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NUMERICAL SIMULATION OF AN INVERSE PROBLEM: TESTING THE INFLUENCE OF DATA

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Abstract. In this paper, we consider the iterative algorithm proposed by Kozlov, Mazya and Fomin (KMF) for solving the inverse problem for Laplace equation which consiste to determine the missing conditions on a part of the boundary from the overspecified coditions on the accessible part. Several formulations are discussed according to the measure of the underspecified boundary and the condition on the other part to conclude the relationschip between the data problems and the rate of convergence. Numerical tests are developed with the software FreeFem with smooth and non-smooth domains.

Keywords: Inverse problems; Data completion; Iterative method; Laplace equation; Freefem.

2000 AMS Subject Classification: 65C20; 65D15; 68U20

1. Introduction

We consider an inverse problem for Laplace equation which consist to recovering missing conditions on some inaccessible part of the boundary (which can not be evaluated due to the physical difficulties or inaccessibility geometric) from the overspecified boundary data on the remaining part of the boundary. This type of problem arises in several areas

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such as corrosion detection[1], mechanical problem's particulary in the areas of identification of boundaries on domains , determination of initial condition and fault location [2], Geophysic [3] and electroencephalography[4].

The ill-posedness of the problem in the sense of Hadamard makes its resolution by direct methods very difficult, and it deals to serious questions including the existence, uniqueness and stability of the solution, that are the three properties required to define well-posed problem according to Hadamard [5]. The existence of the solution of this kind of problem is not always garuanted, but when the conditions on the accessible part of the boundary are compatible then the existence is assured [6]. Thanks to Holmgreen theorem, we know that this problem has at most one solution [7]. Stability is the most delicate problem since a small perturbation of data provides a large difference between the solution obtained by disturbed data and that obtained by undisturbed data [8]. It suffices here to recall the famous example of Hadamard where he showed for a square domain that, with perturbed data the solution is not bounded even if the data problems tend to zero.

In order to solve the inverse problem for the Laplace equation, we have proposed several performing methods to overcome of the ill-posed nature of this kind of problem. The last ancient of them is the one, based on optimization tools, introduced by Kohn and Vogelius [9]. Other methods were experimented, among them, we mention the method of Quasi-reversibility introduced by Lates since 1960 [10] (see also [11],[12]), Thikhonv method [13], Bakus-Gilbert method applied to moment problem [14], the method applied to the minimisation of an energie like fonctional [15], and the KMF iterative algorithm adressed by Kozlov, Mazya and Fomin [16] (see also [17],[18]).

The group of iterative method has the advantage to allow any physical contrainst to be easily taken into account directly in the scheme of the iterative algorithm and simplicity of the implementation shemes. One possible disavantage of this kind of method is the large number of iterations that may be required in order to achieve convergence. Based of these reasons, we have decided in this study to use the KMF algorithm, also called alternating method, for solving the Cauchy problem for Laplace's equation and study in more detail this method. Particulary, the relationship between the rate of convergence of this algorithm and the data of the problem, specialy; the measure of the inaccessible part of the boundary and the different choice of condition on the accessible part by implementing the algorithm by FEM, using the software FreeFem.

2. Mathematical formulation

Let Ω an open set in $\mathbb{R}^2$, with a smooth boundary Γ. We consider a partition of this boundary: $\Gamma = \Gamma_0 \cup \Gamma_1$ and $\Gamma_0 \cap \Gamma_1 = \emptyset$ and $mes(\Gamma_1) \neq 0$.
The problem is to reconstruct a harmonic function u solution of the following problem :

$$(1) \qquad \begin{cases} -\Delta u = 0 & in \quad \Omega \\ u = f & on \quad \Gamma_0 \\ \partial_n u = g & on \quad \Gamma_0 \end{cases}$$

where $\partial_n u$ is the normal derivative of u.
We can notice that no boundary condition is prescribed on the boundary part Γ_1.
For compatible data $(f, g) \in H(\Gamma_0)$ the problem (1) has a unique solution, where $H(\Gamma_0)$ is the space defined as:

$$H(\Gamma_0) = \Big\{(\varphi, \psi) \in H^{\frac{1}{2}}(\Gamma_0) \times H^{-\frac{1}{2}}(\Gamma_0) / \exists v \in H(\Omega) \quad v_{/\Gamma_0} = f \quad and \quad \partial_n v_{/\Gamma_0} = g\}$$

and

$$H(\Omega) = \{v \in H^1(\Omega) / \Delta v = 0 \quad in \quad \Omega\}$$

Principle of the method: The problem is to determine on the part Γ_1 the traces $u_{/\Gamma_1}$ and $\partial_n u_{/\Gamma_1}$that we denote by f^* and g^*. This therefore amounts to determine u, solution of the following problem:

$$(2) \qquad \begin{cases} -\Delta u = 0 & & in \quad \Omega \\ u = f & , \quad \partial_n u = g & on \quad \Gamma_0 \\ u = f^* & , \quad \partial_n u = g^* & on \quad \Gamma_1 \end{cases}$$

This problem can be divided into two well-posed sub-problems, the first with Dirichlet condition in Γ_0 and Neuman condition in Γ_1, and the second with a Neumann condition in Γ_1 and Dirichlet condition in Γ_0 defined as follows:

$$\begin{cases} -\Delta\breve{u} = 0 & in \quad \Omega \\ \breve{u} = f & on \quad \Gamma_0 \\ \partial_n\breve{u} = g^* & on \quad \Gamma_1 \end{cases} \quad and \quad \begin{cases} -\Delta\bar{u} = 0 & in \quad \Omega \\ \partial_n\bar{u} = g & on \quad \Gamma_0 \\ \bar{u} = f^* & on \quad \Gamma_1 \end{cases} \tag{3}$$

The main idea of the KMF method is:

• To solve the Cauchy problem (1), it is necessary to determine u that satisfies the problem (2), what is covered when $\breve{u}$ and $\bar{u}$ coincide.

• From an initial estimate of the solution $u = f^*$ on Γ_1, the method consists in solving alternately two well posed problems of type (3) where each of these problems provides a condition on the part Γ_1 which will be introduced in the other problem to find another condition.

• Thus, a sequence of well-posed problems with mixed boundary conditions is constructed using alternating the given Dirichlet and Neumann on the part of the boundary containing the data, and the iterative procedure stops when a predefined stop criterion is satisfied.

Description of the alternating algorithm: Consider the Cauchy problem (1) with $f \in H^{\frac{1}{2}}(\Gamma_0)$ and $g \in H^{-\frac{1}{2}}(\Gamma_0)$. The iterative algorithm investigated is based on reducing this ill-posed problem to a sequence of mixed well-posed boundary value problems and consists of the following steps :

Step 1: Specify an initial guess u_0 on Γ_1 and solve:

$$\begin{cases} -\Delta u^{(0)} = 0 & in \quad \Omega \\ u^{(0)} = u_0 & on \quad \Gamma_1 \\ \partial_n u^{(0)} = g & on \quad \Gamma_0 \end{cases} \tag{4}$$

to obtain $v_0 = \partial_n u^{(0)}_{/\Gamma_1}$

Step 2: For $n \geq 0$, solving alternatively the following two mixed well-posed boundary value problems :

$$\begin{cases} -\Delta u^{(2n+1)} = 0 & in \quad \Omega \\ \partial_n u^{(2n+1)} = v_n & on \quad \Gamma_1 \\ u^{(2n+1)} = f & on \quad \Gamma_0 \end{cases} \quad and \quad \begin{cases} -\Delta u^{(2n+2)} = 0 & in \quad \Omega \\ u^{(2n+2)} = u_{n+1} & on \quad \Gamma_1 \\ \partial_n u^{(2n+2)} = g & on \quad \Gamma_0 \end{cases} \tag{5}$$

$$\text{to obtain } u_{n+1} = u^{(2n+1)}_{/\Gamma_1} \text{ , to obtain } v_{n+1} = \partial_n u^{(2n+2)}_{/\Gamma_1}$$

Step 3: Repeat the step2 until a prescribed stopping criterion is satisfied.

Numerical formulation: It is clear that the conditions and the two parts of the boundary are involved in the convergence of this algorithm. Hence, the investigation to consider different formulations from same problem, to better understand the behavior of the algorithm (the number of iteration that is the inconvenience of this method) with respect to the changing conditions of the problem. In particular, the measure of the parts of the boundary and the type of condition on each part of the boundary.

In this study, we present three formulations of the Cauchy problem (1), to be solved by the KMF algorithm to determine the unknowns data in the part Γ_1.

- **Formulation 1**

The first formulation considered has the same form as the problem (1) where we have two conditions (Dirichlet and Neuman) in the all accessible part Γ_0.

$$u = f \quad \text{and} \quad \partial_n u = g \quad \text{on} \quad \Gamma_0 \tag{6}$$

- **Formulation 2**

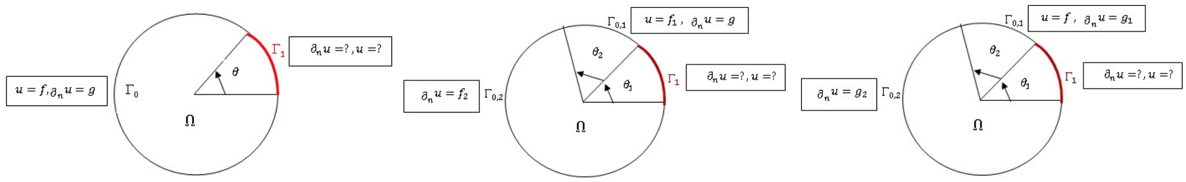
In the second formulation the accessible part is devised in two parts $\Gamma_0 = \Gamma_{0,1} \cup \Gamma_{0,2}$ such as $\Gamma_{0,1} \cap \Gamma_{0,2} = \emptyset$. In the part $\Gamma_{0,1}$ we consider two conditions (the Dirichlet and Newman) and we are satisfied with only one condition (the Dirichlet) in the remaining part $\Gamma_{0,2}$.

$$u = f_2 \quad \text{on} \quad \Gamma_{0,2} \quad , \quad u = f_1 \quad \text{and} \quad \partial_n u = g \quad \text{on} \quad \Gamma_{0,1} \tag{7}$$

- **Formulation 3**

we consider the same formulation as that taken in formulation 2, but the Dirichlet condition in $\Gamma_{0,2}$ is replaced by a Newman condition.

(8) $$\partial_n u = g_2 \quad \text{on} \quad \Gamma_{0,2} \quad , \quad u = f \quad \text{and} \quad \partial_n u = g_1 \quad \text{on} \quad \Gamma_{0,1}$$



formulation 1 *formulation 2* *formulation 3*

The purpose of these three formulations is to complete the missing data on the inaccessible part Γ_1. The three formulations of a problem (1) are solved by the KMF algorithm to examine the rate and accuracy of the convergence.

The reformulation of problem (1) in three forms, the different choices of measurements on the parts of the bonndary and the use of different selection conditions on the accessible part allow us to see the relationship between the data of the problem and the rate and accuracy of convergence, that can be mentioned in the following results :

- The convergence is assured even if the measures are available for a small part of the accessible part of the boundary.
- The measure of accessible and inaccessible parts of boundary affects the rate and accuracy of conergence.
- The convergence is ensured without the need for two conditions throughout the accessible part.
- The type of conditions considered on the accessible part influence the rate of convergence of the algorithm.
- The approximation of the Dirichlet condition is more accurate than the Neumann condition.

3. Numerical result and disccusion

In this section, as a typical example, we consider two examples with smooth and non-smooth boundary using the software FreeFem, which allows the implementation of the iterative algorithm and solve the well-posed problems in the algorithm by the finite element method. In this examples, we use a finit element method with continuous piecewise linear polynomials.

The convergence of the algorithm may be investigated by evaluating at every iteration the error : $e_u = \|u_n - u_{ex}\|_{0,\Gamma_1}$ and $e_v = \|\partial_n u_n - \partial_n u_{ex}\|_{0,\Gamma_1}$, where u_n is the approximation obtained for the function on the boundary Γ_1 after n iterations and u_{ex} is the exact solution of the problem (1). However, in practical applications the error e_u and e_v cannot be evaluated since the analytical solution is not known and therefore the following stoping criterion is addopted $E = \|u_{n+1} - u_n\| \leq 10^{-5}$. Alternative stopping criteria can be found in [19].

Example 1

The following typical benchmark test example in a smooth geometry, such as a disc $\Omega = \{(x, y) \in \mathbb{R}^2/0 \leq x^2 + y^2 \leq 1\}$, namely, the analytical harmonic function to be retrieved is given by: $u_{ex} = x^2 - y^2$.

The under-specified boundary was taken to be $\Gamma_1 = \{(x, y)/x = cos(t), y = sin(t), 0 \leq t < \theta, \text{where}, 0 < \theta < 2\pi\}$ while the overspecified boundary is $\Gamma_0 = \{(x, y)/x = cos(t), y = sin(t), \theta \leq t \leq 2\pi\}$. The unknown data on the under specified boundary Γ_1 are given by: $u(x, y) = 2x^2 - 1$ and $\partial_n u(x, y) = 2(2x^2 - 1)$.

As an initial guess u_0 for the step 1 of the algorithm, we have chosen $u_0 = x^2 - x - \frac{1}{2}$.

We notice that u_0 is not too close to the analytical solution u on the under-specifed boundary Γ_1.

We apply the KMF algorithm to formulation 1 with various choice of angle θ (θ is the parameter that defines the part of the boundary).

Figure 1 and figure 2, show the number of iterations required in order to achieve convergence with respect to the parameter θ.

From figure 1 et figure 2, we observe that when the part of the boundary Γ_1 is small, the algorithm converges rapidly (Two iteration for $\theta = \pi/12$); but we need more iterations to achieve the convergence if the measure of Γ_1 is greater.

For the formulation 2 and formulation 3, we can note that the underspecified boundary is defined as: $\Gamma_1 = \{(x, y)/x = cos(t), y = sin(t), 0 \leq t < \theta_1\}$ and the part $\Gamma_{0,1}$ of the accessible part was taken to be $\Gamma_{0,1} = \{(x, y)/x = cos(t), y = sin(t), \theta_1 \leq t < \theta_1 + \theta_2\}$ while the overspecified boundary is $\Gamma_{0,2} = \{(x, y)/x = cos(t), y = sin(t), \theta_1 + \theta_2 \leq t \leq 2\pi\}$. As initial guess u_0 , we take the same taken in the formulation 1.

The figure 3 (resp. figure 5) representes the convergence of the algorithm for $\theta_1 = \pi/4$, and with different choice of θ_2 with formulation 3 (resp. with formulation 2).

The figure 4 (resp. figure 6) representes the numerical results obtained to calculate the boundary function of u in the point$(cos(\pi/6), sin(\pi/6))$ using the iterative algorithm (KMF) with formulation 3 (with formulation 2) with $\theta_1 = \pi/4$.

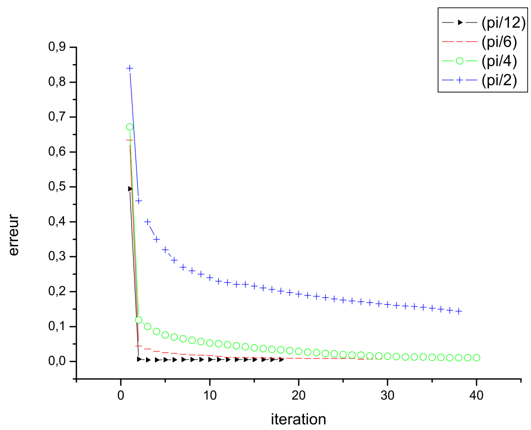


Fig. 1. The error e_u and numbers of iterations for different choices θ with formulation1

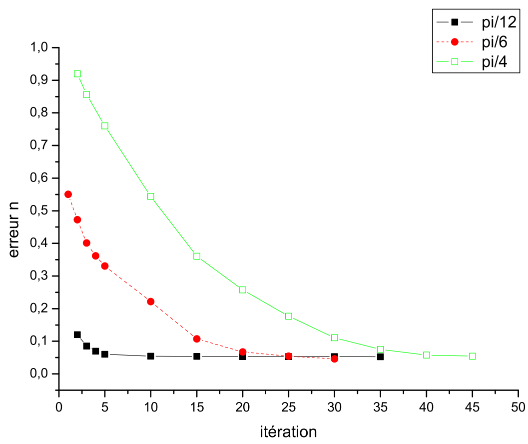


Fig. 2. The error e_v and numbers of iterations for different choices θ with formulation1

We observe from figure 3, figure 4, figure 5 and figure 6 that we do not need two conditions on any part of the boundary for the convergence; However, the convergence becomes faster if we have two conditions (Dirichlet and Neumann) on a substantial part of Γ_0.

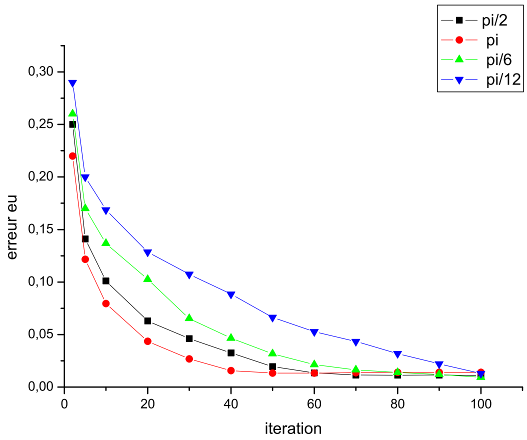


Fig. 3. The error e_u as a function of the number of iterations for different choice θ_2 with the formulation3.

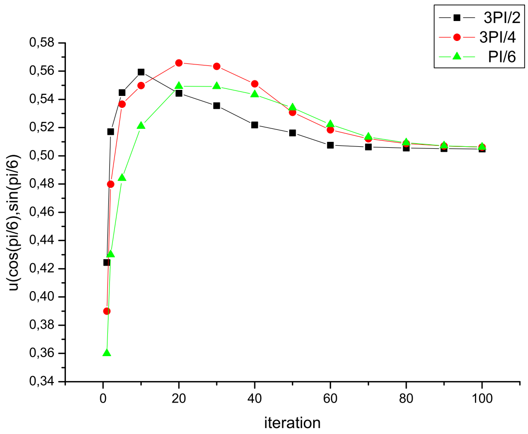


Fig. 4. $u(cos(\pi/6), sin((\pi/6))$ for different choice θ_2 with the formulation3.

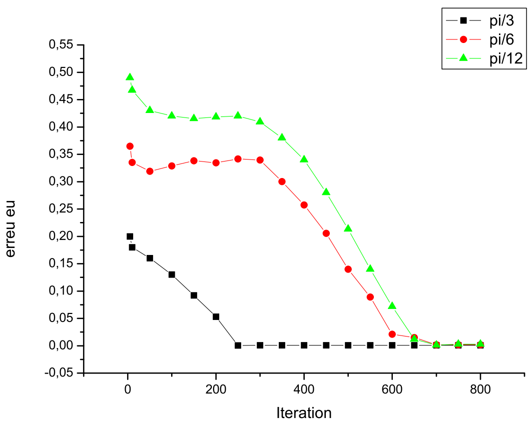


Fig. 5. The erroe e_u as a function of the number of iterations for different choice θ_2 with formulation2.

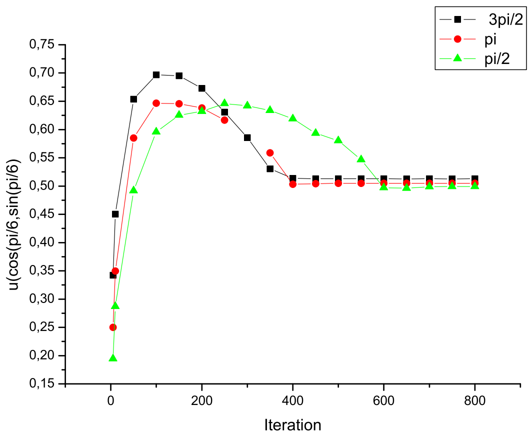


Fig. 6. $u(cos(\pi/6), sin((\pi/6))$ for different choice θ_2 with the formulation2

With two conditions (Dirichlet and Neumann) on a part of Γ_0 and only one condition on the remaining part of this part of boundary, especially with a Neumann condition, one complete data problem on the part of the boundary Γ_1. That can be understood with the fact that the imposition of a Neumann boundary condition contains more information than the imposition of a Direchlet boundary condition.

Figure 7 presents the numerical results obtained for the function u on the boundary $\Gamma_1 = \{(x, y)/x = cos(t), y = sin(t), 0 \leq t \leq \pi/4\}$ by using the three formulations, where in formulation 2 and formulation 3, we take $\theta_2 = \pi/4$. It can be seen that the three

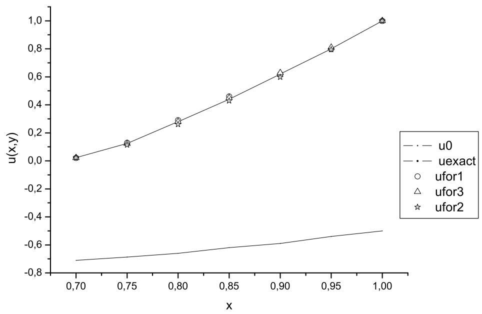


Fig. 7. The numerical results for u on the boundary Γ_1 obtained by using the three formulations in comparison with the analytical solution and the initial guess

formulations considered are equally efficient in producing an accurate numerical solution on the under-specified boundary part on the boundary .

Example 2

We presente here typical benchmark test example in a non-smooth geometry, such as a square $\Omega = (0, L) \times (0, L)$ where $L = 1$, namely, the analytical harmonic function to be retrieved is given by :

$u_{ex}(x) = cos(x)cosh(y) + sin(x)sinh(y)$.

We can take the underspecified boundary as $\Gamma_0 = (0, L) \times \{L\} \cup \{0\} \times (0, L)$ and the overspecified boundary as $\Gamma_1 = (0, L) \times \{0\} \cup \{L\} \times (0, L)$ (see [2]). But, to not violate the hypothesis that Γ_0 and Γ_1 be smooth boundaries on which the mathematical proofs for the convergence of the algorithm (KMF) of Kozlov et al are based, we consider the case when $\Gamma_0 = \{0\} \times (0, L)$ as underspecified boundary, $\Gamma_1 = (0, L) \times \{0\}$, $\Gamma_2 = \{L\} \times (0, L)$ as overspecified boundary and $\Gamma_3 = (0, L) \times \{L\}$.

The known data is given by :

$u_{/\Gamma_1} = cos(x)$,

$u_{/\Gamma_2} = cos(L)cosh(y) + sin(L)sinh(y)$ and $\partial_n u_{/\Gamma_2} = -sin(L)cosh(y) + cos(L)sinh(y)$,

$\partial_n u_{/\Gamma_3} = cos(x)sinh(L) + sin(x)cosh(L)$,

and the unknown data on the underspecified boundary Γ_0 is given by :

$u_{/\Gamma_0} = cosh(y)$ and $\partial_n u_{/\Gamma_0} = -sinh(y)$.

For the step1 of the algorithm, as an initial guess $u_0 \in H^{1/2}(\Gamma_0)$, we have chosen $u_0(y) =$

$1 + y(-L + sinh(L)) + y^2/2$, $y \in [0,1]$, which also ensures the continuity of $\partial u/\partial y$ at the corner $\overline{\Gamma_0} \cap \overline{\Gamma_3}$ and provides that the initial guess is not too close to the exact value of u_{ex}.

In this example, in order to test the influence of lenght of the inaccessible part on the rate of the convergence, we take $\Gamma_0 = \Gamma_{0,1} \cup \Gamma_{0,2}$ where $\Gamma_{0,1} = \{0\} \times (0, a)$ and $\Gamma_{0,2} = \{0\} \times (a, L)$ where $u_{/\Gamma_{0,2}}$ is known with $0 < a \leq L$.

For different choices of lenght a, we use the algorithm (KMF) to obtain the underspecified conditions on $\Gamma_{0,1}$ (see figure 8). We can note that the convergence is also ensured, but; it is faster if $\Gamma_{0,1}$ is more smaller.

Figure 9 presents the numerical results obtained for the function u on the boundary Γ_0 with the three formulations in comparison with the analytical solution and the initial guess. For formulation 2 and formulation 3, the overspecified boundary is devised in two parts $\Gamma_2 = \Gamma_{2,1} \cup \Gamma_{2,2}$ such that $\Gamma_{2,1} \cap \Gamma_{2,2} = \emptyset$, where on $\Gamma_{2,1} = (0, a) \times \{L\}$ we have two conditions and on $\Gamma_{2,2} = (a, L) \times \{L\}$ we have one condition (with $a = 2/3$).

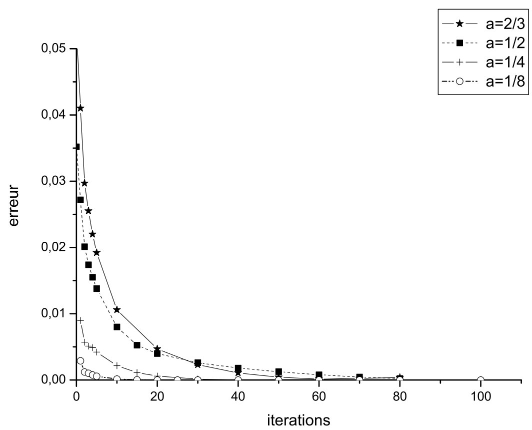


Fig. 8. The numerical results for u on the boundary $\Gamma_{0,1}$ obtained for different choices of lenght with formulation 1

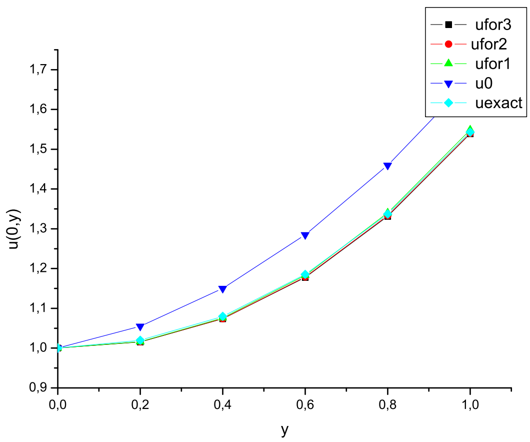


Fig. 9. The numerical results for u on the boundary Γ_0 obtained by using the three formulations in comparison with the analytical solution and the initial guess

Various other tests have been investigated and similar results have been obtained to those obtained for the test example 1.

4. Conclusion

In this paper we have investigated the iterative algorithm (KMF) for a Cauchy problem for Laplace equation. Three formulations with different possibility of boundary conditions are considered to study the influence of data on the convergence of the algorithm. It has been found, by using these three formulations, that the number of iterations necessary to achieve convergence depends on the measure of different parts of the boundary and the type of conditions considered on the accessible part of the boundary. Particularly, the numerical tests performed show that the number of iterations decreases if the inaccessible part of the boundary has a measure smaller; which can be invested to perform the KMF algorithm in order to accelerate the convergence.

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