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ZERO DIVISOR GRAPH OF BOOLEAN LATTICE

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Abstract. The concept of zero divisor graph has been previously studied in algebraic structures like commutative rings, semi groups, semi lattices and ordered sets. In this paper, we investigate the properties of zero divisor graph of boolean lattices. Let L be a lattice and $\Gamma(L)$ be the zero divisor graph of L. We find a relationship between the clique number and chromatic number of $\Gamma(L)$ and some properties of zero divisor graph of boolean lattices. We also study the relationship between $aut(L)$ and $aut(\Gamma(L))$ and establish the isomorphism between them.

Keywords: zero divisor graph; boolean lattice; clique number; chromatic number.

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1. Introduction

Zero divisors of a commutative ring R with unity is $Z(R) = \{x \in R : x \neq 0 \text{ and } xy = 0 \text{ for some } y \neq 0\}$. $\Gamma(R)$ is a simple graph with vertex set $Z(R)^* = Z(R)/\{0\}$ and distinct x and y are adjacent if and only if $xy = 0$. This concept of zero divisor graph of commutative ring was introduced by Beck [1]. The idea of zero divisor graph has been well studied in other algebraic structures like semi groups, semi lattices and ordered sets. E. Estaji and K. Khashyarmanesh [4]

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introduced a simple graph $\Gamma(L)$ associated to a lattice L, whose vertex set is $Z(L)^* = Z(L)/\{0\}$, the non-zero zero divisors of L, with two vertices x and y adjacent if and only if $x \wedge y = 0$. T. T. Chelvam and S. Nithya [2] gave results related to dominating set and independent number of zero divisor graph of lattice. R. Deore and P. Tayde [3] introduced a concept of uniquely complemented ideal B of a lattice L and properties of $\Gamma(L)$ and $\Gamma(B)$. There are also other literature related to zero divisor graph of lattices [6, 7, 5].

In this paper, we investigate the properties of zero divisor graph of boolean lattices. Beck conjectured that both chromatic number and clique number of zero divisor graph of ring are equal. Here we extend it into lattices. We prove that atoms of boolean lattice can be identified from the zero divisor graph and that the atoms have maximum degree. Also we find a relationship between automorphism group of lattices and automorphism group of zero divisor graph of lattices.

First, we recall some definitions and notations from graph theory. A graph $G = (V,E)$ is said to be a connected graph if there is a path between each pair of its distinct vertices. For two vertices x and y, $d(x,y)$ is the length of the shortest path between x and y and $d(x,y) = \infty$ if there is no path. The diameter of G, denoted by $diam(G)$, is defined as $\sup\{d(x,y) : x$ and y are distinct vertices of $G\}$. The degree of a vertex a is the number of the edges of the graph G incident with a. The maximum degree vertex of a graph G is $\Delta(G) = \max\{deg\ v : v$ is a vertex of $G\}$. Pendant vertex is a vertex with degree one. A clique of a graph is a complete sub graph of the graph and the number of vertices in the largest clique is the clique number denoted by $cl(G)$. A cut vertex is a vertex whose deletion along with incident edges results in a graph with more components than the original graph. Similarly, cut edge is an edge whose removal from the graph increases the number of components than the previous. Whenever cut edges exist, cut vertices also exist. But, if cut vertices exist, it is not necessary that cut edges may exist. For a graph G, $\chi(G)$ denotes the chromatic number of the graph G, that is, the minimum number of colors which can be assigned to the vertices of G in such a way that every two adjacent vertices have different colors. For a positive integer r, an r-partite graph is one whose vertex set can be partitioned into r subsets, so that no edge has both ends in any one subset. Complete graph is a simple graph in which every pair of distinct vertices is connected by a unique edge. A complete r-partite graph is one in which each vertex is joined to every vertex that is not in the same partition. A graph G is said to be star graph if G contains at least two vertices and there exists a vertex that is adjacent to all other vertices in G and G has no other edges.

2. Preliminaries

Here we give some important basic definitions and results along with illustrations that are used in the following sections.

Definition 2.1. *A lattice is an algebra $L = (L, \wedge, \vee)$ satisfying the following conditions: for all $a, b \in L$,*

(1) $a \wedge a = a$, $a \vee a = a$,

(2) $a \wedge b = b \wedge a$, $a \vee b = b \vee a$,

(3) $(a \wedge b) \wedge c = a \wedge (b \wedge c)$, $a \vee (b \vee c) = (a \vee b) \vee c$,

(4) $a \wedge (a \vee b) = a \vee (a \wedge b) = a$.

Theorem 2.2. *Let L be a lattice. One can define an order $\leq$ on L as follows:*
For any $a, b \in L$, we set $a \leq b$ if and only if $a \wedge b = a$. Then $(L, \leq)$ is an ordered set in which every pair of elements has a greatest lower bound (glb) and a least upper bound(lub). Conversely, let P be an ordered set such that, for every pair $a, b \in P$, $glb(a,b)$, $lub(a,b) \in P$, we define $a \wedge b = glb(a,b)$ and $a \vee b = lub(a,b)$. Then $(P, \wedge, \vee)$ is a lattice.

Definition 2.3. *Let $(L, \wedge, \vee)$ be a lattice with least element 0. Then $a \in L$ is said to be an atom if there is no $y \in L$ such that $0 < y < a$. The set of all atoms of L is denoted by $A(L)$. The lattice L is said to be atomic if for every $x \in L$, there exists an element $a \in A(L)$ such that $a \leq x$.*
For $a \in L$, $[a]\uparrow = \{x \in L : a \leq x\}$.

Definition 2.4. *Let L be a lattice with 0 element. $x \in L$ is said to have a pseudocomplement, if there exists an element $x^* \in L$, disjoint from x with the property that $x \wedge a = 0$ if and only if $x^* \wedge a = a$ if and only if $a \leq x^*$. A lattice in which every element is a pseudocomplement of some other element is called pseudocomplemented.*

The following observations can also be made:

(1) Pseudocomplemented lattice is bounded.

(2) every finite distributive lattice is pseudocomplemented.

Definition 2.5. *A lattice L is said to be Boolean lattice if*

(1) L is distributive.

(2) L has 0 *and* 1*.*

(3) Each $a \in L$ *has a complement* $a' \in L$*.*

Also every finite Boolean lattice is atomic and lub of the set of atoms is 1.

Proposition 2.6. [4] *The graph* $\Gamma(L)$ *is connected and diam*$(\Gamma(L) \leq 3$.

Theorem 2.7. [4] *The clique number of* $\Gamma(L)$ *is precisely the number of atoms in L.*

Theorem 2.8. *The lattice L has exactly two atoms if and only if* $\Gamma(L)$ *is a complete bipartite graph.*

If Ł has two exactly atoms then the zero divisor graph is complete bipartite. Figure 1 illustrates this fact.

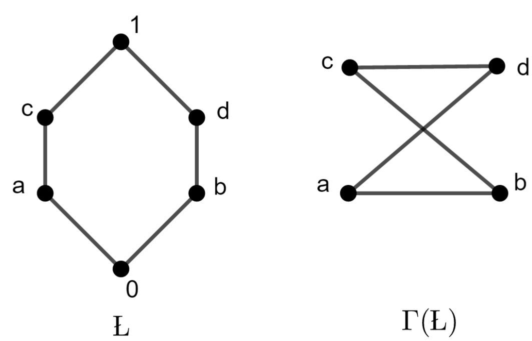

FIGURE 1. Lattice with two atoms

Lemma 2.9. *Let L be a complemented distributive lattice. An element* $b \in L$ *is an atom if and only if* b' *is the unique end adjacent to b.*

In the following section we investigate the interplay between lattice theoretic properties and graph theoretic properties.

3. Atoms of Lattice in Zero Divisor Graphs

We propose the following theorem which makes a relationship between clique number and chromatic number using atoms of the lattice.

Theorem 3.1. *Let L be a finite atomic lattice with n atoms then* $cl(\Gamma(L)) = \chi(\Gamma(L))$.

Proof. Let $a_1, a_2, ..., a_n$ be n atoms in L.
$Z(L) \subseteq [a_1]\uparrow \cup [a_2]\uparrow \cup ... \cup [a_n]\uparrow$.
Since a_i is common in every element in $[a_i]\uparrow$, they are not adjacent in $\Gamma(L)$. Also if $x \in [a_i]\uparrow \cap [a_j]\uparrow$ for $1 \leq i, j \leq n$ then x is not adjacent to a_i and a_j. Thus there exist disjoint sets $A_1, A_2, ..., A_n$ such that

$$Z(L) = A_1 \cup A_2 \cup ... A_n.$$

Then $\Gamma(L)$ is an n-partite graph. n-partite graph is n colorable. Then $\chi(\Gamma(L)) \leq n$. But there are n atoms. So $cl(\Gamma(L)) = n$.

$$cl(\Gamma(L)) \leq \chi(\Gamma(L)) \leq n.$$

This implies $cl(\Gamma(L)) = \chi(\Gamma(L))$. □

The following proposition is used to determine the atoms of Boolean lattce from the zero divisor graph of the lattice.

Proposition 3.2. *Let L be a Boolean lattice then the number of atoms are the number of end vertices.*

Proof. We prove that b is an end vertex of $\Gamma(L)$ if and only if b is a complement of an atom in L. First suppose that b is a complement of atom, say a. That is $a' = b$. Assume there exist $x \neq 0$ such that $b \wedge x = 0$. Since every finite Boolean lattice is pseudocomplemented, $x \leq b' = a$ but a is an atom. So $x = a$, that is L is adjacent to a only.
Conversely assume that b is an end vertex and y is the vertex adjacent to b. $b \wedge y = 0$ and $b \wedge b' = 0$. But b is end vertex so $b' = y$, that is $y' = b$. y is unique adjacent to b. By 2.9, y is an atom. □

Consider the lattice and zero divisor graph of lattice. We can find the number of atoms from the graph.

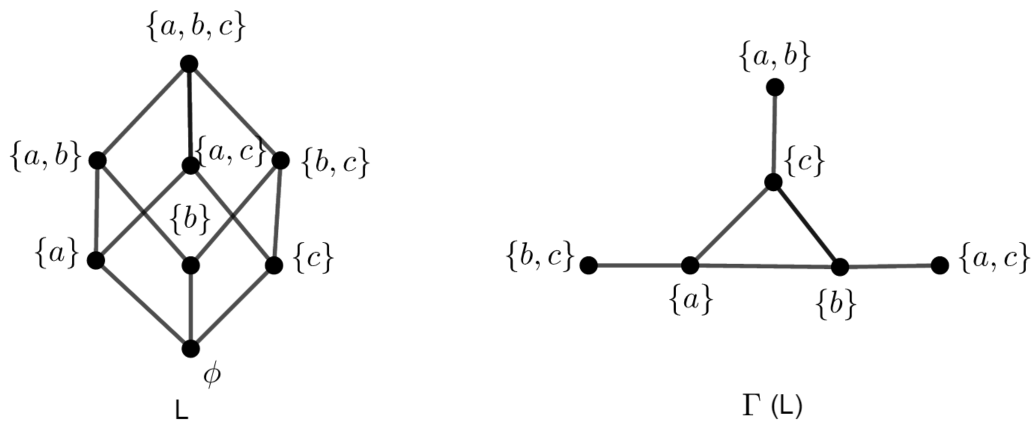

FIGURE 2. count of atoms from the zero divisor graph

In figure 2 both the number of atoms and number of vertices are same.

Diameter of a Boolean lattice never become two and is either equal to one or three. $diam(L) = 1$, when there is exactly two atoms.

Theorem 3.3. *Let L be a Boolean lattice with more than two atoms then* $diam(\Gamma(L)) = 3$.

Proof. Let a_1 and a_2 be any two atoms in L. a_1' and a_2' are the complement of a_1 and a_2 respectively. In zero divisor graph of Boolean lattices complement of atoms are adjacent to the corresponding atoms only. So $d(a_1', a_2') = 3$. Also we have $diam(\Gamma(L) \leq 3$. Hence the theorem. □

But in general there are lattices with $diam(\Gamma(L)) = 2$. The zero divisor graph of the lattice in figure 3 has diameter 2. L is the lattice consisting of the divisors of 24 with order $\leq$, gcd is the meet and lcm is the join.

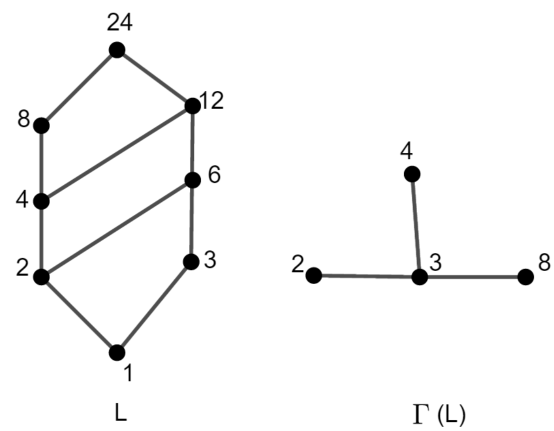

FIGURE 3. lattice with diameter 2

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$\Gamma(L)$ does not have cut vertex in general. But in the case of Boolean lattice there are some cut vertex.

Theorem 3.4. *Zero divisor graph of Boolean lattice L with $|L| > 4$ has cut vertices and number of cut vertices is the number of atoms.*

Proof. Let L be a Boolean lattice. Let a be an atom of L and a' be the complement of a. By lemma 2.9, a' is unique end adjacent to a. By removing a from $\Gamma(L)$, it splits into two components. This implies every atom is a cut vertex.
All other zero divisors other than atoms and their complements are part of cycles so they are not cut vertex. Also if removing complement of an atom, form a connected graph. Therefore atoms are the only cut vertices. □

Not all lattice has cut vertex. Consider the example diamond lattice M_3.

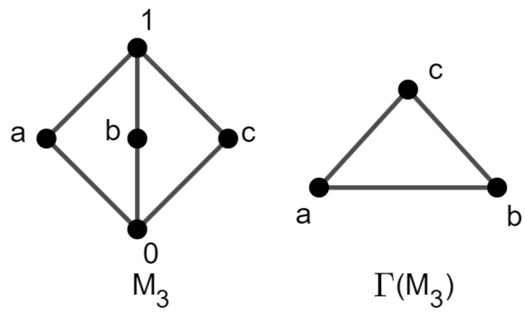

FIGURE 4. lattice without cut vertex.

In figure 4, there are three atoms a,b and c but they are not cut vertices of the zero divisor graph.

Corollary 3.5. *Zero divisor graph of Boolean lattice has cut edges.*

Proof. In the Boolean lattice the edge connecting atom and its complement is a cut edge. □

Theorem 3.6. *Zero divisor graph of a Boolean lattice is star graph if and only if it has exactly two atoms.*

Proof. Let L be a Boolean lattice. First suppose that L has exactly two atoms then $\Gamma(L)$ is star graph.
Conversely suppose that $\Gamma(L)$ is star graph and has more than two atoms, that is there exist a vertex adjacent to all other vertices. Let a be the vertex adjacent to all other vertices. Let $x \in Z(L)/\{a,0\}$. By assumption $x \wedge a = 0$. Since a and $x \in L$, $a \vee x \in L$. Consider $a \wedge (a \vee x) = a \neq 0$, that is $a \vee x$ is not a zero divisor. Then $a \vee x = 1$, that is $x = a'$. So $L = \{0, a, a', 1\}$, a contradiction. Therefore $\Gamma(L)$ has exactly two atoms . □

In the similar manner we can say that $\Gamma(L)$ is complete if and only if it has exactly two atoms.

Automorphism on a Lattice L is an isomorphism that preserves meet and join and set of all automorphism form a group and is denoted by $aut(L)$.

Proposition 3.7. *Automorphism on finite Boolean lattice is determined by its action on atoms.*

Proof. Let L be a Boolean lattice with n atoms and $a_1, a_2, ..., a_n$ be n atoms in L. Let ϕ be an automorphism on L. Suppose that $\phi(a_i) = x$, not an atom for some i such that $1 \leq i \leq n$. Since finite Boolean lattice is atomic there exist $a_j, 1 \leq j \leq n$ such that $a_j \leq x$ and there exist y such that $\phi(y) = a_j$. Then $y \leq a_i$, a contradiction. That is automorphism maps atom to atom.
$\phi(a_i) = a_j$ for $1 \leq i, j \leq n$ and $\phi([a_i]\uparrow) = [a_j]\uparrow$.
Then automorphism of L is the permutation on atoms. □

Graph automorphism are those permutation of vertices which preserves the edges of the graph. Then automorphism on $\Gamma(L)$ is also the permutation on atoms.

Theorem 3.8. *Let L be a Boolean lattice with n atoms, then $aut(L)$ and $aut(\Gamma(L))$ are isomorphic.*

Proof. Let $a_1, a_2, ..., a_n$. By proposition 3.7, $aut(L)$ determined by $a_1, a_2, ..., a_n$. For any $\alpha \in aut(L)$, $\alpha(0) = 0$ and $\alpha(1) = 1$.
Let $\sigma = \begin{pmatrix} 1 & 2 & \cdots & n \\ \sigma(1) & \sigma(2) & \cdots & \sigma(n) \end{pmatrix} \in S_n$. Consider $\phi : S_n \to aut(L)$ defined by $\phi(\sigma) = \alpha_\sigma$ such that $\alpha_\sigma(a_i) = a_{\sigma(i)}$. Then ϕ is a homomorphism and bijective. ϕ is an isomorphism. $aut(L)$ and S_n are isomorphic.
In $\Gamma(L)$ $a_1, a_2, \cdots, a_n$ has maximum degree and are equal. They has same adjacency. Let $\gamma \in aut(\Gamma(L))$. Then $\gamma(a_i) = a_j$ for $1 \leq i, j \leq n$. Also $aut(\Gamma(L))$ depends upon atoms. $aut(\Gamma(L)) \leq S_n$.
For any $\sigma = \begin{pmatrix} 1 & 2 & \cdots & n \\ \sigma(1) & \sigma(2) & \cdots & \sigma(n) \end{pmatrix} \in S_n$ there exist an automorphism mapping $a_1, a_2, \cdots, a_n$ to $a_{\sigma(1)}, a_{\sigma(2)}, \cdots, a_{\sigma(n)}$. Then $aut(\Gamma(L))$ is isomorphic to S_n. That is $aut(L)$ and $aut(\Gamma(L))$ are isomorphic □

In other words, $aut(L)$ and $aut(\Gamma(L))$ are isomorphic to S_n and isomorphism is a transitive relation. then they are isomorphic.

4. Degree Sequence of Zero Divisor Graph

The neighbourhood of a vertex v of graph G is the set $N(v)$ consisting of u adjacent to v. Next theorem shows that atom has the maximum neigbhourhood.

Theorem 4.1. *Let L be a finite Boolean lattice with n atoms and a be an atom. Then* $\Delta(\Gamma(L)) = deg(a)$ *and* $deg(a) = {}^{n-1}C_1 + {}^{n-1}C_2 + ... + {}^{n-1}C_{n-1}$.

Proof. Let $a_1, a_2, ..., a_n$ be the n atoms of L. a_1 is adjacent to $a_2, a_3, ..., a_n$ that contribute ${}^{n-1}C_1$ to $deg(a_1)$. Again a_1 is adjacent to $a_i \vee a_j$, $i, j \neq 1$ and this contribute ${}^{n-1}C_2$. continuing in this way, $a_2 \vee a_3 \vee ... \vee a_n$ contribute ${}^{n-1}C_{n-1}$. Then $deg(a_1) = {}^{n-1}C_1 + {}^{n-1}C_2 + ... + {}^{n-1}C_{n-1}$ and is the maximum degree. □

Corollary 4.2. *Let L be a Boolean lattice. In $\Gamma(L)$, all the atoms have same degree.*

Theorem 4.3. *Degree sequence of zero divisor graph of a Boolean lattice with n atoms is*

${}^{n-1}C_1 + {}^{n-1}C_2 + ... + {}^{n-1}C_{n-1}, {}^{n-1}C_1 + {}^{n-1}C_2 + ... + {}^{n-1}C_{n-1}, ...,$ *(${}^{n}C_1$ times)*
${}^{n-2}C_1 + {}^{n-2}C_2 + ... + {}^{n-2}C_{n-2}, {}^{n-2}C_1 + {}^{n-2}C_2 + ... + {}^{n-2}C_{n-2}, ...,$ *(${}^{n}C_2$ times)*
....
${}^{1}C_1, {}^{1}C_1, ...$ *(${}^{n}C_n$ times)*

Proof. Let L be a Boolean lattice with n atoms $a_1, a_2, ..., a_n$. By above theorem, $a_1, a_2, ..., a_n$ has degree ${}^{n-1}C_1 + {}^{n-1}C_2 + ... + {}^{n-1}C_{n-1}$. ${}^{n}C_1$ vertices have this degree. $a_i \vee a_j$, $i < j$ is adjacent to $a_1, a_2, ..., a_{i-1}, a_{i+1}, ..., a_{j-1}, a_{j+1}, ..., a_n$. This contribute ${}^{n-2}C_1$ to degree of $a_i \vee a_j$, $i < j$. As in the above theorem, degree of $a_i \vee a_j$, $i < j$ is ${}^{n-2}C_1 + {}^{n-2}C_2 + ... + {}^{n-2}C_{n-2}$.In the similar manner, the degree sequence of $\Gamma(L)$ is
${}^{n-1}C_1 + {}^{n-1}C_2 + ... + {}^{n-1}C_{n-1}, {}^{n-1}C_1 + {}^{n-1}C_2 + ... + {}^{n-1}C_{n-1}$ (${}^{n}C_1$ times)
${}^{n-2}C_1 + {}^{n-2}C_2 + ... + {}^{n-2}C_{n-2}, {}^{n-2}C_1 + {}^{n-2}C_2 + ... + {}^{n-2}C_{n-2}$ (${}^{n}C_2$ times)
....
${}^{1}C_1, {}^{1}C_1, ...$ (${}^{n}C_n$ times) □

5. Conclusion

In this article we have shown that some properties of a lattice can be identified from the zero divisor graph. We have identified that in the zero divisor graph of Boolean lattices atoms have the maximum degree and found an expression for degree of atoms. Also we have identified a relationship between the automorphism group of boolean lattice and the automorphism group of its zero divisor graphs. Our future research will be devoted to the development of the relationship between automorphism group of a general lattice and its automorphism group of zero divisor graphs.

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CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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