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SUMUDU TRANSFORM HPM FOR KLEIN-GORDON AND SINE-GORDON EQUATIONS IN ONE DIMENSION FROM AN ANALYTICAL ASPECT

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Abstract: In the present research work, a hybrid algorithm is introduced, which includes an integral transform "Sumudu Transform" and the well-known semi-analytical regime "Homotopy Perturbation Method" named as "Sumudu Transform Homotopy Perturbation Method (STHPM)" to evaluate the exact solution of Klein-Gordon and Sine-Gordon equations. The discussed equations in this research have a prominent role in sciences and engineering. The authenticity and efficacy of this regime are established via a comparison between approximated solutions and exact solutions. Convergence analysis is also provided, which affirms that the solution obtained from STHPM is convergent and unique in nature. The results obtained by STHPM are compared with exact solutions. 2D and 3D plots are also discussed. The present regime is a reliable technique to provide the exact solution to a wide category of non-linear PDEs in an easy way, without any need of discretization, complex computation, linearization, and it is also error-free.

Keywords: Klein-Gordon equation; Sine-Gordon equation; Sumudu transform; HPM; convergence analysis; 2D plots.

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1. INTRODUCTION

1.1. Klein-Gordon Equation

Klein-Gordon equations are one of the most common equations in the modeling of systems in different fields like applied physics. These equations have an important role not only from a mathematical aspect but as well as from a Physics aspect also. For instance, its significance is shown in [1-3]. The general form of the Klein-Gordon equation is as follows:

$$u_{tt} - k\, u_{xx} + g(u) = 0 \qquad (1)$$

A number of analytical methods have been proposed to solve these specific equations. In literature, researchers have contributed a lot in the latest years regarding the analytical solution of partial differential equations. One of the most recent examples is the multi-step differential reduction regime [4, 5]. Elzaki transform is also a tool to deal with the analytical solution, which was proposed by Elzaki [6]. Elzaki transform has been proved as a very useful technique to solve different differential equations.

One another form of the Klein-Gordon equation can be as follows:

$$u_{tt} - u_{xx} + N[u(x,t)] = f(x,t) \qquad (2)$$

I.C.s: $u(x,0) = f(x)$ and $u_t(x,0) = g(x)$

Where, $u(x,t)$ is the function of x. $N[u(x,t)]$ is a non-linear term. $f(x,t)$ is the unknown function. Due to the importance of the Klein-Gordon equation in quantum mechanics, several regimes have been developed, such as Homotopy Perturbation Method, Sumudu Transform Method, New Perturbation Iteration Transform Method, and many others.

1.2. Sine-Gordon Equation

The Sine-Gordon equation is considered one of the most crucial evolution equations (non-linear in nature), which has an important role in engineering and physical science. Sine-Gordon equation has the property of non-linear Hyperbolic PDE, which contains the Sine of an unknown function as well as the d' Alembert operator.

The Sine-Gordon equation was initially proposed in the 19th century to deal with various problems of differential geometry [7]. In the early 1970s, it was first considered that the Sine-Gordon equation might be helpful in generating the Kink and Anti-Kink solutions (Soliton solutions) [8]. The concept of the Sine-Gordon equation is implemented in many physical applications like physical applications in relativistic field theory, mechanical transmission line: Josephson junction, and others.

The standard form of the Sine-Gordon equation is as follows:

$$u_{tt}(x,t) - \alpha^2 u_{xx}(x,t) - \beta sin[u(x,t)] = 0 \tag{3}$$

I.C.s: $u(x,0) = f(x)$ and $u_t(x,0) = g(x)$

Where α and β are the constant values.

Different analytical methods have come into existence to deal with solution of PDEs. ADM [9-12], VIM [13-17], LDM [18], NDM [19-24], HPM [25-27], Tanh method [TM] [28], Exp-function method [EFM] [29], NHPM [30-32], the reduced differential transform method [RDTM] [33-35], and so on. Other useful references are available in [36-38].

1.3. Sumudu Transform

The Sumudu transform, whose basic properties are presented in the present section, is still not widely in consideration as well as nor implemented. In 2003, Belgacem et al. claimed Sumudu transform as a theoretical dual to the Laplace transform.

Due to its simplicity and easy-to-implement properties, Sumudu transform is considered one of the most promising techniques. It can be widely applicable to deal with the problems of engineering mathematics and applied sciences.

Watugala [39] implemented the notion of Sumudu transform in 1993 to deal with the engineering control problems. Weerakoon [41] presented applications of Sumudu transform to the PDEs by following Watugala's [39] work.

Watugala [40] presented the notion that the Sumudu transform is a useful technique to solve different ODEs and engineering control problems. Weerakoon [42] followed one more concept given by Watugala regarding the complex natured inversion formula of the Sumudu transform.

In [43-45], it is presented how to solve the partial-integro-differential equations. Watugala [46] extended the concept of Sumudu transform from one variable to two variables regarding the solution of PDEs. Belgacem et al. [47] implemented an application for the integral equations of convolution type.

Definition

Considered the set of functions:

$$A = \{f(t): there\ exists\ , \eta_1, \eta_2 > 0, |f(t)| < Me^{\left(\frac{|t|}{\eta_j}\right)}, if\ t \in (-1)^j \times [0, \infty)\} \tag{4}$$

Sumudu transform is defined as follows:

$$S[f(t)] = \int_0^\infty f(ut)e^{-t}\, dt, u \in (-\eta_1, \eta_2) \tag{5}$$

One of the most interesting properties regarding Sumudu transform is that it preserves the linear function as well as is itself linear, and therefore it does not change the units [48], [47].

Formulae regarding Sumudu Transform

- $S[1] = 1$
- $S[t] = u$
- $S\left[\frac{t^{n-1}}{\angle n-1}\right] = u^{n-1}, n = 1, 2, 3,\backslash dots$
- $S[e^{at}] = \frac{1}{1-au}$
- $S[e^{-at}] = \frac{1}{1+au}$
- $S[sinat] = \frac{au}{1+a^2u^2}$
- $S[cosat] = \frac{1}{1+a^2u^2}$
- $S[sinh(at)] = \frac{au}{1-a^2u^2}$
- $S[cosh(at)] = \frac{1}{1-a^2u^2}$
- $S[u'(x,t)] = \frac{1}{u}[S\big(u(x,t)\big) - u(x,0)]$
- $S[u''(x,t)] = \frac{1}{u^2}S[u(x,t)] - \frac{u(x,0)}{u^2} - \frac{u'(x,0)}{u}$
- $S\left[u^{(n)}(x,t)\right] = \frac{1}{u^n}S[u(x,t)] - \frac{u(x,0)}{u^n} - \cdots - \frac{u^{(n-1)}(x,0)}{u}$

Formulae regarding Inverse Sumudu Transform.

- $S^{-1}[1] = 1$
- $S^{-1}[u] = t$
- $S^{-1}[u^{n-1}] = u^{n-1}\frac{t^{n-1}}{\angle n-1}, n = 1, 2, 3, \ldots$
- $S^{-1}\left[\frac{1}{1-au}\right] = e^{at}$
- $S^{-1}\left[\frac{1}{1+au}\right] = e^{-at}$
- $S^{-1}\left[\frac{au}{1+a^2u^2}\right] = sinat$
- $S^{-1}\left[\frac{1}{1+a^2u^2}\right] = cosat$
- $S^{-1}\left[\frac{au}{1-a^2u^2} sinh(at)\right] = sinh(at)$
- $S^{-1}\left[\frac{1}{1-a^2u^2}\right] = cosh(at)$

Linearity Property of Sumudu Transform

If $$S[f(t)] = u_1 \; and \; S[g(t)] = u_2$$

then $$S[\alpha f(t) + \beta g(t)] = \alpha S[f(t)] + \beta S[g(t)]$$

$$\Rightarrow S[\alpha f(t) + \beta g(t)] = \alpha\, u_1 + \beta u_2$$

Linearity Property of Inverse Sumudu Transform

If $$S^{-1}[u_1] = f(t) \; and \; S^{-1}[u_2] = g(t)$$

then $$S^{-1}[\alpha u_1 + \beta u_2] = \alpha S^{-1}[u_1] + \beta S^{-1}[u_2])]$$

$$\Rightarrow S^{-1}[\alpha u_1 + \beta u_2] = \alpha\, f(t) + \beta g(t)$$

1.4. HPM

In the last some decades, non-linear science has emerged as an interest of scientists and engineers regarding analytical techniques for the non-linear problems.

The most common applied techniques are known as perturbation techniques, but Perturbation methods have their own limits.

1. Most of the perturbation methods considered that there must be an existence of a small parameter.

2. The determination of this small parameter is not an easy task to evaluate, which needs some special techniques to implement.

3. An unsuitable choice of the small parameter may lead to a wrong result.

But still, the approximated solutions by Perturbation methods are valid. Different Perturbation methods have been applied regarding non-linear problems.

Many novel techniques have been generated based upon the Perturbation concept. He [51] proposed HAM. In [52], a review regarding the recent development in non-linear sciences is provided. He [49, 50] provided HPM to tackle different differential and integral equations. The regime which is generated by coupling of the conventional perturbation method and Homotopy in topology reforms a problem that can be tackled easily. This method does not need any small parameter in the equation.

The HPM provides a rapid convergent solution in non-linear approach. A very less number of iterations is required to get an accurate solution. He' HPM is one of the most well-known regimes which can solve numerous PDEs.

Definition

Let X and Y are topological spaces.

$f, g \quad \rightarrow \quad$ continuous maps from X to Y.

f is homotopic to g if there exists a continuous map.

$$F : X \times [0,1] \rightarrow Y \tag{6}$$

s. t. $\quad F(x,0) = f(x)$ and $F(x,1) = g(x)$

HPM is a hybrid method generated by classical perturbation and Homotopy map used in topology.

Basic notion of HPM

Considered a non-linear differential equation as follows:

$$A(u) - f(r) = 0, r \in \mathrm{D} \tag{7}$$

B.C.: $\quad B\left(u, \frac{\partial u}{\partial \eta}\right) = 0, r \in \Gamma$

Where A is considered as a general differential operator, B is known as a boundary operator, $f(r)$ is the given analytic function, and Γ is the boundary of the provided domain D. A can be splitted into two parts, where L and N are the respective parts, L is considered as the linear part, and N is considered as the non-linear part.

$$A(u) - f(r) = 0$$

will be transformed into,

$$L(u) + N(u) - f(r) = 0$$

In the concept of HPM, construction is Homotopy is as follows:

$$H(r, p): D \times [0,1] \rightarrow R$$

Satisfying,

$$H(\nu, p) = (1 - p)[L(\nu) - L(\nu_0)] + p[A(\nu) - f(r)] = 0$$

and $$p \in [0,1], r \in D$$

or $$H(\nu, p) = L(\nu) - L(\nu_0) + p\, L(u_0) + p[N(\nu) - f(r)] = 0$$

Where, $p \in [0,1]$ is given parameter and u_0 is considered as the initial approximation.

along with, $$H(\nu, 0) = L(\nu) - L(u_0) = 0$$

$$H(\nu, 1) = A(\nu) - f(u_0) = 0$$

Where, $$\nu = \nu_0 + p\,\nu_1 + p^2\nu_2 + p^3\nu_3 + \cdots$$

Approximated solution is as follows:

$$u = \lim_{p \rightarrow 1} \nu$$

$$\Rightarrow u = \nu_0 + \nu_1 + \nu_2 + \nu_3 + \cdots$$

Outline of Paper

- In Section 2, detail regarding the implementation of the proposed scheme is provided.
- In Subsection 2.1, Implementation of STHPM is given regarding the Klein-Gordon equation.
- In Subsection 2.1, the Implementation of STHPM is given regarding the Sine-Gordon equation.
- In Section 3, a notion regarding convergence analysis is provided.
- In Section 4, six examples are discussed regarding the testing of the proposed scheme as well as graphical plots are provided.
- Section 5 is given as concluding remarks.

2. IMPLEMENTATION PROCESS OF STHPM

2.1. Implementation of STHPM upon Klein Gordon Equation

Applying Sumudu transform in Equation (1):

$$S[u_{tt}] = S[k\, u_{xx} - g(u)]$$
$$\frac{1}{u^2} S[u(x,t)] - \frac{u(x,0)}{u^2} - \frac{u'(x,0)}{u} = S[k\, u_{xx} - g(u)]$$
$$\frac{S[u(x,t)]}{u^2} = \frac{u(x,0)}{u^2} + \frac{u'(x,0)}{u} + S[k\, u_{xx} - g(u)]$$
$$S[u(x,t)] = u(x,0) + u'(x,0)u + u^2 S[k\, u_{xx} - g(u)]$$
$$u(x,t) = u(x,0) + u'(x,0) S^{-1}[u] + S^{-1}[u^2 S[k\, u_{xx} - g(u)]\,]$$
$$u(x,t) = u(x,0) + u'(x,0)[t] + \; S^{-1}[u^2 S[k\, u_{xx} - g(u)]\,]$$

Applying HPM:

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = u(x,0) + u'(x,0)[t] + p\, S^{-1}\left[u^2 S\left\{k\left(\sum_{n=0}^{\infty} p^n u_n\right)_{xx} - \sum_{n=0}^{\infty} p^n H_n(u)\right\}\right]$$

Where, $\quad g(u) = H_n(u)$

Comparing powers of p:

$$\boldsymbol{p^0}: u_0(x,t) = u(x,0) + u'(x,0)[t]$$
$$\boldsymbol{p^1}: u_1(x,t) = S^{-1}[u^2 S\{k\,(u_0)_{xx} - H_0(u)\}]$$
$$\boldsymbol{p^2}: u_2(x,t) = S^{-1}[u^2 S\{k\,(u_1)_{xx} - H_1(u)\}]$$
$$\boldsymbol{p^3}: u_3(x,t) = S^{-1}[u^2 S\{k\,(u_2)_{xx} - H_2(u)\}]$$
$$\boldsymbol{p^n}: u_n(x,t) = S^{-1}[u^2 S\{k\,(u_{n-1})_{xx} - H_{n-1}(u)\}]$$

A general solution is provided as follows:

$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + u_3(x,t) + \cdots + u_n(x,t)$

2.2. Implementation of STHPM upon Sine-Gordon Equation

Applying Sumudu transform in Equation (3):

$$S[u_{tt}] = S[\alpha^2\, u_{xx} + \beta sin(u)] \tag{8}$$
$$\frac{1}{u^2} S[u(x,t)] - \frac{u(x,0)}{u^2} - \frac{u'(x,0)}{u} = S[\alpha^2\, u_{xx} + \beta sin(u)]$$
$$S[u(x,t)] - u(x,0) - u'(x,0)u = u^2 S[\alpha^2\, u_{xx} + \beta sin(u)]$$

$$S[u(x,t)] = u(x,0) + u'(x,0)u + u^2 S[\alpha^2\, u_{xx} + \beta sin(u)]$$

$$u(x,t) = u(x,0) + u'(x,0)S^{-1}[u] + S^{-1}[u^2 S[\alpha^2\, u_{xx} + \beta sin(u)]]$$

$$u(x,t) = u(x,0) + u'(x,0)t + S^{-1}[u^2 S[\alpha^2\, u_{xx} + \beta sin(u)]] \tag{9}$$

Applying HPM in Equation (9):

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = u(x,0) + u'(x,0)t + p\, S^{-1}[u^2 S\{\alpha^2(\sum_{n=0}^{\infty} p^n u_n)_{xx} + \beta(\sum_{n=0}^{\infty} p^n H_n(u))\}] \tag{10}$$

Where, $H_n(u) = sin(u)$ [non-linear term]

Comparing powers of p in Equation (10):

$$\boldsymbol{p^0}: u_0(x,t) = u(x,0) + u'(x,0)t$$

$$\boldsymbol{p^1}: u_1(x,t) = S^{-1}[u^2 S\{\alpha^2(u_0)_{xx} + \beta\, H_0(u)\}]$$

$$\boldsymbol{p^2}: u_2(x,t) = S^{-1}[u^2 S\{\alpha^2(u_1)_{xx} + \beta\, H_1(u)\}]$$

$$\boldsymbol{p^3}: u_3(x,t) = S^{-1}[u^2 S\{\alpha^2(u_2)_{xx} + \beta\, H_2(u)\}]$$

$$\dots$$

$$\boldsymbol{p^n}: u_n(x,t) = S^{-1}[u^2 S\{\alpha^2(u_{n-1})_{xx} + \beta\, H_{n-1}(u)\}]$$

General solution is provided as follows:

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + u_3(x,t) + \dots + u_n(x,t)$$

3. CONVERGENCE ANALYSIS

Theorem 1.

Let F defined is an operator from Hilbert space. The series solution is considered as $u_n(x,t) = \sum_{i=0}^{n} u_i(x,t)$, which converges if there exists $0 < \eta < 1$ such that,

$$||F(u_0 + u_1 + u_2 + u_3 + \dots + u_{i+1})|| \le k\, ||F(u_0 + u_1 + u_2 + u_3 + \dots + u_i)||$$

for all $i = 0, 1, 2, 3, \dots$

Proof. From [55] and [56].

Theorem 2.

If there is a series solution $u(x,t) = \sum_{i=0}^{\infty} u_i(x,t)$ considered as convergent, then the given series solution will denote the exact solution of the given linear/non-linear problem.

Proof. From [57], [58] and [59].

Theorem 3.

Considered the series solution $\sum_{i=0}^{\infty} u_i(x,t)$, is convergent to $u(x,t)$ if the truncated series $\sum_{i=0}^{n} u_i(x,t)$ is considered as an approximate solution of the given problem then $Max.Trunc.Error \leq \frac{1}{1-k}\ k\ ||u_0||$.

Proof. From [57], [58] and [59].

In short, As per Theorem 1 and Theorem 2, a solution obtained converges to the exact solution if $0 < \eta < 1$ such that

$$\left||F[u_0 + u_1 + u_2 + u_3 + \cdots + u_{i+1}]|\right| \leq \eta\left||F[u_0 + u_1 + u_2 + u_3 + \cdots + u_{i+1}]|\right|$$

i.e. $\left||u_{i+1}|\right| \leq \eta\left||u_i|\right|$

for all $i = 0, 1, 2, 3, \ldots$

4. Examples and Applications of STHPM

In the present section, six examples are discussed. The first five examples are related to the notion of the Klein-Gordon equation, and the sixth example is related to the concept of the Sine-Gordon equation. Accuracy of present schemes is shown with the aid of matching between approximated and exact solutions via graphs. In Figure 1, Comparison of approximated and exact results is provided at t = 0.001, 0.002, 0.003 and 0.004 for N = 51 regarding Example 1. Comparison of approximated and exact results is provided at t = 0.001, 0.002, 0.003 and 0.004 for N = 21 regarding Example 2. In Figure 3, comparison of approximated and exact solutions is given at t = 0.1, 0.2, 0.3, 0.4 and 0.5 for N = 51 regarding Example 6. In Figure 4, a comparison of approximated and exact Solutions is provided at t = 1, 2, 3, 4, and 5 for N = 101.

Example 1. [Klein-Gordon Equation] [53]

Considered one dimensional Klein-Gordon equation is as follows:

$$u_{tt} - u_{xx} - u = 0 \tag{11}$$

Initial conditions: $u(x,0) = 1 + sinx$ and $u_t(x,0) = 0$

Applying Sumudu transform:

$$S[u_{tt}] = S[u_{xx} + u]$$

$$\frac{1}{u^2} S[u(x,t)] - \frac{u(x,0)}{u^2} - \frac{u'(x,0)}{u} = S[u_{xx} + u]$$

$$\frac{1}{u^2} S[u(x,t)] - \frac{1 + sinx}{u^2} = S[u_{xx} + u]$$

$$\frac{S[u(x,t)]}{u^2} = \frac{1 + sinx}{u^2} + S[u_{xx} + u]$$

$$S[u(x,t)] = (1 + sinx) + u^2 S[u_{xx} + u]$$

$$u(x,t) = (1 + sinx) + S^{-1}[u^2 S\{u_{xx} + u\}] \tag{12}$$

Applying HPM in Equation (12):

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = (1 + sinx) + p\, S^{-1}\left[u^2 S\left\{\left(\sum_{n=0}^{\infty} p^n u_n\right)_{xx} + \left(\sum_{n=0}^{\infty} p^n u_n\right)\right\}\right] \tag{13}$$

Comparing powers of p in Equation (13):

$$\boldsymbol{p^0}: u_0(x,t) = (1 + sinx)$$

$$\boldsymbol{p^1}: u_1(x,t) = S^{-1}[u^2 S\{(u_0)_{xx} + u_0\}]$$

$$u_1(x,t) = S^{-1}[u^2 S\{-sinx + 1 + sinx\}]$$

$$u_1(x,t) = S^{-1}[u^2 S\{1\}]$$

$$u_1(x,t) = S^{-1}[u^2]$$

$$u_1(x,t) = \frac{t^2}{\angle 2}$$

$$\boldsymbol{p^2}: u_2(x,t) = S^{-1}[u^2 S\{(u_1)_{xx} + u_1\}]$$

$$u_2(x,t) = S^{-1}[u^2 S\{\frac{t^2}{\angle 2}\}]$$

$$u_2(x,t) = S^{-1}[u^4]$$

$$u_2(x,t) = \frac{t^4}{\angle 4}$$

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + u_3(x,t) + \cdots$$

$$u(x,t) = (1 + sinx) + \frac{t^2}{\angle 2} + \frac{t^4}{\angle 4} + \cdots$$

$$u(x,t) = sinx + \left[1 + \frac{t^2}{\angle 2} + \frac{t^4}{\angle 4} + \cdots\right] = sinx + \sum_{i=0}^{\infty} \frac{t^{2i}}{\angle i}$$

$$u(x,t) = sinx + cosht$$

Convergence regarding Example 1

$$\mu_0 = \frac{\|u_1\|}{\|u_0\|} = 4.5461e - 05 < 1$$

$$\mu_1 = \frac{\|u_2\|}{\|u_1\|} = 8.3333e - 06 < 1$$

$$\mu_2 = \frac{\|u_3\|}{\|u_2\|} = 3.3333e - 06 < 1$$

$$\mu_3 = \frac{\|u_4\|}{\|u_3\|} = 1.7857e - 06 < 1$$

$\mu_i's$ for $i \geq 0$ and $t \leq (\frac{j}{2})$ for $0 < j < 1$ are less than one. Hence convergence condition is completed.

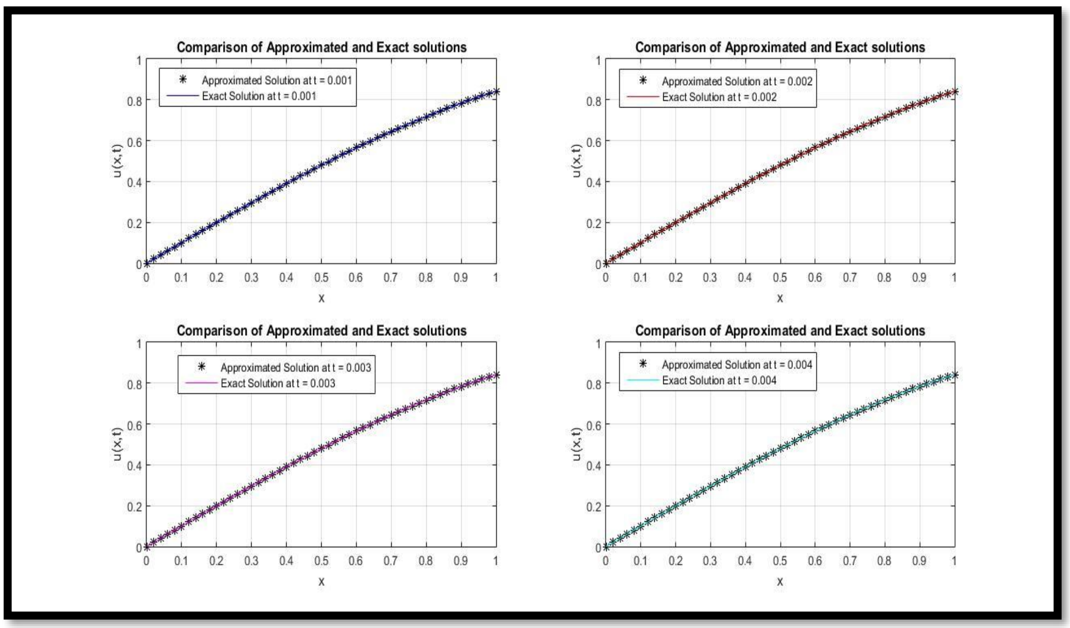

Figure 1. Comparison of approximated and exact results at t = 0.001, 0.002, 0.003 and 0.004 for N = 51 regarding Example 1.

Example 2. [Klein-Gordon Equation] [53]

Considered one dimensional Klein-Gordon equation is as follows:

$$u_{tt} - u_{xx} + u = 0 \tag{14}$$

I.C.s: $u(x,0) = 0$ and $u_t(x,0) = 0$

Applying Sumudu transform in Equation (14):

$$S[u_{tt}] = S[u_{xx} - u]$$

$$\frac{1}{u^2}\left[S\big(u(x,t)\big)\right] - \frac{1}{u^2}u(x,0) - \frac{1}{u}u'(x,0) = S[u_{xx} - u]$$

$$\frac{1}{u^2}S[u(x,t)] - \frac{x}{u^2} = S[u_{xx} - u]$$

$$\frac{S[u(x,t)]}{u^2} = \frac{x}{u^2} + S[u_{xx} - u]$$

$$S[u(x,t)] = x + u^2 S[u_{xx} - u]$$

$$u(x,t) = x + S^{-1}[u^2\, S\{u_{xx} - u\}] \tag{15}$$

Applying HPM in Equation (15):

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = x + p\, S^{-1}\left[u^2 S\left\{\left(\sum_{n=0}^{\infty} p^n u_n\right)_{xx} + \left(\sum_{n=0}^{\infty} p^n u_n\right)\right\}\right] \tag{16}$$

Comparing powers of p in Equation (16):

$$\boldsymbol{p^0}: u_0(x,t) = x$$

$$\boldsymbol{p^1}: u_1(x,t) = S^{-1}[u^2 S\{(u_0)_{xx} - u_0\}]$$

$$u_1(x,t) = S^{-1}[u^2 S\{-u_0\}]$$

$$u_1(x,t) = -x\, S^{-1}[u^2]$$

$$u_1(x,t) = -x\frac{t^2}{\angle 2}$$

$$\boldsymbol{p^2}: u_2(x,t) = S^{-1}[u^2 S\{(u_1)_{xx} - u_1\}]$$

$$u_2(x,t) = S^{-1}[u^2 S\{-u_1\}]$$

$$u_2(x,t) = S^{-1}\left[u^2 S\left\{x\frac{t^2}{2}\right\}\right]$$

$$u_2(x,t) = \frac{x}{2}S^{-1}[u^2 S\{t^2\}]$$

$$u_2(x,t) = \frac{x}{2}S^{-1}[2\, u^4]$$

$$u_2(x,t) = x\ \frac{t^4}{\angle 4}$$

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + u_3(x,t) + \cdots$$

$$u(x,t) = x\ - x\frac{t^2}{\angle 2} +\ x\ \frac{t^4}{\angle 4} - \cdots$$

$$u(x,t) = x\left[1 - \frac{t^2}{\angle 2} + \frac{t^4}{\angle 4} - \cdots\right] = x\ \sum_{i=0}^{\infty}(-1)^i\frac{t^{2i}}{\angle 2i}$$

$$u(x,t) = x\ cos(t)$$

Convergence regarding Example 2.

$$\mu_0 = \frac{\|u_1\|}{\|u_0\|} = 5.0000e - 05 < 1$$

$$\mu_1 = \frac{\|u_2\|}{\|u_1\|} = \ \ 8.3333e - 06 < 1$$

$$\mu_2 = \frac{\|u_3\|}{\|u_2\|} = \ \ 3.3333e - 06 < 1$$

$$\mu_3 = \frac{\|u_4\|}{\|u_3\|} = \ 1.7857e - 06 < 1$$

$\mu_i' s$ for $i \geq 0$ and $t \leq (\frac{j}{2})$ for $0 < j < 1$ are less than one. Hence convergence condition is completed.

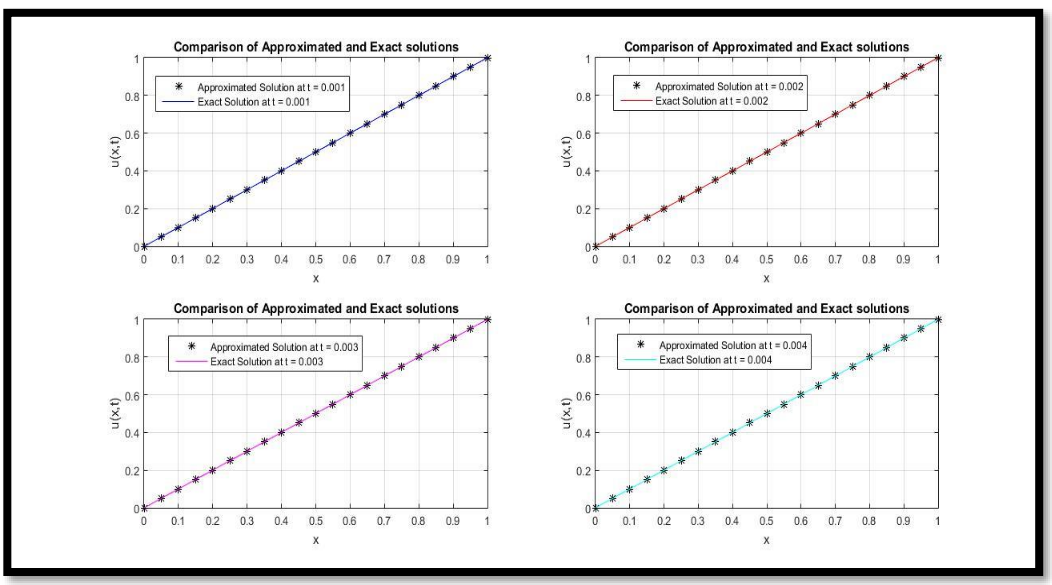

Figure 2. Comparison of approximated and exact results at t = 0.001, 0.002, 0.003 and 0.004 for N = 21 regarding Example 2.

Example 3. [Klein-Gordon Equation] [53]

Considered one dimensional Klein-Gordon equation is as follows:

$$u_{tt} - u_{xx} + u = 2\ sinx \tag{17}$$

I.C.s: $u(x,0) = sinx$ and $u_t(x,0) = 1$

Applying Sumudu transform in Equation (17):

$$S[u_{tt}] = S[2\ sinx + \ u_{xx} - u]$$

$$\frac{1}{u^2}\ S[u(x,t)] - \frac{u(x,0)}{u^2} - \frac{u'(x,0)}{u} = S[2\ sinx + \ u_{xx} - u]$$

$$\frac{1}{u^2}\ S[u(x,t)] - \frac{sinx}{u^2} - \frac{1}{u} = \ S[2\ sinx + \ u_{xx} - u]$$

$$\frac{1}{u^2} S[u(x,t)] = \frac{sinx}{u} + \frac{1}{u} + S[2\ sinx + \ u_{xx} - u]$$

$$S[u(x,t)] = sinx + u + u^2 S[2\ sinx + \ u_{xx} - u]$$

$$u(x,t) = sinx + S^{-1}[u] + S^{-1}[u^2 S\{2\ sinx + \ u_{xx} - u\}]$$

$$u(x,t) = sinx + t + S^{-1}[u^2 S\{2\ sinx + \ u_{xx} - u\}] \tag{18}$$

Applying HPM in Equation (18):

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = sinx + t + p\ S^{-1}\left[u^2 S\left\{2\ sinx + \left(\sum_{n=0}^{\infty} p^n u_n\right)_{xx} - \left(\sum_{n=0}^{\infty} p^n u_n\right)\right\}\right] \tag{19}$$

Comparing powers of p in Equation (19):

$$\boldsymbol{p^0}: u_0(x,t) = sinx + t$$

$$\boldsymbol{p^1}: u_1(x,t) = S^{-1}[u^2 S\{2\ sinx + (u_0)_{xx} - (u_0)\}]$$

$$u_1(x,t) = S^{-1}[u^2 S\{2\ sinx - sinx - (sinx + t)\}]$$

$$u_1(x,t) = S^{-1}[u^2 S\{-t\}]$$

$$u_1(x,t) = \ -S^{-1}[u^2 S(t)]$$

$$u_1(x,t) = \ -S^{-1}[u^3]$$

$$u_1(x,t) = \ -\frac{t^3}{\angle 3}$$

$$\boldsymbol{p^2}: u_2(x,t) = S^{-1}[u^2 S\{2\ sinx + (u_1)_{xx} - (u_1)\}]$$

$$u_2(x,t) = S^{-1}[u^2 S\{-u_1\}]$$

$$u_2(x,t) = S^{-1}\left[u^2 S\left\{\frac{t^3}{\angle 3}\right\}\right]$$

$$u_2(x,t) = S^{-1}[u^5]$$

$$u_2(x,t) = \frac{t^5}{\angle 5}$$

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + u_3(x,t) + \cdots$$

$$u(x,t) = sinx + t - \frac{t^3}{\angle 3} + \frac{t^5}{\angle 5} - \cdots = \sin x \; + \; \sum_{i=0}^{\infty} \frac{(-1)^i \, t^{2i+1}}{\angle(2i+1)}$$

$$u(x,t) = sinx + sint$$

Example 4. [Klein-Gordon Equation] [53]

Considered one dimensional Klein-Gordon equation is as follows:

$$u_{tt} - u_{xx} + u^2 = x^2 t^2 \tag{20}$$

I.C.s: $u(x,0) = 0$ and $u_t(x,0) = x$

Applying Sumudu transform in Equation (20):

$$S[u_{tt}] = S[x^2t^2 + u_{xx} - u^2]$$

$$\frac{1}{u^2}S[u(x,t)] - \frac{u(x,0)}{u^2} - \frac{u'(x,0)}{u} = S[x^2t^2 + u_{xx} - u^2]$$

$$\frac{S[u(x,t)]}{u^2} - \frac{x}{u} = S[x^2t^2 + u_{xx} - u^2]$$

$$\frac{S[u(x,t)]}{u^2} = \frac{x}{u} + S[x^2t^2 + u_{xx} - u^2]$$

$$S[u(x,t)] = xu + u^2 S[x^2t^2 + u_{xx} - u^2]$$

$$u(x,t) = x\,S^{-1}[u] + S^{-1}[u^2 S[x^2t^2 + u_{xx} - u^2]]$$

$$u(x,t) = x\,t + S^{-1}[u^2 S[x^2t^2 + u_{xx} - u^2]] \tag{21}$$

Applying HPM in Equation (21):

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = x\,t + p\,S^{-1}[u^2 S\{x^2t^2 + \left(\sum_{n=0}^{\infty} p^n u_n\right)_{xx} - (\sum_{n=0}^{\infty} p^n H_n(u))\}] \tag{22}$$

Where, $H_n(u) = u^2$

$$H_0(u) = u_0^2 = x^2 t^2$$

$$H_1(u) = 2\,u_0 u_1 = 0$$

Comparing powers of p in Equation (22):

$$\boldsymbol{p^0}: u_0(x,t) = xt$$

$$\boldsymbol{p^1}: u_1(x,t) = S^{-1}[u^2 S\{x^2t^2 + (u_0)_{xx} - H_0(u)\}]$$

$$u_1(x,t) = S^{-1}[u^2 S\{x^2t^2 - x^2t^2\}]$$

$$u_1(x,t) = 0$$

$$\boldsymbol{p^2}: u_2(x,t) = S^{-1}[u^2 S\{x^2t^2 + (u_1)_{xx} - H_1(u)\}]$$

$$u_2(x,t) = 0$$

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + u_3(x,t) + \cdots$$

$$u(x,t) = xt$$

Example 5. [Klein-Gordon Equation] [53]

Considered one dimensional Klein-Gordon equation is as follows:

$$u_{tt} - u_{xx} - u + u^2 = xt + x^2t^2 \tag{23}$$

I.C.s: $u(x,0) = 1$ and $u_t(x,0) = x$

Applying Sumudu transform in Equation (23):

$$S[u_{tt}] = S[xt + x^2t^2 + u_{xx} + u - u^2]$$

$$\frac{1}{u^2}S[u(x,t)] - \frac{u(x,0)}{u^2} - \frac{u'(x,0)}{u} = S[xt + x^2t^2 + u_{xx} + u - u^2]$$

$$\frac{S[u(x,t)]}{u^2} - \frac{1}{u^2} - \frac{x}{u} = S[xt + x^2t^2 + u_{xx} + u - u^2]$$

$$\frac{S[u(x,t)]}{u^2} = \frac{1}{u^2} + \frac{x}{u} + S[xt + x^2t^2 + u_{xx} + u - u^2]$$

$$S[u(x,t)] = 1 + x\,u + u^2 S[xt + x^2t^2 + u_{xx} + u - u^2]$$

$$u(x,t) = 1 + x\,S^{-1}[u] + S^{-1}[u^2 S\{xt + x^2t^2 + u_{xx} + u - u^2\}]$$

$$u(x,t) = 1 + xt + S^{-1}[u^2 S\{xt + x^2t^2 + u_{xx} + u - u^2\}] \tag{24}$$

Applying HPM in Equation (24):

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = 1 + xt + p\,S^{-1}\left[u^2 S\left\{xt + x^2t^2 + \left(\sum_{n=0}^{\infty} p^n u_n\right)_{xx} + \left(\sum_{n=0}^{\infty} p^n u_n\right) - \sum_{n=0}^{\infty} p^n H_n(u)\right\}\right] \tag{25}$$

Where, $$H_n(u) = u^2$$

$$H_0(u) = u_0^2 = (1 + xt)^2$$

$$H_1(u) = 2\, u_0 u_1$$

Comparing powers of p in Equation (25):

$$\boldsymbol{p^0}: u_0(x,t) = (1 + xt)$$

$$\boldsymbol{p^1}: u_1(x,t) = S^{-1}[u^2 S\{xt + x^2t^2 + (u_0)_{xx} + u_0 - H_0(u)\}]$$

$$u_1(x,t) = S^{-1}[u^2 S\{0\}]$$

$$u_1(x,t) = 0$$

$$\boldsymbol{p^2}: u_2(x,t) = S^{-1}[u^2 S\{xt + x^2t^2 + (u_1)_{xx} + u_1 - H_1(u)\}]$$

$$u_2(x,t) = 0$$

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + u_3(x,t) + \cdots$$

$$u(x,t) = (1 + xt)$$

Example 6. [Sine-Gordon Equation] [54]

Considered one dimensional Sine-Gordon equation is as follows:

$$u_{tt} + u_t = u_{xx} - 2\,sinu + \pi^2 e^{-t} sin(\pi x) + 2\,sin[e^{-t} sin(\pi x)] \qquad (26)$$

I.C.: $u(x,0) = sin(\pi x)$ and $u_t(x,0) = -sin(\pi x)$

Applying Sumudu Transform in Equation (26):

$$S[u_{tt}] + S[u_t] = S[u_{xx} - 2\,sinu + \pi^2 e^{-t} sin(\pi x) + 2\sin[e^{-t}\sin(\pi x)]]$$

$$\left[\frac{1}{u^2} S[u(x,t)] - \frac{u(x,0)}{u^2} - \frac{u'(x,0)}{u}\right] + \frac{1}{u}[S[u(x,t) - u(x,0)]]$$

$$= S[u_{xx} - 2\,sinu + \pi^2 e^{-t} sin(\pi x) + 2\sin[e^{-t}\sin(\pi x)]]$$

$$S[u(x,t)]\left[\frac{1}{u^2} + \frac{1}{u}\right] = \frac{sin(\pi x)}{u^2} + S[u_{xx} - 2\,sinu + \pi^2 e^{-t} sin(\pi x) + 2\sin[e^{-t}\sin(\pi x)]]$$

$$S[u(x,t)]\left[\frac{1+u}{u^2}\right] = \frac{sin(\pi x)}{u^2} + S[u_{xx} - 2\,sinu + \pi^2 e^{-t} sin(\pi x) + 2\sin[e^{-t}\sin(\pi x)]]$$

$$S[u(x,t)] = \frac{sin(\pi x)}{1+u} + \frac{u^2}{1+u} S[u_{xx} - 2\,sinu + \pi^2 e^{-t} sin(\pi x) + 2\sin[e^{-t}\sin(\pi x)]]$$

$$u(x,t) = \sin(\pi x)\, S^{-1}\left[\frac{1}{1+u}\right] + S^{-1}[\frac{u^2}{1+u} S[u_{xx} - 2\,sinu + \pi^2 e^{-t} sin(\pi x)$$
$$+ 2\sin[e^{-t}\sin(\pi x)]]]$$

$$u(x,t) = \sin(\pi x)\, e^{-t} + S^{-1}[\tfrac{u^2}{1+u} S[u_{xx} - 2\, sinu + \pi^2 e^{-t} sin(\pi x) + 2 \sin\, [e^{-t} \sin\, (\pi x)]]] \tag{27}$$

Applying HPM in Equation (27):

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = \sin(\pi x)\, e^{-t} + p\, S^{-1}\left[\frac{u^2}{1+u} S\left\{\left(\sum_{n=0}^{\infty} p^n u_n\right)_{xx} - 2\sum_{n=0}^{\infty} p^n H_n(u) + \pi^2 e^{-t} \sin(\pi x) + 2 \sin(e^{-t}\, sin(\pi x))\right\}\right] \tag{28}$$

Where, $H_n(u) = \sin\,(u)$

$$H_0(u) = \sin\,(u_0)$$

$$H_1(u) = \cos(u_0)\, u_1$$

Comparing powers of p in Equation (28):

$$\boldsymbol{p^0}: u_0(x,t) = \sin(\pi x)\, e^{-t}$$

$$\boldsymbol{p^1}: u_1(x,t) = S^{-1}[\frac{u^2}{1+u} S\{(u_0)_{xx} - 2\, H_0(u) + \pi^2 e^{-t} \sin(\pi x) + 2 \sin\,(e^{-t} \sin\,(\pi x))\}]$$

$$u_1(x,t) = S^{-1}[\frac{u^2}{1+u}\, S\{-\pi^2 sin(\pi x) e^{-t} - 2\, sin(sin(\pi x) e^{-t}) + \pi^2 e^{-t} sin(\pi x) + 2 sin(e^{-t} sin(\pi x))\}]$$

$$u_1(x,t) = S^{-1}[0]$$

$$u_1(x,t) = 0$$

$$\boldsymbol{p^2}: u_2(x,t) = S^{-1}[\frac{u^2}{1+u} S\{(u_1)_{xx} - 2\, H_1(u)\}]$$

$$u_2(x,t) = S^{-1}[\frac{u^2}{1+u} S\{0\}]$$

$$u_2(x,t) = 0$$

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + u_3(x,t) + \cdots = \sin(\pi x) \sum_{i=0}^{\infty} \frac{(-t)^i}{\angle i}$$

$$u(x,t) = e^{-t} sin(\pi x)$$

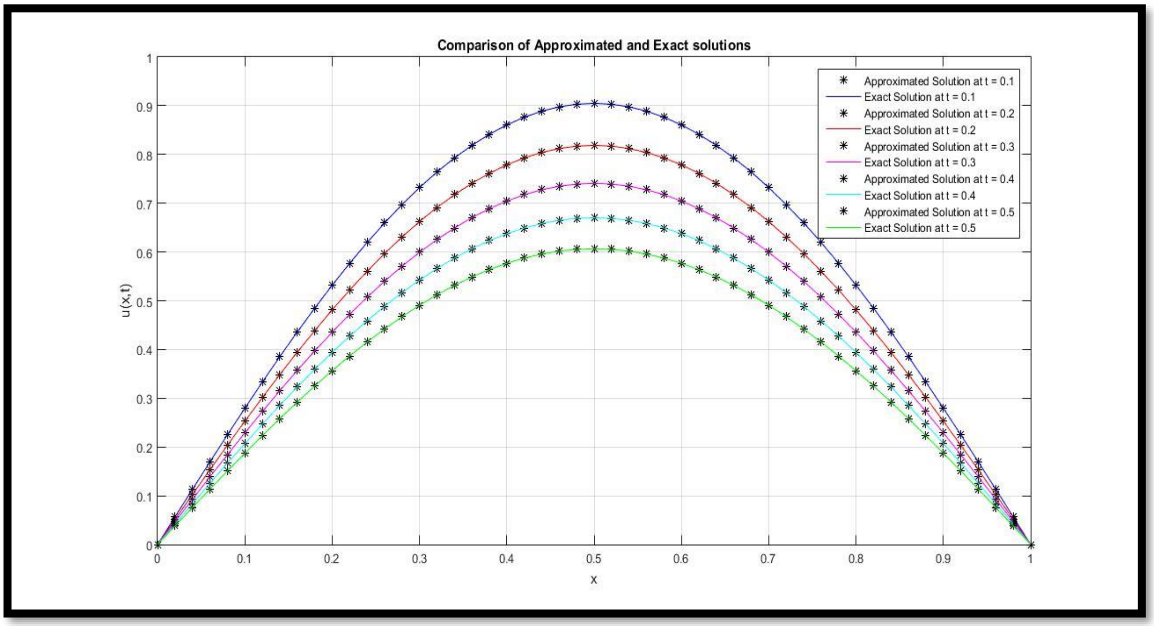

Figure 3. Comparison of Approximated and Exact Solutions at t = 0.1, 0.2, 0.3, 0.4 and 0.5 for N = 51 regarding Example 6.

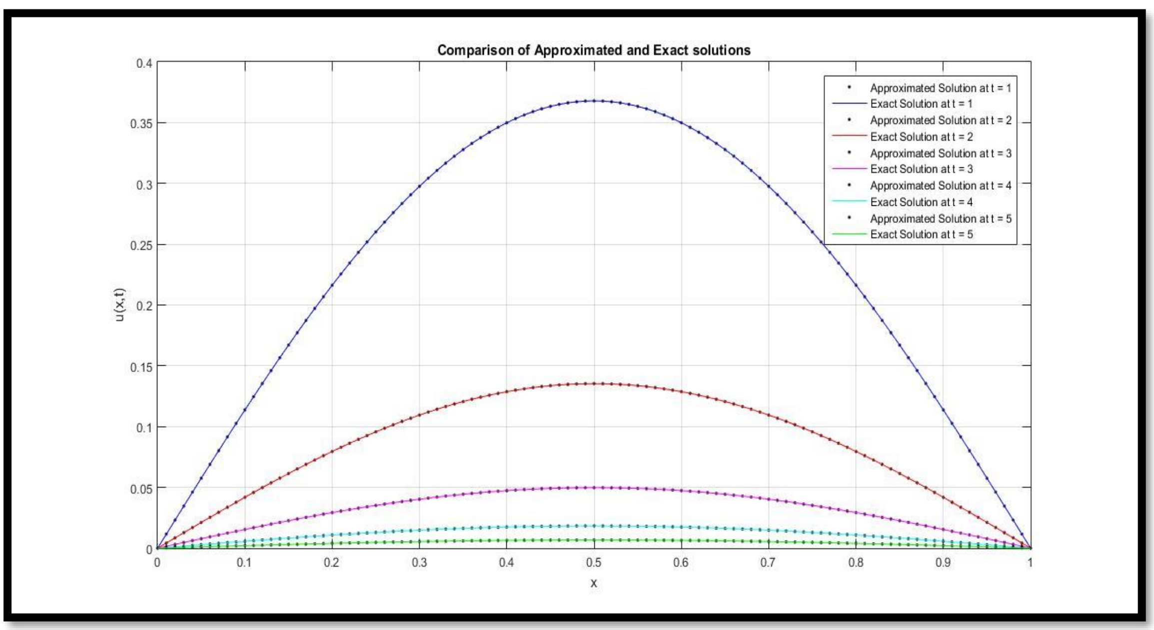

Figure 4. Comparison of Approximated and Exact Solutions at t = 1, 2, 3, 4 and 5 for N = 101.

5. Conclusion

In the present study, a hybrid regime named "Sumudu Transform Homotopy Perturbation Method" (STHPM) is implemented to solve Klein-Gordon and Sine-Gordon equations analytically. Convergence analysis of the scheme is also provided. Graphical plots are provided to test the compatibility between the approximated and exact results. The present scheme is easy to implemented and has produced acceptable results. With the aid of the present regime, complex-natured differential equations can also be solved.

Conflict of Interests

The author(s) declare that there is no conflict of interests.

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