \langle S_\Lambda^\sigma f, f\rangle + \langle S_\Gamma^{\sigma^c} f, f\rangle$$

This imply that

$$AKK^* \leq T_\Lambda^\sigma (T_\Lambda^\sigma)^* + T_\Gamma^{\sigma^c} (T_\Gamma^{\sigma^c})^*$$

by lemma (2.6), there exist two bounded operators $L_1, L_2 : \mathscr{U} \Longrightarrow \bigoplus_{i\in I} \mathscr{V}_i$ such that

$$Kf = T_\Lambda^\sigma L_1 f + T_\Gamma^{\sigma^c} L_2 f, \quad \forall f \in \mathscr{U}.$$

Let $L_1 f = \{f_i\}_{i\in I} \in \bigoplus_{i\in I} \mathscr{V}_i$ and $L_2 f = \{g_i\}_{i\in I} \in \bigoplus_{i\in I} \mathscr{V}_i$, then

$$\begin{aligned} Kf &= T_\Lambda^\sigma L_1 f + T_\Gamma^{\sigma^c} L_2 f \\ &= T_\Lambda^\sigma \{f_i\}_{i\in I} + T_\Gamma^{\sigma^c} \{g_i\}_{i\in I}. \\ &= \sum_{i\in\sigma} \Lambda_i^* f_i + \sum_{i\in\sigma^c} \Gamma_i^* g_i. \end{aligned}$$

and

$$\|\{f_i\}_{i\in I}\| = \|L_1 f\| \leq \|L_1\| \|f\|$$

$$\|\{g_i\}_{i\in I}\| = \|L_2 f\| \leq \|L_2\| \|f\|.$$

So $\{\Lambda_i\}_{i\in\sigma} \cup \{\Gamma_i\}_{i\in\sigma^c}$ is an atomic system for K.

$ii) \Longrightarrow i)$. Suppose that the family $\{\Lambda_i\}_{i\in\sigma}\cup\{\Gamma_i\}_{i\in\sigma^c}$ is an atomic system for K, for any partition $[\sigma,\sigma^c]$ of I, then the family $\{\Lambda_i\}_{i\in\sigma}\cup\{\Gamma_i\}_{i\in\sigma^c}$ is a g-Bessel sequence for $\mathscr{U}$, then for any $g\in\mathscr{U}$, there exist $\{f_i\}_{i\in I}\in\bigoplus_{i\in I}\mathscr{V}_i$ such that

$$Kg=\sum_{i\in\sigma}\Lambda_i^* f_i+\sum_{i\in\sigma^c}\Gamma_i^* f_i$$

where

$$\|\{f_i\}_{i\in I}\|\leq C\|g\|.$$

Then

$$\begin{aligned}\|K^*f\|^2 &= sup_{g\in\mathscr{U},\|g\|=1}\|\langle K^*f,g\rangle\|\\ &= sup_{g\in\mathscr{U},\|g\|=1}\|\langle f,Kg\rangle\|.\\ &= sup_{g\in\mathscr{U},\|g\|=1}\|\langle f,\sum_{i\in\sigma}\Lambda_i^* f_i+\sum_{i\in\sigma^c}\Gamma_i^* f_i\rangle\|.\\ &= sup_{g\in\mathscr{U},\|g\|=1}\|\langle\sum_{i\in\sigma}\Lambda_i f+\sum_{i\in\sigma^c}\Gamma_i f,f_i\rangle\|.\\ &\leq sup_{g\in\mathscr{U},\|g\|=1}\|\sum_{i\in I}\langle f_i,f_i\rangle\|\|\langle\sum_{i\in\sigma}\Lambda_i f+\sum_{i\in\sigma^c}\Gamma_i f,\sum_{i\in\sigma}\Lambda_i f+\sum_{i\in\sigma^c}\Gamma_i f\rangle\|.\\ &\leq C\|\langle\sum_{i\in\sigma}\Lambda_i f+\sum_{i\in\sigma^c}\Gamma_i f,\sum_{i\in\sigma}\Lambda_i f+\sum_{i\in\sigma^c}\Gamma_i f\rangle\|.\\ &\leq C\|\langle\sum_{i\in\sigma}\Lambda_i f,\sum_{i\in\sigma}\Lambda_i f\rangle+\langle\sum_{i\in\sigma^c}\Gamma_i f,\sum_{i\in\sigma^c}\Gamma_i f\rangle\|.\end{aligned}$$

Hence

$$\frac{1}{C}\|K^*f\|^2\leq\|\langle\sum_{i\in\sigma}\Lambda_i f,\sum_{i\in\sigma}\Lambda_i f\rangle+\langle\sum_{i\in\sigma^c}\Gamma_i f,\sum_{i\in\sigma^c}\Gamma_i f\rangle\|.$$

Therefore, the family $\{\Lambda_i\}_{i\in\sigma}\cup\{\Gamma_i\}_{i\in\sigma^c}$ is a g-Bessel sequence, then t he families $\{\Lambda_i\}_{i\in I}$ and $\{\Gamma_i\}_{i\in I}$ are woven K-g-frames. □

Proposition 3.8. *Let $\Lambda=\{\Lambda_i\}_{i\in I}$ and $\Gamma=\{\Gamma_i\}_{i\in I}$ be two g-Bessel sequences in $\mathscr{U}$ with respect to $\{\mathscr{V}_i : i\in I\}$ with g-Bessel bounds B_1, B_2, respectively. If for $J\subset I$; $\Lambda_J=\{\Lambda_i\}_{i\in J}$ and $\Gamma_J=\{\Gamma_i\}_{i\in J}$ are woven K-g-frames, then Λ and Γ are woven K-g-frames for $\mathscr{U}$.*

Proof. Let A be universal lower bound for the woven K-g-frame Λ_J and Γ_J, and let $\sigma\subset I$ be a subset of I. Then,

$$\begin{aligned} A\langle K^*f,K^*f\rangle &\leq \sum_{j\in\sigma\cap J}\langle\Lambda_j f,\Lambda_j f\rangle+\sum_{j\in\sigma^c\cap J}\langle\Gamma_j f,\Gamma_j f\rangle \\ &\leq \sum_{j\in\sigma}\langle\Lambda_j f,\Lambda_j f\rangle+\sum_{j\in\sigma^c}\langle\Gamma_j f,\Gamma_j f\rangle. \\ &\leq (B_1+B_2)\langle f,f\rangle. \end{aligned}$$

Hence, Λ and Γ are woven K-g-frames for $\mathscr{U}$. □

Theorem 3.9. *Let $\Lambda=\{\Lambda_i\}_{i\in I}$ and $\Gamma=\{\Gamma_i\}_{i\in I}$ be woven K-g-frames for $\mathscr{U}$ with respect to $\{\mathscr{V}_i : i\in I\}$ with universal K-g-frame bounds A and B. If for all $f\in\mathscr{U}$ $\sum_{j\in J}\langle\Lambda_j f,\Lambda_j f\rangle\leq D\langle K^*f,K^*f\rangle$ for some $0<D<A$ and some $J\subset I$ Then $\Lambda_0=\{\Lambda_i\}_{i\in I\setminus J}$ and $\Gamma_0=\{\Gamma_i\}_{i\in I\setminus J}$ are woven K-g-frames for $\mathscr{U}$ with universal K-g-frame boundsA $-D$ and B.*

Proof. Let σ be a subset of $I\setminus J$, then

$$\begin{aligned} \sum_{j\in\sigma}\langle\Lambda_j f,\Lambda_j f\rangle+\sum_{j\in\setminus(J\cup\sigma)}\langle\Gamma_j f,\Gamma_j f\rangle &= (\sum_{j\in\sigma\cup J}\langle\Lambda_j f,\Lambda_j f\rangle-\sum_{j\in J}\langle\Lambda_j f,\Lambda_j f\rangle)+\sum_{j\in\setminus(J\cup\sigma)}\langle\Gamma_j f,\Gamma_j f\rangle. \\ &= (\sum_{j\in\sigma\cup J}\langle\Lambda_j f,\Lambda_j f\rangle+\sum_{j\in\setminus(J\cup\sigma)}\langle\Gamma_j f,\Gamma_j f\rangle)-\sum_{j\in J}\langle\Lambda_j f,\Lambda_j f\rangle. \\ &\geq A\langle K^*f,K^*f\rangle-D\langle K^*f,K^*f\rangle \\ &= (A-D)\langle K^*f,K^*f\rangle, \quad \forall f\in\mathscr{U}. \end{aligned}$$

And for the upper bound

$$\begin{aligned} \sum_{j\in\sigma}\langle\Lambda_j f,\Lambda_j f\rangle+\sum_{j\in\setminus(J\cup\sigma)}\langle\Gamma_j f,\Gamma_j f\rangle &\leq \sum_{j\in\sigma\cup J}\langle\Lambda_j f,\Lambda_j f\rangle+\sum_{j\in\setminus(J\cup\sigma)}\langle\Gamma_j f,\Gamma_j f\rangle \\ &\leq B\langle f,f\rangle. \end{aligned}$$

It follows that, Λ_0 and Γ_0 are woven K-g-frames for U with the universal lower and upper K-g-frame bounds $A-D$ and B, respectively □

Theorem 3.10. *Let $\Lambda=\{\Lambda_i\}_{i\in I}$ and $\Gamma=\{\Gamma_i\}_{i\in I}$ be a pair of K-g-frames for $\mathscr{U}$ with respect to $\{\mathscr{V}_i : i\in I\}$ with universal K-g-frame bounds A_1, B_1 and A_2, B_2, respectively. Assume that there are constants $0<\alpha,\beta,\mu<1$ such that*

$$\alpha\sqrt{B_1}+\beta\sqrt{B_2}+\mu<\frac{A_1}{2(\sqrt{B_1}+\sqrt{B_1})}$$

and

$$\|\sum_{i\in I}\langle(\Lambda_i^*-\Gamma_i^*)f_i,(\Lambda_i^*-\Gamma_i^*)f_i\rangle\|^{\frac{1}{2}}\leq\alpha\|\sum_{i\in I}\langle\Lambda_i^*f,\Lambda_i^*f\rangle\|^{\frac{1}{2}}+\beta\|\sum_{i\in I}\langle\Gamma_i^*f,\Gamma_i^*f\rangle\|^{\frac{1}{2}}+\mu\|\langle\{f_i\},\{f_i\}\rangle\|^{\frac{1}{2}}$$

for all $\{f_i\}\in(\oplus\mathscr{V}_i)_{i\in I}$. *Then,* Λ *and* Γ *are woven K-g-frames with universal lower and upper frame bounds* $A_1-\frac{A_1}{2}\|K^\dagger\|$ *and* B_1+B_2*, respectively.*

Proof.

$$\begin{aligned}
\|\sum_{i\in\sigma}\Lambda_i^*\Lambda_if-\sum_{i\in\sigma}\Gamma_i^*\Gamma_if\| &= \|T_\Lambda^\sigma(\{\Lambda_if\}_{i\in\sigma})-T_\Gamma^\sigma(\{\Gamma_if\}_{i\in\sigma})\|\\
&= \|T_\Lambda^\sigma(T_\Lambda^\sigma)^*f-T_\Gamma^\sigma(T_\Gamma^\sigma)^*f\|.\\
&= \|T_\Lambda^\sigma(T_\Lambda^\sigma)^*f-T_\Lambda^\sigma(T_\Gamma^\sigma)^*f+T_\Lambda^\sigma(T_\Gamma^\sigma)^*f-T_\Gamma^\sigma(T_\Gamma^\sigma)^*f\|.\\
&\leq \|T_\Lambda^\sigma(T_\Lambda^\sigma)^*f-T_\Lambda^\sigma(T_\Gamma^\sigma)^*f\|+\|T_\Lambda^\sigma(T_\Gamma^\sigma)^*f-T_\Gamma^\sigma(T_\Gamma^\sigma)^*f\|.\\
&\leq \|T_\Lambda^\sigma(T_\Lambda^\sigma)^*f-T_\Lambda^\sigma(T_\Gamma^\sigma)^*f\|+\|T_\Lambda^\sigma(T_\Gamma^\sigma)^*f-T_\Gamma^\sigma(T_\Gamma^\sigma)^*f\|.\\
&\leq \|T_\Lambda^\sigma\|\|(T_\Lambda^\sigma)^*f-(T_\Gamma^\sigma)^*f\|+\|T_\Lambda^\sigma-T_\Gamma^\sigma\|\|(T_\Gamma^\sigma)^*\|\|f\|.\\
&\leq \|T_\Lambda\|\|T_\Lambda-T_\Gamma\|\|f\|+\|T_\Lambda-T_\Gamma\|\|(T_\Gamma)\|\|f\|.\\
&\leq \|T_\Lambda\|\|T_\Lambda-T_\Gamma\|\|K^\dagger\|\|K^*f\|+\|T_\Lambda-T_\Gamma\|\|T_\Gamma\|\|K^\dagger\|\|K^*f\|.\\
&\leq (\alpha\|T_\Lambda\|+\beta\|T_\Gamma\|+\mu)(\|T_\Lambda\|+\|T_\Gamma\|)\|K^\dagger\|\|K^*f\|.\\
&< \frac{A_1}{2(\sqrt{B_1}+\sqrt{B_1})}(\sqrt{B_1}+\sqrt{B_1})\|K^\dagger\|\|K^*f\|.\\
&= \frac{A_1}{2}\|K^\dagger\|\|K^*f\|.
\end{aligned}$$

On the other hand

$$\begin{aligned}
\|\sum_{i\in\sigma^c}\Lambda_i^*\Lambda_if+\sum_{i\in\sigma}\Gamma_i^*\Gamma_if\| &= \|\sum_{i\in\sigma^c}\Lambda_i^*\Lambda_if+\sum_{i\in\sigma}\Lambda_i^*\Lambda_if-\sum_{i\in\sigma}\Lambda_i^*\Lambda_if+\sum_{i\in\sigma}\Gamma_i^*\Gamma_if\|\\
&= \|\sum_{i\in I}\Lambda_i^*\Lambda_if+\sum_{i\in\sigma}\Gamma_i^*\Gamma_if-\sum_{i\in\sigma}\Lambda_i^*\Lambda_if\|.\\
&\geq \|\sum_{i\in I}\Lambda_i^*\Lambda_if\|-\|\sum_{i\in\sigma}\Lambda_i^*\Lambda_if-\sum_{i\in\sigma}\Gamma_i^*\Gamma_if\|.\\
&\geq A_1\|K^*f\|-\|\sum_{i\in\sigma}\Lambda_i^*\Lambda_if-\sum_{i\in\sigma}\Gamma_i^*\Gamma_if\|.\\
&\geq A_1\|K^*f\|-\frac{A_1}{2}\|K^\dagger\|\|K^*f\|.
\end{aligned}$$

$$= (A_1 - \frac{A_1}{2}\|K^\dagger\|)\|K^* f\|.$$

So $(A_1 - \frac{A_1}{2}\|K^\dagger\|)$ is an universal lower bound, and one can see that $B_1 + B_2$ is an universal upper bound. □

Theorem 3.11. *Let $\Lambda = \{\Lambda_i\}_{i\in I}$ and $\Gamma = \{\Gamma_i\}_{i\in I}$ be woven K-g-frames for $\mathscr{U}$ with respect to $\{\mathscr{V}_i : i \in I\}$ with universal K-g-frame bounds A_1, B_1 and A_2, B_2, respectively. Assume that there are constants $0 < \alpha, \beta, \mu < 1$ such that*

$$\alpha B_1\|K^\dagger\| + \beta B_2\|K^\dagger\| + \mu\|K^\dagger\| < A_1$$

and

$$\|\sum_{i\in\sigma}\langle(\Lambda_i^*\Lambda_i - \Gamma_i^*\Gamma_i)f_i, (\Lambda_i^*\Lambda_i - \Gamma_i^*\Gamma_i)f_i\rangle\|^{\frac{1}{2}} \le \alpha\|\sum_{i\in\sigma}\langle\Lambda_i^*\Lambda_i f, \Lambda_i^*\Lambda_i f\rangle\|^{\frac{1}{2}} + \beta\|\sum_{i\in\sigma}\langle\Gamma_i^*\Gamma_i f, \Gamma_i^*\Gamma_i f\rangle\|^{\frac{1}{2}}$$
$$+ \mu(\sum_{i\in\sigma}\|\Lambda_i f\|)^{\frac{1}{2}}$$

for all $f \in \mathscr{U}$ and $\sigma \subset I$. Then, Λ and Γ are woven K-g-frames with universal lower and upper frame bounds $(A_1 - \alpha B_1\|K^\dagger\| - \beta B_2\|K^\dagger\| - \mu\|$ and $(B_1 + \alpha B_1 + \beta B_2 + \mu\sqrt{B_1})$, respectively.

Proof. For any $\sigma \in I$, we have by hypothesis

$$\|\sum_{i\in\sigma}(\Lambda_i^*\Lambda_i\| \le \alpha\|\sum_{i\in\sigma}\Lambda_i^*\Lambda_i f\| + \beta\|\sum_{i\in\sigma}\Gamma_i^*\Gamma_i f\| + \mu(\sum_{i\in\sigma}\|\Lambda_i f\|)^{\frac{1}{2}}$$

then

$$\begin{aligned}
\|\sum_{i\in\sigma^c}\Lambda_i^*\Lambda_i f + \sum_{i\in\sigma}\Gamma_i^*\Gamma_i f\| &= \|\sum_{i\in\sigma^c}\Lambda_i^*\Lambda_i f + \sum_{i\in\sigma}\Lambda_i^*\Lambda_i f - \sum_{i\in\sigma}\Lambda_i^*\Lambda_i f + \sum_{i\in\sigma}\Gamma_i^*\Gamma_i f\| \\
&= \|\sum_{i\in I}\Lambda_i^*\Lambda_i f + \sum_{i\in\sigma}\Gamma_i^*\Gamma_i f - \sum_{i\in\sigma}\Lambda_i^*\Lambda_i f\|. \\
&\ge \|\sum_{i\in I}\Lambda_i^*\Lambda_i f\| - \|\sum_{i\in\sigma}\Lambda_i^*\Lambda_i f - \sum_{i\in\sigma}\Gamma_i^*\Gamma_i f\|. \\
&\ge A_1\|K^* f\| - \|\sum_{i\in\sigma}\Lambda_i^*\Lambda_i f - \sum_{i\in\sigma}\Gamma_i^*\Gamma_i f\|. \\
&\ge A_1\|K^* f\| - \alpha\|\sum_{i\in\sigma}\Lambda_i^*\Lambda_i f\| - \beta\|\sum_{i\in\sigma}\Gamma_i^*\Gamma_i f\| - \mu(\sum_{i\in\sigma}\|\Lambda_i f\|)^{\frac{1}{2}}. \\
&\ge (A_1 - \alpha B_1\|K^\dagger\| - \beta B_2\|K^\dagger\| - \mu\|K^\dagger\|)\|K^* f\|
\end{aligned}$$

.

On the other hand

$$\begin{aligned}\|\sum_{i\in\sigma^c}\Lambda_i^*\Lambda_i f+\sum_{i\in\sigma}\Gamma_i^*\Gamma_i f\| &= \|\sum_{i\in I}\Lambda_i^*\Lambda_i f+\sum_{i\in\sigma}\Gamma_i^*\Gamma_i f-\sum_{i\in\sigma}\Lambda_i^*\Lambda_i f\|.\\ &\leq \|\sum_{i\in I}\Lambda_i^*\Lambda_i f\|+\|\sum_{i\in\sigma}\Lambda_i^*\Lambda_i f-\sum_{i\in\sigma}\Gamma_i^*\Gamma_i f\|.\\ &\leq B_1\|f\|+\alpha\|\sum_{i\in\sigma}\Lambda_i^*\Lambda_i f\|+\beta\|\sum_{i\in\sigma}\Gamma_i^*\Gamma_i f\|+\mu(\sum_{i\in\sigma}\|\Lambda_i f\|)^{\frac{1}{2}}.\\ &\leq (B_1+\alpha B_1+\beta B_2+\mu\sqrt{B_1})\|f\|.\end{aligned}$$

Then, Λ and Γ are woven k-g-frames with the universal lower and upper bounds $(A_1 - \alpha B_1\|K^\dagger\| - \beta B_2\|K^\dagger\| - \mu\|K^\dagger\|)$ and $(B_1+\alpha B_1+\beta B_2+\mu\sqrt{B_1})$, respectively. □

Conflict of Interests

The author(s) declare that there is no conflict of interests.

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