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STUDY OF SOME PROPERTIES OF COMPLEMENT OF OPEN SUBSET INCLUSION GRAPH OF A TOPOLOGICAL SPACE

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Abstract. In the recent paper, authors introduced a graph topological structure, called as open subset inclusion graph $\jmath(\tau)$ of a topological space (X,τ) on a finite set X and discussed some important properties of this graph. In this paper, we discuss some properties of the graph $\jmath(\tau)^c$. It is shown that, if τ is a discrete topology defined on nonempty set X with $|X| \leq 3$, then the graph $\jmath(\tau)^c$ is bipartite, and if $|X| = 2$, then the graph $\jmath(\tau)^c$ is regular & complete bipartite. Moreover, if τ is a discrete topology defined on nonempty set X with $|X| = 2$ or $|X| = 3$ then it is shown that the graph $\jmath(\tau)^c$ is Hamiltonian, vertex-transitive, edge-transitive and has a perfect matching. We also provide exact value of the independence number, vertex connectivity and edge connectivity of the graph $\jmath(\tau)^c$ of a discrete topology defined on nonempty set$X$ with $|X| = 2$ or $|X| = 3$. Main finding of this work is that, if (X,τ) is a discrete topological space with $|X| = 2$ or $|X| = 3$ then it is shown that $\jmath(\tau)^c$ is distance-transitive graph and distance regular graph.

Keywords: Discrete Topology, Graph, Clique, Chromatic Number, Domination Set, Independence Set.
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1. Introduction

If R is a commutative ring with unity then the zero divisor graph of R was firstly introduced by Beck[2], which is defined as, if R is any ring then $G(R)$ denotes the zero divisor graph of R whose vertex set is $V = R$, such that any distinct vertices x and y are adjacent if and only if $x \cdot y = 0$.

In the recent decades, graphs of several algebraic structures were defined which can be found in[1, 3]. Among these graphs, zero divisor graphs of ring and module are more attractive for many researchers. A. Das[4, 5, 6], introduced a graphs of a vector space & he also discussed some results on these graphs.

The graphs of a vector space were also studied independently by some authors which can be found in [7] and [11]. Some properties on incomparability graphs $\Gamma(L)$ of lattices L were discussed by Wasadikar, M. and Survase P[12]. They classified lattice L by using the graph $\Gamma(L)$ of a lattice L. As Graph theory has wide range of applications in various fields this motivated us to introduce new concept of graphs of topological space (X, τ) with some important properties of these graphs which can be found in [8, 9, 10]. In[8], authors indroduced the graph $\jmath(\tau)$ of τ, which is defined as follows.

Definition 1.1.[8]**Open Subset Inclusion Graph of a Topological Space:** Let X be a finite set and τ be a topology defined on X then a graph $\jmath(\tau) = (V(\tau), E(\tau))$ is called as an open subset inclusion graph of (X, τ), where $V(\tau) = \{P \in \tau \mid P \neq \phi, P \neq X\}$ and for $P, Q \in V(\tau)$, $(P, Q) \in E(\tau)$ iff $P \subset Q$ or $Q \subset P$.

Example 1.1 Let (X, τ) be the discrete topological space with $X = \{a_1, a_2, a_3\}$ and $\tau = \{\phi, X, U_1 = \{a_1\}, U_2 = \{a_2\}, U_3 = \{a_3\}, U_{12} = \{a_1, a_2\}, U_{13} = \{a_1, a_3\}, U_{23} = \{a_2, a_3\}\}$, then an open subset inclusion graph of (X, τ) and compliment of open subset inclusion graph of a discrete topological space (X, τ) with $|X| = 3$. are shown in fig. 1 and fig. 2

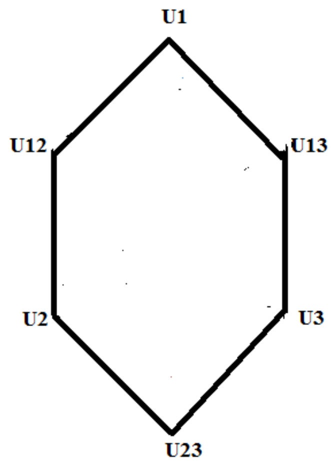

FIGURE 1. Open Subset Inclusion Graph of a Discrete Topological Space (X, τ) with $|X| = 3$.

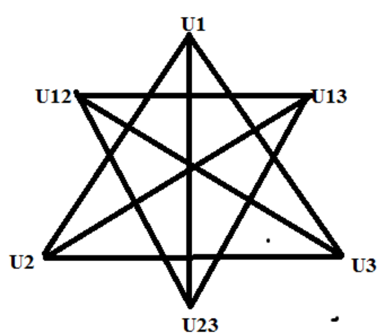

FIGURE 2. Complement of Open Subset Inclusion Graph of a Discrete Topological Space (X, τ) with $|X| = 3$.

2. Some Results on Complement of Inclusion Graph of a Topological Space

Theorem 2.1. If U_1 and U_2 are any two distinct open subsets of X of same cardinality k, then $U_1 \sim U_2$ in $\jmath(\tau)^c$.

Proof: Let U_1 and U_2 are two distinct open subsets of X of same cardinality k then $U_1 \not\subset U_2$ and $U_2 \not\subset U_1$ and hence $U_1 \sim U_2$ in $\jmath(\tau)^c$.

Theorem 2.2. If τ is a discrete topology defined on a nonempty set X and $|X| \geq 3$, then $\jmath(\tau)^c$ is connected and diam($\jmath(\tau)^c$) ≤ 2.

Proof: Let U and V are any two non empty proper open subsets of X

Case 1: If $|X| = 2$ then by Theorem 2.1, $\jmath(\tau)^c$ is a complete graph with 2 vertices. Hence the graph $\jmath(\tau)^c = K_2$ is connected and $dist(U, V) = 1$ with diam($\jmath(\tau)^c$) = 1 in this case.

Case 2: If $|X| \geq 3$ and $|U| = |V| = k$ then by Theorem 2.1, $U \sim V$ and hence $U \sim V$ be a required path from U to V .

Case 3: If $|X| \geq 3$ and $|U| \neq |V|$.

Sub Case I: If $U \not\subset V$ or $V \not\subset U$ then $U \sim V$ in the graph $\jmath(\tau)^c$ and hence $U \sim V$ be a required path from U to V in the graph $\jmath(\tau_S)^c$ and diam($\jmath(\tau)^c$) = 1 in this case.

Sub Case II: If $U \subset V$ or $V \subset U$ then $U \nsim V$ in the graph $\jmath(\tau)^c$. If $U \cup V \neq X$ then there exists an open set $W = (U \cap V)^c$ such that $U \not\subset V$ and $V \not\subset U$ and hence $U \sim W$, $W \sim V$. Thus, $U \sim W \sim V$ be a required path from U to V and $dist(U, V) = 2$ in the graph $\jmath(\tau)^c$.

Sub Case III: If $U \subset V$ or $V \subset U$ then $U \nsim V$ in the graph $\jmath(\tau)^c$. If $U \cup V = X$ and $U \cap V = \phi$ then $U \sim V$. Hence $U \sim V$ be a required path from U to V and $U \cup V = X$ and $U \cap V \neq \phi$ then there exists an open set $W = (U \cap V)^c$ such that $U \sim W$ and $V \sim W$ in the graph $\jmath(\tau)^c$. Hence $U \sim W \sim V$ be a required path from U to V and $dist(U, V) = 2$ in the graph $\jmath(\tau)^c$.

Thus, diam($\jmath(\tau)^c$) ≤ 2.

Theorem 2.3. Let τ is a discrete topology defined on nonempty set X and $|X| = n$. If $n \geq 3$, $girth(\jmath(\tau)^c)$ is 3.

Proof: If $n \geq 3$ and a, b, c be three distinct elements in X then there exist three open subsets U_1, U_2 and U_3 of same cardinality k, for $k \geq 1$ and none of them is equal to X. Then by Theorem 2.1, $U_1 \sim U_2 \sim U_3 \sim U_1$, which is triangle and hence $girth(\jmath(\tau)^c)$ is 3.

Note: The above theorem guarantees that there always exist at least one three cycle in $\jmath(\tau)^c$, when $|X| \geq 3$.

3. Special Properties of an Inclusion Graph of a Discrete Topological Space (X, τ), when $|X| = 3$.

In this section, we will discuss some special properties like bipartiteness, Hamiltonicity, vertex and edge transitivity, independence and domination number etc. of $\jmath(\tau)^c$, when $|X| = 3$.

Theorem 3.1. If τ is a discrete topology defined on nonempty set X with $|X| = n$ then $\jmath(\tau)^c$ is bipartite if and only if $n = 2$. Moreover, $\jmath(\tau)^c$ is complete bipartite.

Proof: As τ is a discrete topology defined on nonempty set X with $|X| = n$. Case I: If $|X| = n \geq 3$

then by Theorem 2.3, girth of the graph $\jmath(\tau)^c$ is 3. This shows the presence of cycles of odd length 3 and hence $\jmath(\tau)^c$ is not bipartite graph, for $n \geq 3$.

Case II: If $n = 2$, now consider $U_k = a_k | k = 1, 2$ be the collection of open subsets of X of cardinality 1. Then the vertex set of $\jmath(\tau)^c$ can be partitioned as $V(\tau) = U_1 \cup U_2$. Moreover, due to Theorem 2.1, vertices of same cardinality are adjacent. Thus, in this case, the graph $\jmath(\tau)^c = K_2$ is bipartite graph and hence $\jmath(\tau)^c$ is complete bipartite graph.

Theorem 3.2. If τ is a discrete topology defined on nonempty set X with $|X| = n$ then the graph $\jmath(\tau)^c$ is a regular graph if and only if $n \leq 3$

Proof: Let τ is a discrete topology defined on nonempty set X with $|X| = n$. Case-I: If $n = 2$, then graph $\jmath(\tau)^c$ is complete graph K_2 and hence, the graph $\jmath(\tau)^c$ is 1-regular graph.

Case-II: If $n = 3$ then vertices of the graph $\jmath(\tau)^c$ are of open subsets of cardinality 1 or 2. By Theorem 2.1, all the vertices (namely, open subsets of cardinality 1 and 2) in $\jmath(\tau)^c$ are of same degree 3 and hence the graph is regular and therefore $\jmath(\tau)^c$ is 3 - regular graph.

On other hand, if $n \geq 4$, the degree of 1-element open subset is of degree $deg(U) = |Y| = 2^n - 2^{n-1} + 3$ and that of 2 - element open subset is of degree $deg(U) = |Y| = 2^n - 2^{n-2} + 1$ which are not same and hence the graph $\jmath(\tau)^c$ is not regular and hence we are through.

Theorem 3.3. If τ is a discrete topology defined on nonempty set X with $|X| = 3$ then order and size of the graph $\jmath(\tau)^c$ are 9 and 18 respectively.

Proof: If $|X| = 3$ then order of $\jmath(\tau)^c$ is equals to number of vertices of the graph $\jmath(\tau)^c$ and hence 6. Now by Theorem 3.2, $\jmath(\tau)^c$ is 3-regular graph. Thus, if m is the number of edges in $\jmath(\tau_S)^c$, then by degree-sum formula we have, order and size of the graph $\jmath(\tau)^c$ is $m = 18$.

Theorem 3.4. If τ is a discrete topology defined on nonempty set X with $|X| = 3$ then vertex connectivity of $\jmath(\tau)^c$ is 3.

Proof: If X is nonempty set with $|X| = 3$ then by the Theorem 3.2, $\jmath(\tau_S)^c$ is a 3-regular graph and hence by [[8], Example 17], vertex connectivity of $\jmath(\tau_S)^c$ is 3.

Theorem 3.5. Let τ is a discrete topology defined on nonempty set X with $|X| = 3$. If W_1 and W_2 be non-trivial proper open subsets of X of same cardinality then there exists a graph automorphism ϕ on $\jmath(\tau_S)^c$ such that $\phi(W_1) = W_2$.

Proof: Since W_1 and W_2 be non-trivial proper open subsets of same cardinality, then we have either both of them are of cardinality 1 or of cardinality 2.

Case-I If $|W_1| = |W_2| = 1$, for simplicity suppose that $W_1 = a_1$ and $W_2 = a_2$. Let ϕ be the unique automorphism from X to itself which maps a_1 to a_2, a_2 to a_1 and a_3 to a_1. Then clearly, ϕ induces a graph automorphism on $\jmath(\tau)^c$ and it maps W_1 to W_2.

Case-II If $|W_1| = |W_2| = 2$, for simplicity suppose that $W_1 = a_1, a_2$ and $W_2 = a_1, a_3$. By the proceeding analogously, we get a graph automorphism ϕ on $\jmath(\tau)^c$ which maps a_1 to a_1, a_2 to a_3, a_3 to a_2. Thus ϕ be an graph automorphism which maps W_1 to W_2. Hence the theorem follows.

4. Open Subsets of X

It is to be noted that if (X, τ) is a discrete topological space with $|X| = n = 3$ and the $\{X = a_1, a_2, a_3\}$ then

Open subsets of cardinality 1 is: $(1) U_k = \{a_k\} : k = 1, 2, 3$. (Number of such open subsets is 3)

Open subsets of cardinality 2 is: $(2) U_{ij} = \{a_i, a_j\} : i, j = 1, 2, 3; i \neq j$. (Number of such open subsets is 3)

Define a map $\phi : \jmath(\tau) \rightarrow \jmath(\tau)$ as following:

$\phi(U_k) = U_{ij}$ where $i, j, k \in 1, 2, 3$ and are all distinct

$\phi(U_{ij}) = U_k$ where $i, j, k \in 1, 2, 3$ and are all distinct.

Remark 4.1: From the definition of ϕ, it is a bijection. Moreover, it can be checked that ϕ preserves both adjacency and non-adjacency in $\jmath(\tau)^c$. Thus ϕ is a graph automorphism. is vertex-transitive

Theorem 4.1. If τ is a discrete topology defined on nonempty set X with $|X| = 2$ then the graph $\jmath(\tau)^c$ is vertex-transitive.

Proof: If W_1 and W_2 be nonempty proper open subsets of X of same cardinality 1 then by Theorem 3.5, there exists a graph automorphism ψ which maps W_1 to W_2 and vice versa. Thus, there exists a graph automorphism which maps W_1 to W_2 and hence $\jmath(\tau)^c$ is vertex-transitive graph.

Theorem 4.2. Let τ is a discrete topology defined on nonempty set X with $|X| = 3$ then the graph $\jmath(\tau)^c$ is vertex-transitive.

Proof: If W_1 and W_2 be nonempty proper open subsets of X. If both are of same cardinality, then by Theorem 3.5, there exists a graph automorphism ψ which maps W_1 to W_2.

Case-I: If $|W_1| = 1$ and $|W_2| = 2$ and $W_2 = W_1^c$ then by by Remark 4.1, there exists graph automorphism ϕ which maps W_1 to W_2.

Case-II: If $|W_1 = a_1| = 1$ and $|W_2| = 2$ and $W_2 = a_1, a_2 \neq W_1^c = a_2, a_3$ then by Remark 4.1, ϕ is a graph automorphism which maps W_1 to two-elements open subset W_1^c of X. Now, by Theorem 3.5, there exists a graph automorphism ψ which maps a_1 to a_2, a_2 to a_2, a_3 to a_1 which maps W_1^c to $W_2 = a_1, a_2$. Thus $\psi \circ \phi$ is a graph automorphism which maps$W_1 \rightarrow W_2$.

Case-III: If $|W_1| = 2$ and $|W_2| = 1$ then by the same manner in that of the previous case. We can find $\phi \circ \psi$ is a graph automorphism which maps W_2 to W_1.. Thus in any case, there exists a graph automorphism which maps W_1 to W_2 and hence $\jmath(\tau)^c$ is vertex-transitive graph.

Theorem 4.3. Let τ is a discrete topology defined on nonempty set X with $|X| = 2$ then $\jmath(\tau)^c$ is a retract of a Cayley graph.

Proof: As $|X| = 2$ then $\jmath(\tau)^c$ is connected vertex transitive graph and hence by Theorem 3.9.1[8], $\jmath(\tau)^c$ is a retract of a Cayley graph.

Theorem 4.4. Let τ is a discrete topology defined on nonempty set X with $|X| = 3$ then $\jmath(\tau)^c$ is a retract of a Cayley graph .

Proof: As $|X| = 3$ then is connected vertex transitive graph and hence by Theorem 3.9.1[8], $\jmath(\tau)^c$ is a retract of a Cayley graph

Theorem 4.5. Let τ is a discrete topology defined on nonempty set X with $|X| = 3$ then the independence number of the graph $\jmath(\tau)^c$ is 1 .

Proof: As $|X| = 2$, then the graph $\jmath(\tau)^c$ is 1 - regular, connected, vertex-transitive graph and hence by Lemma 3.3.3[8], edge connectivity of $\jmath(\tau)^c$ is equal to its minimum degree, i.e. 1. Now, we turn towards finding the independence number of the graph $\jmath(\tau)^c$ when (X, τ) is a discrete topological space and $|X| = 2$. From Theorem 3.1, it is clear that $\jmath(\tau)^c$ is a bipartite graph with each partite set with 1 vertices and as each partite set is an independent set. Thus we have the independence number of the graph $\jmath(\tau)^c$, that is $\alpha(\jmath(\tau)^c) = 1$

Theorem 4.6. [[14] , Corollary 1.3]: If G is a bipartite graph with $2n$ vertices, then $\alpha(G) = n$ if and only if G has a perfect matching.

Theorem 4.7. If τ is a discrete topology defined on nonempty set X with $|X| = 2$ then independence number of the graph $\jmath(\tau)^c$ is 1.

Proof: As $\jmath(\tau)^c$ is a bipartite graph with 2 . 1 = 2 vertices. Using Theorem 4.6, it suffices to show that $\jmath(\tau)^c$ admits a perfect matching. For this consider the vertices of $\jmath(\tau)^c$ as describe in beging of Section 4. We explicitly describe the perfect matching which consists of following type of edges: $a_1 \sim a_2$, $a_2 \sim a_1$. It is quite clear that the above is indeed a perfect matching on the graph $\jmath(\tau)^c$. The result follows from Theorem 4.6.

Next, we study the Hamiltonicity of $\jmath(\tau)^c$, when (X, τ) is a discrete topological space with $|X| = n = 3$.

Theorem 4.8. If τ is a discrete topology defined on nonempty set X with |X| = 2 then $\jmath(\tau)^c$ is Hamiltonian.

Proof: As $|X| = 2$ then the order of the graph $\jmath(\tau)^c$ is 2 = 2.1. As a vertex-transitive graph of order 2.p, where p is a prime, is Hamiltonian graph (See [1]) and hence the result follows.

Theorem 4.9. If τ is a discrete topology defined on nonempty set X with $|X| = 3$ then $\jmath(\tau)^c$ is Hamiltonian.

Proof: As $|X| = 3$ then the order of the graph $\jmath(\tau)^c$ is 6 = 2.3. As a vertex-transitive graph of order 2.p, where p is a prime, is Hamiltonian graph (See [1]) and hence the result follows

Theorem 4.10. If τ is a discrete topology defined on nonempty set X with $|X| = 3$ then edge connectivity of $\jmath(\tau)^c$ is 3 .

Proof: As $|X| = 3$, then the graph $\jmath(\tau)^c$ is 3 - regular, connected, vertex-transitive graph and hence by Lemma 3.3.3[8], edge connectivity of $\jmath(\tau)^c$ is equal to its minimum degree, i.e. 3.

Theorem 4.11. If τ is a discrete topology defined on nonempty set X with $|X| = 2$ then the graph $\jmath(\tau)^c$ is edge-transitive.

Proof: Let (X, τ) be the discrete topological space with $|X| = 2$. Let $U_1 \sim U_2$ be edge in $\jmath(\tau)^c$. Without loss of generality, let us suppose that U_1, U_2 be an open subsets of X of cardinality 1. Let ψ be the bijection from vertex set $V(\tau)$ to itself which maps U_i to U_i^c for i = 1, 2, 3. Clearly, ψ induces a graph automorphism on $\jmath(\tau)^c$ and it maps U_1 to U_2, U_2 to U_1. Therefore, it maps the edge $U_1 \sim U_2$ to the edge $U_2 \sim U_1$ and hence $\jmath(\tau)^c$ is an edge-transitive graph.

Theorem 4.12. If τ is a discrete topology defined on nonempty set X with $|X| = 3$ then the graph $\jmath(\tau)^c$ is edge-transitive.

Proof: Let (X, τ) be the discrete topological space with $|X| = 3$. Let $U_1 \sim U_2$ and $U_{23} \sim U_{13}$ be two edges in $\jmath(\tau)^c$. Without loss of generality, let us suppose that U_1, U_2 be an open subsets of X of cardinality 1 and U_{23}, U_{13} be an open subsets of X of cardinality 2. Then we have $U_1 \not\subset U_2$ and $U_{23} \not\subset U_{13}$. Again, without loss of generality, let us consider $U_1 = a_1, U_{23} = a_2, a_3, U_2 = a_2$ and $U_{13} = a_1, a_3$, $U_3 = a_3$ and $U_{12} = a_1, a_2$. Let ψ be the bijection from vertex set $V(\tau)$ to itself which maps U_i to $U_i^c$for i = 1, 2, 3. Clearly, ψ induces a graph automorphism on $\jmath(\tau)$ and it maps U_1 to U_{23}, U_2 to U_{13} and U_3 to U_{12}. Therefore, it maps the edge $U_1 \sim U_2$ to the edge $U_{23} \sim U_{13}$, edge $U_2 \sim U_{12}$ to the edge $U_{13} \sim U_3$, edge $U_2 \sim U_3$ to the edge $U_{13} \sim U_{12}$ and hence $\jmath(\tau)^c$ is an edge-transitive graph.

5. The Graph $\jmath(\tau)^c$ is Distance Regular, when $n = 3$

In this section, we prove that $\jmath(\tau)^c$ is distance transitive.

Involution of $\jmath(\tau)^c$: Let Ψ be the mapping from $V(\tau)$, the vertex set of $\jmath(\tau)^c$, to itself, which sends any $W \in V(\tau)$ to W^c. Then Ψ^2 is the identity mapping on $V(\tau)$, which implies that Ψ is a bijection on $V(\tau)$. Noting that $W_1 \subset W_2$ if and only if $W_1^c \subset W_2^c$, we have $W_1 \sim W_2$ if and only if $\Psi(W_1) \sim \Psi(W_2)$. Consequently, Ψ is an automorphism of $\jmath(\tau)^c$, which is called the involution of $\jmath(\tau)^c$.

Invertible Function: Let σ be an invertible function on X and $W \in V(\tau)$. Then $\sigma(W) = \{\sigma(v)|v \in W\}$ also lies in $V(\tau)$ and $\sigma(W_1) \subset \sigma(W_2)$ if and only if $W_1 \subset W_2$. Thus, this invertible function σ induces an automorphism of$\jmath(\tau)^c$ which is also denoted by σ.

Theorem 5.1. If τ is a discrete topology defined on nonempty set X with $|X| = 2$ then the graph $\jmath(\tau)^c$ is distance transitive.

Proof: If $|X| = 2$, then by Theorem 4.1, the graph $\jmath(\tau)^c$ is vertex-transitive. As the graph $\jmath(\tau)^c$ is vertex-transitive connected graph with 2 vertices then the graph $\jmath(\tau)^c$ is distance transitive.

Theorem 5.2. If (X, τ) is the discrete topological space with $|X| = 3$ then the graph $\jmath(\tau)^c$ is distance transitive.

Proof: Suppose, U, V and $U_1, V_1 \in V(\tau)$ are two pairs of vertices in the graph $\jmath(\tau)^c$ that satisfy the condition $dist(U_1, V_1) = dist(U, V)$. To complete the proof, it is suffices to prove that there is an automorphism σ on $V(\tau)$ of $\jmath(\tau)^c$ such that $\sigma(U_1) = U$ and $\sigma(V_1) = V$. As $\jmath(\tau)^c$ is vertex-transitive then by Theorem 4.2, we have there is an automorphism σ_1 on $V(\tau)$ such that

$\sigma_1(U_1) = U$. Suppose $\sigma_1(V_1) = V_2$. If $V_2 = V$, then the proof is completed. Assume $V_2 \neq V$. If we can further find an automorphism σ_2 on $V(\tau)$ such that $\sigma_2(U) = U$ and $\sigma_2(V_2) = V$, then $\sigma_2 \circ \sigma_1$ will send U_1 to U and send V_1 to V , and thus we complete the proof. For finding such an automorphism, consider the following three separate cases.

Case I: If $dist(U, V) = 1$ and $|U| = 2$.

If both the cardinality of V and V_2 are 2, and hence $U \not\subset V$, $U \not\subset V_2$. Suppose $U = \{a, b\}$ and $V = \{a, c\}$, $V_2 = \{b, c\}$. Since, $V \neq V_2$, there is a unique invertible transformation σ_2 on $V(\tau)$ such that $\sigma_2(a) = b$, $\sigma_2(b) = a$, $\sigma_2(c) = c$. This induces automorphism on $\jmath(\tau)^c$ by σ_2 which fixes U and sends V_2 to V. If both the cardinality of V and V_2 are 1, then $V = V_2$, otherwise $dist(U, V) = 2$, this implies that there is an automorphism σ_0 on $\jmath(\tau)^c$ which fixes U and sends V_2 to V .

Case II: If $dist(U, V) = 2$ and $|U| = 1$.

$\{U = \{a\} \sim Z = \{b\} \sim V = \{a, c\}$ and $U = \{a\} \sim Z_2 = \{c\} \sim V_2 = \{a, b\}\}$. In this case, there are vertices $Z, Z_2 \in V$ such that $U \sim Z \sim V$ and $U \sim Z_2 sim V_2$. If $|U| = 1$ then $|V| = |V_2| = 2$ and Z and Z_2 are both open subsets of X of cardinality 1. Suppose that $Z = \{b\}, Z_2 = \{c\}$. Let $U = \{a\}$, $V = \{a, c\}$ and $V_2 = \{a, b\}$ be an open subsets of X. As $U \neq V$, it is easy to see that $c \notin U$. There is an invertible transformation σ_2 on $V(\tau)$ such that $\sigma_2(a) = a$; $\sigma_2(b) = c$, $\sigma_2(c) = b$. The induced automorphism on $\jmath(\tau)^c$ by σ_2 fixes U and sends V_2 to V .

Case III: If $dist(U, V) = 2$ and $|U| = 2$.

Let $\{U = \{a, b\} \sim Z = \{a, c\} \sim V = \{b\}$ and $U = \{a, b\} \sim Z_2 = \{b, c\} \sim V_2 = \{a\}\}$. In this case, there are vertices $Z, Z_2 \in V$ such that $U \sim Z \sim V$ and $U \sim Z_2 sim V_2$. If$|U| = 1$ then $|V| = |V_2| = 2$ and Z and Z_2 are both open subsets of X of cardinallity 1. Suppose that $Z = \{a, c\}, Z_2 = \{b, c\}$. Let $U = \{a, b\}$, $V = \{b\}$ and $V_2 = \{a\}$be an open subsets of X. As $U \neq V$, it is easy to see that$b \notin U$ There is an invertible transformation σ_2 on $V(\tau)$ such that $\sigma_2(a) = b$; $\sigma_2(b) = a$, $\sigma_2(c) = c$. The induced automorphism on $\jmath(\tau)^c$ by σ_2 fixes U and sends V_2 to V .

Remark 5.3 : It is well known (see [[8]]) that a distance-transitive graph is distance regular, which leads to the following theorem.

Theorem 5.3. If (X, τ) is the discrete topological space with $|X| = n = 2$ or $n = 3$ then the graph $\jmath(\tau)^c$ is a distance regular.

Proof: If (X, τ) is the discrete topological space with $|X| = n = 2$ or $n = 3$ then by Theorem 5.1, and Theorem 5.2, a graph $\jmath(\tau)^c$ is distance transitive graph and hence by Remark 5.3, a graph $\jmath(\tau)^c$ is distance regular graph.

6. Conclusion

In this present work, we studied the open subset inclusion graph of a topological space $\jmath(\tau)^c$ on a finite set X. It is found that, if τ is a discrete topology defined on nonempty set X with $|X| \leq 3$ then the graph $\jmath(\tau)^c$ is bipartite. We also provide exact value of the independence number, vertex connectivity and edge connectivity of the graph $\jmath(\tau)^c$ for $|X| = 2, 3$. Moreover, if τ is a discrete topology defined on nonempty set X with$|X| = 2$ or $|X| = 3$ then it is shown that the graph $\jmath(\tau)^c$ is Hamiltonian, regular, complete bipartite, vertex-transitive, edge-transitive and has a perfect matching. Main finding of this work is that, if (X, τ) is a discrete topological space with$|X| = 2$ or $|X| = 3$then it is shown that $\jmath(\tau)^c$ is distance-transitive graph and distance regular graph.

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Conflict of Interests

The author(s) declare that there is no conflict of interests.

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