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GENERALIZED STEPANOV-LIKE PSEUDO ALMOST AUTOMORPHIC SOLUTIONS TO SOME CLASSES OF NONAUTONOMOUS EVOLUTION EQUATIONS

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Abstract. In this paper, Under Acquistapace-Terreni conditions, we make extensive use of interpolation spaces and exponential dichotomy techniques to obtain the existence of generalized Stepanov-like pseudo almost automorphic solutions to some classes of nonautonomous partial evolution equations. Our results extend many recent known ones on this topics. An interesting example is presented at the end of the paper to illustrate the main findings.

Keywords: Fixed point theorem; Generalized Stepanov-like pseudo almost automorphy; Neutral evolution equations.

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1. Introduction

Let $(\mathbb{X}, \|\cdot\|)$ be a Banach space. The main impetus of the present paper is the fundamental work by Diagana [5], in which the concept of $\mathbb{S}^p_\gamma$-pseudo almost automorphy (or generalized Stepanov-like pseudo almost automorphy) was introduced as a natural generalisation of the concept of pseudo almost automorphy as well as the one of Stepanov-like pseudo almost automorphy. Among other things, under some reasonable assumptions,

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the author obtained the existence and uniqueness theorem of pseudo almost automorphic solutions to the class of Sobolev type evolution equations given by

$$\frac{d}{dt}\big[u(t) + f(t, u(t))\big] = A(t)u(t) + g(t, u(t)), \quad t \in \mathbb{R}, \tag{1}$$

where $A(t) : D \subset \mathbb{X} \to \mathbb{X}$ for $t \in \mathbb{R}$ is a family of densely defined closed linear operator on a domain D, independent of t, and $f, g : \mathbb{R} \times \mathbb{X} \to \mathbb{X}$ are suitable functions.

In this work, we consider a more general setting and use slightly different techniques to study the existence of generalized Stepanov-like pseudo almost automorphic solutions to some class of abstract nonautonomous differential equations

$$\frac{d}{dt}\big[u(t) + f(t, B(t)u(t))\big] = A(t)u(t) + g(t, C(t)u(t)), \quad t \in \mathbb{R}, \tag{2}$$

where $A(t)$ for $t \in \mathbb{R}$ is a family of closed linear operators on $D(A(t))$ satisfying the well-known Acquistapace-Terreni conditions, $B(t), C(t) \quad (t \in \mathbb{R})$ are families of (possibly unbounded) linear operators, and $f : \mathbb{R} \times \mathbb{X} \to \mathbb{X}^t_\beta \quad (0 < \alpha < \beta < 1)$ and $g : \mathbb{R} \times \mathbb{X} \to \mathbb{X}$ are jointly continuous satisfying some additional assumptions.

In this parer, we consider a general intermediate spaces $\mathbb{X}_\alpha$ and $\mathbb{X}_\beta$ between $D(A)$ and $\mathbb{X}$. In contrast with the fractional power spaces, the interpolation and the Hölder spaces depend only on $D(A)$ and $\mathbb{X}$ and can be explicitly expressed in many concrete cases. The literature related to those intermediate spaces is very extensive, in particular, we refer the reader to the excellent book by Lunardi [13], which contains a comprehensive presentation on this topic and related issue.

The existence of pseudo almost automorphic solutions to differential equations constitutes one of the most attractive topics in qualitative theory of differential equations due to its applications in control theory or engineering for instance. Some contributions on the existence of pseudo almost automorphic solutions to differential equations have been made, among them are [9, 12, 16, 17, 18, 19, 20] and the references therein. However, the existence of pseudo almost automorphic solutions to Eq. (2) with perturbations and interpolation spaces is quite new and untreated original problem, which constitutes one of the main motivations of this paper. We point out that pseudo almost periodic mild solution to Eq. (2) was investigated in [2] without generalized Stepanov-like pseudo almost automorphic terms.

2. Preliminaries

Let $(\mathbb{X}, \|\cdot\|)$ be a Banach space. If L is a linear operator on $\mathbb{X}$, then $D(L)$, $\rho(L)$, and $\sigma(L)$ stand respectively for the domain, resolvent, and spectrum of L. Similarly, one sets $R(\lambda, L) := (\lambda I - L)^{-1}$ for all $\lambda \in \rho(L)$ where I is the identity operator for $\mathbb{X}$. Furthermore, we set $Q = I - P$ for a projection P. We denote the Banach algebra of bounded linear operators on $\mathbb{X}$ equipped with its natural norm by $B(\mathbb{X})$.

If $\mathbb{Y}$ is another Banach space, we then let $BC(\mathbb{R}, \mathbb{X})$ (respectively, $BC(\mathbb{R}\times\mathbb{Y}, \mathbb{X})$) denote the collection of all $\mathbb{X}$-valued bounded continuous functions and equip it with the sup norm (respectively, the space of jointly bounded continuous functions $F : \mathbb{R} \times \mathbb{Y} \to \mathbb{X}$).

The space $BC(\mathbb{R}, \mathbb{X})$ equipped with the sup norm is a Banach space. Furthermore, $C(\mathbb{R}, \mathbb{Y})$ (respectively, $C(\mathbb{R} \times \mathbb{Y}, \mathbb{X})$) denotes the class of continuous functions from $\mathbb{R}$ into $\mathbb{Y}$ (respectively, the class of jointly continuous functions $F : \mathbb{R} \times \mathbb{Y} \to \mathbb{X}$).

2.1. Evolution Families

The setting of this subsection follows that of Baroun *et al.* [1] and Diagana [3]. Fix once and for all a Banach space $(\mathbb{X}, \|\cdot\|)$.

Definition 2.1. *A family of closed linear operators $A(t)$ for $t \in \mathbb{R}$ on $\mathbb{X}$ with domain $D(A(t))$ (possibly not densely defined) satisfy the so-called Acquistapace and Terreni conditions, if there exist constants $\omega \in \mathbb{R}$, $\theta \in (\pi/2, \pi)$, $L > 0$ and $\mu, \nu \in (0, 1]$ with $\mu + \nu > 1$ such that*

$$\Sigma_\theta \cup \{0\} \subset \rho(A(t) - \omega) \ni \lambda, \quad \|R(\lambda, A(t) - \omega)\| \leq \frac{K}{1 + |\lambda|} \quad \textit{for all } t \in \mathbb{R}, \tag{3}$$

and

$$\|(A(t) - \omega)R(\lambda, A(t) - \omega)\,[R(\omega, A(t)) - R(\omega, A(s))]\| \leq L\,\frac{|t - s|^\mu}{|\lambda|^\nu} \tag{4}$$

for $t, s \in \mathbb{R}$, $\lambda \in \Sigma_\theta := \{\lambda \in \mathbb{C} \setminus \{0\} : |\arg \lambda| \leq \theta\}$.

For a given family of linear operators $A(t)$, the existence of an evolution family associated with it is not always guaranteed. However, if $A(t)$ satisfies Acquistapace-Terreni conditions, then there exists a unique evolution family

$$\mathcal{U} = \{U(t,s) : t,s \in \mathbb{R} \text{ such that } t \geq s\}$$

on $\mathbb{X}$ associated with $A(t)$ such that $U(t,s)\mathbb{X} \subseteq D(A(t))$ for all $t,s \in \mathbb{R}$ with $t \geq s$, and

(a) $U(t,s)U(s,r) = U(t,r)$ for $t,s \in \mathbb{R}$ such that $t \geq s \geq r$;

(b) $U(t,t) = I$ for $t \in \mathbb{R}$ where I is the identity operator of $\mathbb{X}$;

(c) $(t,s) \to U(t,s) \in B(\mathbb{X})$ is continuous for $t > s$;

(d) $U(\cdot,s) \in C^1((s,\infty), B(\mathbb{X}))$, $\frac{\partial U}{\partial t}(t,s) = A(t)U(t,s)$ and

$$\|A(t)^k U(t,s)\| \leq K\,(t-s)^{-k}$$

for $0 < t-s \leq 1$ and $k = 0,1$.

Definition 2.2. *An evolution family $\mathcal{U} = \{U(t,s) : t,s \in \mathbb{R}$ such that $t \geq s\}$ is said to have an exponential dichotomy (or is hyperbolic) if there are projections $P(t)$ ($t \in \mathbb{R}$) that are uniformly bounded and strongly continuous in t and constants $\delta > 0$ and $N \geq 1$ such that*

(e) *$U(t,s)P(s) = P(t)U(t,s)$;*

(f) *the restriction $U_Q(t,s) : Q(s)\mathbb{X} \to Q(t)\mathbb{X}$ of $U(t,s)$ is invertible (we then set $\widetilde{U}_Q(s,t) := U_Q(t,s)^{-1}$);*

(g) *$\|U(t,s)P(s)\| \leq Ne^{-\delta(t-s)}$ and $\|\widetilde{U}_Q(s,t)Q(t)\| \leq Ne^{-\delta(t-s)}$ for $t \geq s$ and $t,s \in \mathbb{R}$.*

Definition 2.3. ([14, 15]) *Given an hyperbolic evolution family, we define its so-called Green's function by*

$$\Gamma(t,s) := \begin{cases} U(t,s)P(s), & \text{for } t \geq s, \quad t,s \in \mathbb{R} \\ \\ U_Q(t,s)Q(s), & \text{for } t < s, \quad t,s \in \mathbb{R} \end{cases} \tag{5}$$

This setting requires some estimates related to $\mathcal{U} = \{U(t,s)\}_{t \geq s}$. For that, we introduce the interpolation spaces for $A(t)$.

Let A be a sectorial operator on $\mathbb{X}$ (in Definition , replace $A(t)$ with A) and let $\alpha \in (0,1)$. Define the real interpolation space

$$\mathbb{X}_\alpha^A := \left\{x \in \mathbb{X} : \|x\|_\alpha^A := \sup_{r>0} \|r^\alpha (A-\omega)R(r, A-\omega)x\| < \infty\right\},$$

which, by the way, is a Banach space when endowed with the norm $\|\cdot\|_\alpha^A$. For convenience we further write

$$\mathbb{X}_0^A := \mathbb{X}, \quad \|x\|_0^A := \|x\|, \quad \mathbb{X}_1^A := D(A)$$

and $\|x\|_1^A := \|(\omega - A)x\|$. Moreover, let $\hat{\mathbb{X}}^A := \overline{D(A)}$ of $\mathbb{X}$. In particular, we will frequently be using the following continuous embedding

$$D(A) \hookrightarrow \mathbb{X}_\beta^A \hookrightarrow D((\omega - A))^\alpha \hookrightarrow \mathbb{X}_\alpha^A \hookrightarrow \hat{\mathbb{X}}^A \hookrightarrow \mathbb{X},$$

for all $0 < \alpha < \beta < 1$, where the fractional powers are defined in the usual way.

In general, $D(A)$ is not dense in the spaces ${\mathbb{X}_\alpha}^A$ and $\mathbb{X}$. However, we have the following continuous injection:

$$\mathbb{X}_\beta^A \hookrightarrow \overline{D(A)}^{\|\cdot\|_\alpha^A} \quad \text{for} \quad 0 < \alpha < \beta < 1.$$

Definition 2.4. *Given a family of linear operators $A(t)$ for $t \in \mathbb{R}$ satisfying the Acquistapace-Terreni conditions, we set $\mathbb{X}_\alpha^t := \mathbb{X}_\alpha^{A(t)}$ and $\hat{\mathbb{X}}^t := \hat{\mathbb{X}}^{A(t)}$ for $0 \le \alpha \le 1$ and $t \in \mathbb{R}$, with the corresponding norms.*

Proposition 2.5. ([1]) *For $x \in \mathbb{X}$, $0 \le \alpha \le 1$ and $t > s$, the following hold:*

(i) *There is a constant $n(\alpha)$, such that*

$$\|U(t,s)P(s)x\|_\alpha^t \le n(\alpha)e^{-\frac{\delta}{2}(t-s)}(t-s)^{-\alpha}\|x\|. \tag{6}$$

(ii) *There is a constant $m(\alpha)$, such that*

$$\|\widetilde{U}_Q(s,t)Q(t)x\|_\alpha^s \le m(\alpha)e^{-\delta(t-s)}\|x\|, \quad t \le s. \tag{7}$$

2.2. $\mathbb{S}_\gamma^p$-pseudo almost automorphy

Definition 2.6. ([6, 7]) *(i) A function $f \in C(\mathbb{R}, \mathbb{X})$ is said to be almost automorphic if for every sequence of real numbers $(s'_n)_n$ there exists a subsequence $(s_n)_n$ such that*

$$\lim_{m\to\infty}\lim_{n\to\infty} f(t + s_n - s_m) = f(t) \text{ for each } t \in \mathbb{R}.$$

this limit means that

$$g(t) := \lim_{n\to\infty} f(t + s_n)$$

is well defined for each $t \in \mathbb{R}$, and

$$\lim_{n\to\infty} g(t - s_n) = f(t)$$

for each $t \in \mathbb{R}$. The collection of all such functions will be denoted by $AA(\mathbb{X})$.

(ii) A function $f \in C(\mathbb{R} \times \mathbb{Y}, \mathbb{X})$ is said to be almost automorphic if $f(t, u)$ is almost automorphic in $t \in \mathbb{R}$ uniformly for all $u \in K$, where K is any bounded subset of $\mathbb{Y}$. The collection of all such functions will be denoted by $AA(\mathbb{Y}, \mathbb{X})$.

Theorem 2.7. ([7]) *Assume $f, g : \mathbb{R} \to \mathbb{X}$ are almost automorphic and λ is any scalar. Then the following holds true:*

(1) *$f + g, \lambda f, f_\tau(t) := f(t + \tau)$ and $\widehat{f}(t) := f(-t)$ are almost automorphic.*

(2) *The range R_f of f is precompact, so f is bounded.*

(3) *If $\{f_n\}$ is a sequence of almost automorphic functions and $f_n \to f$ uniformly on $\mathbb{R}$, then f is almost automorphic.*

Definition 2.8.(Xiao et al. [11]) *A continuous function $L : \mathbb{R}\times\mathbb{R} \to \mathbb{X}$ is called bi-almost automorphic if for every sequence of real numbers $(s'_n)_{n\in\mathbb{N}}$, we can extract a subsequence $(s_n)_{n\in\mathbb{N}}$ such that*

$$H(t, s) := \lim_{n\to\infty} L(t + s_n, s + s_n)$$

is well defined in $t, s \in \mathbb{R}$, and

$$\lim_{n\to\infty} H(t - s_n, s - s_n) = L(t, s)$$

for each $t, s \in \mathbb{R}$. The collection of such functions will be denoted $bAA(\mathbb{R} \times \mathbb{R}, \mathbb{X})$.

We now recall positively bi-almost automorphic functions. For that, let $\mathbb{T}$ be the set defined by:

$$\mathbb{T} := \big\{(t,s) \in \mathbb{R} \times \mathbb{R} : t \geq s\big\}.$$

Definition 2.9.([5]) *A continuous function $L : \mathbb{T} \to \mathbb{X}$ is called positively bi-almost automorphic if for every sequence of real numbers $(s'_n)_{n\in\mathbb{N}}$, we can extract a subsequence $(s_n)_{n\in\mathbb{N}}$ such that*

$$H(t,s) := \lim_{n\to\infty} L(t+s_n, s+s_n)$$

is well defined in $t, s \in \mathbb{T}$, and

$$\lim_{n\to\infty} H(t-s_n, s-s_n) = L(t,s)$$

for each $(t,s) \in \mathbb{T}$. The collection of such functions will be denoted $bAA(\mathbb{T}, \mathbb{X})$.

Define the classes of functions

$$PAP_0(\mathbb{X}) := \big\{f \in BC(\mathbb{R}, \mathbb{X}) : \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \|f(\sigma)\| d\sigma = 0\big\}.$$

and

$$PAP_0(\mathbb{R}\times\mathbb{Y}, \mathbb{X}) := \{f \in BC(\mathbb{R}\times\mathbb{Y}, \mathbb{X}) : \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \|f(\sigma,u)\| d\sigma = 0$$
$$\text{uniformly for } u \text{ in any bounded subset of } \mathbb{Y}\}$$

Definition 2.10.([4]) *A function $f \in C(\mathbb{R}, \mathbb{X})$ is called pseudo almost automorphic if it can be expressed as $f = h + \varphi$, where $h \in AA(\mathbb{X})$ and $\varphi \in PAP_0(\mathbb{X})$. The collection of such functions will be denoted by $PAA(\mathbb{X})$.*

Definition 2.11.([4]) *A function $F \in C(\mathbb{R}\times\mathbb{Y}, \mathbb{X})$ is said to pseudo almost automorphic if it can be expressed as $F = G + \Phi$, where $G \in AA(\mathbb{R}\times\mathbb{Y}, \mathbb{X})$ and $\varphi \in PAP_0(\mathbb{R}\times\mathbb{Y}, \mathbb{X})$. The collection of such functions will be denoted by $PAA(\mathbb{R}\times Y, \mathbb{X})$.*

Now, let us recall the following definitions and basic results about $\mathbb{S}^p_\gamma$-almost automorphic functions. For more details, we refer the reader to Diagana [5].

Let $p \in [1, \infty)$ and let $\mathbb{U}$ denote the collection of all measurable functions $\gamma : (0, \infty) \mapsto (0, \infty)$ satisfying the following condition:

$$\gamma_0 := \lim_{\varepsilon \to 0} \int_{\varepsilon}^{1} \gamma(\sigma) d\sigma = \int_{0}^{1} \gamma(\sigma) d\sigma < \infty. \tag{8}$$

Definition 2.12.([10]) *The Bochner transform $f^b(t, s)$, $t \in \mathbb{R}$, $s \in [0, 1]$ of a function $f : \mathbb{R} \to \mathbb{X}$ is defined by $f^b(t, s) := f(t + s)$.*

Remark 2.13. *(i) A function $\varphi(t, s)$, $t \in \mathbb{R}$, $s \in [0, 1]$, is the Bochner transform of a certain function f, $\varphi(t, s) = f^b(t, s)$, if and only if $\varphi(t + \tau, s - \tau) = \varphi(s, t)$ for all $t \in \mathbb{R}$, $s \in [0, 1]$ and $\tau \in [s - 1, s]$.*
(ii) Note that if $f = h + \varphi$, then $f^b = h^b + \varphi^b$. Moreover, $(\lambda f)^b = \lambda f^b$ for each scalar λ.

Definition 2.14.([8]) *The Bochner transform $F^b(t, s, u)$, $t \in \mathbb{R}$, $s \in [0, 1]$, $u \in \mathbb{X}$ of a function $F(t, u)$ on $\mathbb{R} \times \mathbb{X}$, with values in $\mathbb{X}$, is defined by $F^b(t, s, u) := F(t + s, u)$ for each $u \in \mathbb{X}$.*

Definition 2.15.([5]) *Let $p \in [1, \infty)$ and let $\gamma \in \mathbb{U}$. The space $BS^p_\gamma(\mathbb{X})$ of all generalized Stepanov spaces, with the exponent p and weight γ, consists of all $\gamma d\tau$-measurable functions $f : \mathbb{R} \to \mathbb{X}$ such that $f^b \in L^\infty\big(\mathbb{R}, L^p((0, 1), \gamma d\tau)\big)$. This is a Banach space when it is equipped with the norm*

$$\|f\|_{S^p_\gamma(\mathbb{X})} := \sup_{t \in \mathbb{R}} \Big(\int_t^{t+1} \gamma(\tau - t) \|f(\tau)\|^p \, d\tau \Big)^{1/p} = \sup_{t \in \mathbb{R}} \Big(\int_0^1 \gamma(\tau) \|f(\tau + t)\|^p \, d\tau \Big)^{1/p}.$$

Remark 2.16. Under assumption (8), the identically constant functions belong to $BS^p_\gamma(\mathbb{X})$. Of course, if $\gamma(t) = 1$ for all $t \in (0, \infty)$, then $BS^p_1(\mathbb{X}) = BS^p(\mathbb{X})$.

Definition 2.17.([5]) *Let $p \geq 1$ and let $\gamma \in \mathbb{U}$. The space $AS^p_\gamma(\mathbb{X})$ of generalized Stepanov-like almost automorphic functions (or $\mathbb{S}^p_\gamma$-almost automorphic) consists of all $f \in BS^p_\gamma(\mathbb{X})$ such that for every sequence of real numbers $(s'_n)_{n \in \mathbb{N}}$, there exists a subsequence $(s_n)_{n \in \mathbb{N}}$ and a function $g \in L^p_{\mathrm{loc}}(\mathbb{R}, \gamma ds)$ such that*

$$\begin{aligned} &\Big[\int_t^{t+1} \gamma(s - t) \|f(s_n + s) - g(s)\|^p ds \Big]^{1/p} \\ &= \Big[\int_0^1 \gamma(s) \|f(s_n + s + t) - g(s + t)\|^p ds \Big]^{1/p} \to 0, \end{aligned}$$

and

$$\begin{aligned}&\Big[\int_t^{t+1}\gamma(s-t)\|g(s-s_n)-f(s)\|^p ds\Big]^{1/p}\\&=\Big[\int_0^1\gamma(s)\|g(s+t-s_n)-f(s+t)\|^p ds\Big]^{1/p}\to 0\end{aligned}$$

as $n\to\infty$ for each $t\in\mathbb{R}$.

Remark 2.18. Let $\gamma\in\mathbb{U}$. If $1\le p<q<\infty$ and $f\in L^q_{\text{loc}}(\mathbb{R},\gamma ds)$ is $\mathbb{S}^q_\gamma$-almost automorphic, then f is $\mathbb{S}^p_\gamma$-almost automorphic. Also using (8), one can show that if $f\in AA(\mathbb{X})$, then f is $\mathbb{S}^p_\gamma$-almost automorphic for any $1\le p<\infty$.

Definition 2.19.([5]) *Let $\gamma\in\mathbb{U}$. A function $F:\mathbb{R}\times\mathbb{Y}\to\mathbb{X},(t,u)\to F(t,u)$ with $F(\cdot,u)\in L^p_{\text{loc}}(\mathbb{R},\gamma ds)$ for each $u\in\mathbb{Y}$, is said to be $\mathbb{S}^p_\gamma$-almost automorphic in $t\in\mathbb{R}$ uniformly in $u\in\mathbb{Y}$ if $t\to F(t,u)$ is $\mathbb{S}^p_\gamma$-almost automorphic for each $u\in\mathbb{Y}$, that is, for every sequence of real numbers $(s'_n)_{n\in\mathbb{N}}$, there exists a subsequence $(s_n)_{n\in\mathbb{N}}$ and a function $G(\cdot,u)\in L^p_{\text{loc}}(\mathbb{R},\gamma ds)$ such that*

$$\begin{aligned}&\Big[\int_t^{t+1}\gamma(s-t)\|F(s_n+s,u)-G(s,u)\|^p ds\Big]^{1/p}\to 0,\\&\Big[\int_t^{t+1}\gamma(s-t)\|G(s-s_n,u)-F(s,u)\|^p ds\Big]^{1/p}\to 0\end{aligned}$$

as $n\to\infty$ pointwise on $\mathbb{R}$ for each $u\in\mathbb{Y}$.

The collection of those $\mathbb{S}^p_\gamma$-almost automorphic functions $F:\mathbb{R}\times\mathbb{Y}\to\mathbb{X}$ will be denoted by $AS^p_\gamma(\mathbb{R}\times\mathbb{Y},\mathbb{X})$.

Definition 2.20.([5]) *A function $f\in BS^p_\gamma(\mathbb{X})$ is called $\mathbb{S}^p_\gamma$-pseudo almost automorphic (or generalized Stepanov-like pseudo almost automorphic) if it can be expressed as*

$$f=h+\varphi,$$

where $h^b\in AA\big(L^p((0,1),\gamma ds)\big)$ and $\varphi^b\in PAP_0\big(L^p((0,1),\gamma ds)\big)$. The collection of such functions will be denoted by $PAA^p_\gamma(\mathbb{X})$.

Clearly, a function $f\in L^p_{\text{loc}}(\mathbb{R},\gamma ds)$ is said to be $\mathbb{S}^p_\gamma$-pseudo almost automorphic if its Bochner transform $f^b:\mathbb{R}\to L^p((0,1),\gamma ds)$ is pseudo almost automorphic in the sense that there exist two functions $h, \varphi : \mathbb{R} \to \mathbb{X}$ such that $f = h + \varphi$, where $h^b \in AA(L^p((0,1), \gamma ds))$ and $\varphi^b \in PAP_0(L^p((0,1), \gamma ds))$.

Definition 2.21.([5]) *Let $\gamma \in \mathbb{U}$. A function $F : \mathbb{R} \times \mathbb{Y} \to \mathbb{X}, (t,u) \to F(t,u)$ with $F(\cdot, u) \in L^p(\mathbb{R}, \gamma ds)$ for each $u \in \mathbb{Y}$, is said to be $\mathbb{S}^p_\gamma$-pseudo almost automorphic if there exists two functions $H, \Phi : \mathbb{R} \times \mathbb{Y} \to \mathbb{X}$ such that $F = H + \Phi$, where $H^b \in AA(\mathbb{R} \times \mathbb{Y}, L^p((0,1), \gamma ds))$ and $\Phi^b \in PAP_0(\mathbb{R} \times \mathbb{Y}, L^p((0,1), \gamma ds))$. The collection of those $\mathbb{S}^p_\gamma$-pseudo almost automorphic functions will be denoted by $PAA^p_\gamma(\mathbb{R} \times \mathbb{Y}, \mathbb{X})$.*

Remark 2.22. By definition, the decomposition of $\mathbb{S}^p_\gamma$-pseudo almost automorphic functions is unique. Furthermore, $\mathbb{S}^p_\gamma$-pseudo almost automorphic spaces are translation-invariant.

A significant result is the next theorem, which is due to Diagana [5].

Theorem 2.23. *Let $\gamma \in \mathbb{U}$. The space $PAA^p_\gamma(\mathbb{X})$ equipped with the norm $\| \cdot \|_{\mathbb{S}^p_\gamma}$ is a Banach space.*

We also have the following composition result.

Theorem 2.24.([5]) *Let $F = G + \Phi \in PAA^p_\gamma(\mathbb{R} \times \mathbb{Y}, \mathbb{X})$ such that $H^b \in AA(\mathbb{R} \times \mathbb{Y}, L^p((0,1), \gamma(s)ds))$ and $\Phi^b \in PAP_0(\mathbb{R} \times \mathbb{Y}, L^p((0,1), \gamma(s)ds))$. Moreover, we suppose that G satisfies the Lipschitz condition; that is, there exists $L > 0$ such that for all $u, v \in \mathbb{Y}$ and $t \in \mathbb{R}$*

$$\|f(t,u) - f(t,v)\| \leq L\|u - v\|_{\mathbb{Y}}. \tag{9}$$

and that Φ satisfies:

$$\begin{aligned} &\Big(\int_0^1 \gamma(s) \|\Phi(t+s, u(s)) - \Phi(t+s, v(s))\|^p ds \Big)^{1/p} \\ &\leq L \Big(\int_0^1 \gamma(s) \|u(s) - v(s)\|^p ds \Big)^{1/p}. \end{aligned} \tag{10}$$

for all $u, v \in L^p_{\mathrm{loc}}(\mathbb{R}, \gamma ds)$ and $t \in \mathbb{R}$.
Furthermore, if $h = g + \varphi \in PAA^p_\gamma(\mathbb{Y})$ with $h^b \in AA(L^p((0,1), \gamma(s)ds))$ and $\varphi^b \in PAP_0(L^p((0,1), \gamma(s)ds))$ such that $K = \overline{\{g(t) : t \in \mathbb{R}\}}$ is compact, then $t \mapsto F(t, h(t))$ belongs to $PAA^p_\gamma(\mathbb{X})$.

3. Main results

Fix $\gamma \in \mathbb{U}$ and $p > 1$. Throughout the rest of the paper, we suppose that $\gamma \in \mathbb{U}$ satisfies

$$\inf_{t\in(0,\infty)} \gamma(t) = c_0 > 0.$$

To study the existence of a pseudo almost automorphic solution to Eq. (1) with $\mathbb{S}^p_\gamma$-pseudo almost automorphic coefficients we will assume that the following assumptions hold:

(H1) The family of closed linear operators $A(t)$ for $t \in \mathbb{R}$ on $\mathbb{X}$ with domain $D(A(t))$ (possibly not densely defined) satisfy the Acquistapace and Terreni conditions, the evolution family of operators $\mathcal{U} = \{U(t,s)\}_{t\geq s}$ generated by $A(\cdot)$ has an exponential dichotomy with constants $N, \delta > 0$ and dichotomy projections $P(t)$ $(t \in \mathbb{R})$. Moreover, $0 \in \rho(A)$ for each $t \in \mathbb{R}$ and the following hold

$$\sup_{t,s\in\mathbb{R}} \|A(s)A^{-1}(t)\|_{B(\mathbb{X},\mathbb{X}_\beta)} < c_1 \tag{11}$$

(H2) There exists $0 \leq \alpha < \beta < 1$ such that $\mathbb{X}^t_\alpha = \mathbb{X}_\alpha$ and $\mathbb{X}^t_\beta = \mathbb{X}_\beta$ for all $t \in \mathbb{R}$, with uniform equivalent norms. Let $c_2(\alpha), c_3, c_4$ be the bounds of the continuous injections $\mathbb{X}_\beta \hookrightarrow \mathbb{X}_\alpha, \mathbb{X}_\alpha \hookrightarrow \mathbb{X}, \mathbb{X}_\beta \hookrightarrow \mathbb{X}$.

(H3) The function $\mathbb{R}\times\mathbb{R} \to \mathbb{X}$, $(t,s) \to A(s)\Gamma(t,s)y \in bAA(\mathbb{T},\mathbb{X}_\alpha)$ uniformly for $y \in \mathbb{X}_\beta$.

(H4) The function $\mathbb{R}\times\mathbb{R} \to \mathbb{X}$, $(t,s) \to \Gamma(t,s)y \in bAA(\mathbb{T},\mathbb{X}_\alpha)$ uniformly for $y \in \mathbb{X}$.

(H5) The linear operators $B(t), C(t) : \mathbb{X}_\alpha \to \mathbb{X}$ are bounded uniformly in $t \in \mathbb{R}$. Moreover, both $t \mapsto B(t)$ and $t \mapsto C(t)$ belong to $AA(B(\mathbb{X}_\alpha,\mathbb{X}))$. We then set

$$c_5 := \max\Big(\sup_{t\in\mathbb{R}}\|B\|_{B(\mathbb{X}_\alpha,\mathbb{X})}, \sup_{t\in\mathbb{R}}\|C\|_{B(\mathbb{X}_\alpha,\mathbb{X})}\Big)$$

(H6) The function $f \in PAA^p_\gamma(\mathbb{R}\times\mathbb{X},\mathbb{X}_\beta)$ and $g \in PAA^p_\gamma(\mathbb{R}\times\mathbb{X},\mathbb{X}) \cap C(\mathbb{R}\times\mathbb{X},\mathbb{X})$. Moreover, there exists $L > 0$ such that

$$\|f(t,u) - f(t,v)\|_\beta \leq L\|u - v\|$$

for all $u, v \in \mathbb{X}$ and $t \in \mathbb{R}$, and

$$\|g(t,u) - g(t,v)\| \leq L\|u - v\|$$

for all $u, v \in \mathbb{X}$ and $t \in \mathbb{R}$.

Definition 3.1. *A continuous function $u : \mathbb{R} \to \mathbb{X}_\alpha$ is said to be a mild solution to* (1) *provided that the functions $s \to A(s)U(t,s)P(s)f(s, B(s)u(s))$ and $s \to A(s)U(t,s)Q(s)f(s, B(s)u(s))$ are integrable on (t,s) and*

$$\begin{aligned} u(t) &= -f(t, B(t)u(t)) + U(t,s)(u(s) + f(s, B(s)u(s))) \\ &- \int_s^t A(s)U(t,s)P(s)f(s, B(s)u(s))ds + \int_t^s A(s)U(t,s)Q(s)f(s, B(s)u(s))ds \\ &+ \int_s^t U(t,s)P(s)g(s, C(s)u(s))ds - \int_t^s U(t,s)Q(s)g(s, C(s)u(s))ds \end{aligned}$$

for $t \geq s$ and for all $t, s \in \mathbb{R}$.

Under previous assumptions (H1)–(H6), it can be easily shown that (1) has a unique mild solution given by

$$\begin{aligned} u(t) &= -f(t, B(t)u(t)) - \int_{-\infty}^t A(s)U(t,s)P(s)f(s, B(s)u(s))ds \\ &+ \int_t^\infty A(s)U_Q(t,s)Q(s)f(s, B(s)u(s))ds + \int_{-\infty}^t U(t,s)P(s)g(s, C(s)u(s))ds \\ &- \int_t^\infty U_Q(t,s)Q(s)g(s, C(s)u(s))ds \end{aligned}$$

for each $t \in \mathbb{R}$.

The proof of our main result requires the next technical lemmas

Lemma 3.2. *Under assumption* (H5)*, if $u \in PAA(\mathbb{X}_\alpha)$, then $B(\cdot)u(\cdot)$ and $C(\cdot)u(\cdot)$ belong to $PAA(\mathbb{X})$.*

Proof. Let $u = h + \varphi \in PAA(\mathbb{X}_\alpha)$ where $h \in AA(\mathbb{X}_\alpha)$ and $\varphi \in PAP_0(\mathbb{X}_\alpha)$, then $B(\cdot)u(\cdot) = B(\cdot)h(\cdot) + B(\cdot)\varphi(\cdot)$. First, it is easy to see that $B(\cdot)u(\cdot) \in BC(\mathbb{R}, \mathbb{X}_\alpha)$. Since $h \in AA(\mathbb{X}_\alpha)$, for every sequence of real numbers $(s'_n)_{n\in\mathbb{N}}$, there exists a subsequence $(s_n)_{n\in\mathbb{N}}$ and a measurable function g_1 such that

$$\lim_{n\to\infty} \|h(s_n + s) - g_1(s)\|_\alpha = 0,$$

and

$$\lim_{n\to\infty} \|g_1(s - s_n) - h(s)\|_\alpha = 0$$

for each $t \in \mathbb{R}$.

Since $B(\cdot) \in AA(B(\mathbb{X}_\alpha, \mathbb{X}))$, there exists a subsequence $(s_{n_k})_{k\in\mathbb{N}}$ of $(s_n)_{n\in\mathbb{N}}$ and a measurable function g_2 such that

$$\|B(s_{n_k}+s) - g_2(s)\|_{B(\mathbb{X}_\alpha,\mathbb{X})} \to 0,$$

and

$$\|g_2(s - s_{n_k}) - B(s)\|_{B(\mathbb{X}_\alpha,\mathbb{X})} \to 0$$

as $k \to \infty$ for each $t \in \mathbb{R}$.

By using the triangle inequality, One has

$$\begin{aligned}
\|B(s_{n_k}+s)h(s_{n_k}+s) - g_2(s)g_1(s)\| &\le \|B(s_{n_k}+s)h(s_{n_k}+s) - B(s_{n_k}+s)g_1(s)\| \\
&+ \|B(s_{n_k}+s)g_1(s) - g_2(s)g_1(s)\| \\
&\le c_5\|h(s_{n_k}+s) - g_1(s)\|_{X_\alpha} + \|g_1\|_\infty \|B(s_{n_k}+s) - g_2(s)\|_{B(\mathbb{X}_\alpha,\mathbb{X})}
\end{aligned}$$

Then

$$\lim_{n\to\infty} \|B(s_{n_k}+s)h(s_{n_k}+s) - g_2(s)g_1(s)\| = 0,$$

Analogously, one can prove that

$$\lim_{n\to\infty} \|g_2(s - s_{n_k})g_1(s - s_{n_k}) - B(s)h(s)\| = 0,$$

hence, $B(\cdot)h(\cdot) \in AA(\mathbb{X})$

To complete the proof, it suffices to notice that

$$\frac{1}{2T}\int_{-T}^{T} \|B(s)\varphi(s)\| ds \le \frac{c_5}{2T}\int_{-T}^{T} \|\varphi(s)\|_{X_\alpha} ds$$

and hence

$$\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \|B(s)\varphi(s)\| ds = 0.$$

Lemma 3.3. ([2])*For each $x \in \mathbb{X}$, suppose that Assumptions* (H1)–(H2) *hold and let α, β be real numbers such that $0 < \alpha < \beta < 1$ with $2\beta > \alpha + 1$, then there are constants $r(\alpha,\beta), r'(\alpha,\beta), d(\beta) > 0$ such that*

$$\|A(t)U(t,s)P(s)x\|_\beta \le r'(\alpha,\beta)e^{\frac{-\delta}{4}(t-s)}(t-s)^{-\beta}\|x\|, \quad t > s \tag{12}$$

$$\|A(s)U(t,s)P(s)x\|_\beta \le r(\alpha,\beta)e^{\frac{-\delta}{4}(t-s)}(t-s)^{-\beta}\|x\|, \quad t>s \tag{13}$$

and

$$\|A(s)\widetilde{U}_Q(s,t)Q(t)x\|_\beta \le d(\beta)e^{-\delta(s-t)}\|x\|, \quad t\le s \tag{14}$$

Lemma 3.4. *Under assumptions* (H1)–(H6)*, the integral operators* Γ_1 *and* Γ_2 *defined by*

$$(\Gamma_1 u)(t) := \int_{-\infty}^{t} A(s)U(t,s)P(s)f(s,B(s)u(s))ds$$

and

$$(\Gamma_2 u)(t) := \int_{t}^{\infty} A(s)U_Q(t,s)Q(s)f(s,B(s)u(s))ds$$

map $PAA(\mathbb{X}_\alpha)$ *into itself.*

Proof. Let $u\in PAA(\mathbb{X}_\alpha)$. By Lemma (3.2) one has $B(\cdot)u(\cdot)\in PAA(\mathbb{X})\subset PAA^p_\gamma(\mathbb{X})$. Using the composition theorem on generalized Stepanov-like pseudo almost automorphic functions (Theorem (2.24)), we deduce that $F(t) := f(t,B(t)u(t))\in PAA^p_\gamma(\mathbb{X}_\beta)$. In particular,

$$\|F\|_{\infty,\beta} = \sup_{t\in\mathbb{R}}\|f(t,B(t)u(t))\|_\beta < \infty$$

Now write $F=\phi+\psi$, where $\phi^b\in AA(L^p((0,1),\gamma(s)ds))$ and $\psi^b\in PAP_0(L^p((0,1),\gamma(s)ds))$, then Γ_1 can be decomposed as

$$(\Gamma_1 u)(t) = \Phi(t)+\Psi(t)$$

where

$$\Phi(t) = \int_{-\infty}^{t} A(s)U(t,s)P(s)\phi(s)ds \quad\text{and}\quad \Psi(t) = \int_{-\infty}^{t} A(s)U(t,s)P(s)\psi(s)ds,$$

Next we show that $\Phi\in AA(\mathbb{X}_\alpha)$. For that; for each $t\in\mathbb{R}$ and $k\in\mathbb{N}$; we set

$$\Phi_k(t) := \int_{k-1}^{k} A(t-s)U(t,t-s)P(t-s)\phi(t-s)ds = \int_{t-k}^{t-k+1} A(s)U(t,s)P(s)\phi(s)ds,$$

Let $q>1$ such that $\frac{1}{p}+\frac{1}{q}=1$, where $p>1$. Using Eq. (13) and the Hölder's inequality, it follows that

$$\begin{aligned}
\|\Phi_k(t)\|_\alpha &\le c_2(\alpha)\|\Phi_k(t)\|_\beta \le c_2(\alpha)r(\alpha,\beta)\int_{t-k}^{t-k+1} e^{\frac{-\delta}{4}(t-s)}(t-s)^{-\beta}\|\phi(s)\|_\beta ds \\
&= c_2(\alpha)r(\alpha,\beta)\int_{t-k}^{t-k+1}\gamma^{-1/p}(s-t+k)e^{\frac{-\delta}{4}(t-s)}(t-s)^{-\beta}\|\phi(s)\|_\beta\gamma^{1/p}(s-t+k)ds \\
&\le c_2(\alpha)r(\alpha,\beta)\Big[\int_{t-k}^{t-k+1}\gamma^{-q/p}(s-t+k)e^{\frac{-q\delta}{4}(t-s)}(t-s)^{-q\beta}ds\Big]^{1/q} \\
&\quad\times\Big[\int_{t-k}^{t-k+1}\gamma(s-t+k)\|\phi(s)\|_\beta^p ds\Big]^{1/p} \\
&\le c_0^{-1/p}c_2(\alpha)r(\alpha,\beta)\Big[\int_{k-1}^{k} e^{\frac{-q\delta}{4}s}s^{-q\beta}ds\Big]^{1/q}\|\phi\|_{\mathbb{S}_\gamma^p(\mathbb{X}_\beta)} \\
&\le c_0^{-1/p}c_2(\alpha)r(\alpha,\beta)\sqrt[q]{\frac{1+e^{\frac{q\delta}{4}}}{\frac{q\delta}{4}}}(k-1)^{-\beta}e^{\frac{-\delta}{4}k}\|\phi\|_{\mathbb{S}_\gamma^p(\mathbb{X}_\beta)} \\
&:= C_q(\alpha,\beta,\delta)\|\phi\|_{\mathbb{S}_\gamma^p(\mathbb{X}_\beta)}.
\end{aligned}$$

Since the series $\sum_{k=1}^{\infty}\Big((k-1)^{-\beta}e^{\frac{-\delta}{4}k}\Big)$ is convergent, we deduce from the well-known Weierstrass test that the series $\sum_{k=1}^{\infty}\Phi_k(t)$ is uniformly convergent on $\mathbb{R}$. Furthermore

$$\Phi(t)=\int_{-\infty}^{t}A(s)U(t,s)P(s)\phi(s)ds=\sum_{k=1}^{\infty}\Phi_k(t),$$

$\Phi\in C(\mathbb{R},\mathbb{X}_\alpha)$ and

$$\|\Phi(t)\|_\alpha\le\sum_{k=1}^{\infty}\|\Phi_k(t)\|\le K_1\|\phi\|_{\mathbb{S}_\gamma^p(\mathbb{X}_\beta)}.$$

Fix $k\in\mathbb{N}$, let us take a sequence $(s_n')_n$ of real numbers. Since $\phi\in AS_\gamma^p(\mathbb{X}_\beta)$ and $A(s)U(t,s)P(s)y\in bAA(\mathbb{T},\mathbb{X}_\alpha)$ uniformly for $y\in X_\beta$, then for every sequence $(s_n')_n$ there exists a subsequence $(s_n)_n$ and functions θ,h such that

$$\lim_{n\to\infty}A(s+s_n)U(t+s_n,s+s_n)P(s+s_n)x=\theta(t,s)x\quad\text{for each}\quad t,s\in\mathbb{R},x\in\mathbb{X}_\beta. \tag{15}$$

$$\lim_{n\to\infty}\theta(t-s_n,s-s_n)x=A(s)U(t,s)P(s)x\quad\text{for each}\quad t,s\in\mathbb{R},x\in\mathbb{X}_\beta. \tag{16}$$

$$\lim_{n\to\infty}\|\phi(t+s_n+\cdot)-h(t+\cdot)\|_{S_\gamma^p(\mathbb{X}_\beta)}=0,\quad\text{for each}\quad t\in\mathbb{R}. \tag{17}$$

$$\lim_{n\to\infty} \|h(t-s_n+\cdot)-\phi(t+\cdot)\|_{S^p_\gamma(\mathbb{X}_\beta)} = 0 \quad \text{for each} \quad t\in\mathbb{R}. \tag{18}$$

We set

$$G_k(t) := \int_{k-1}^{k} \theta(t,t-s)h(t-s)ds$$

Using triangle inequality, we obtain that

$$\|\Phi_k(t+s_n)-G_k(t)\|_\alpha \le a_n^k(t)+b_n^k(t)$$

where

$$a_n^k(t) := \int_{k-1}^{k} \left\|A(t+s_n-s)U(t+s_n,t+s_n-s)P(t+s_n-s)\left(\phi(t+s_n-s)-h(t-s)\right)\right\|_\alpha ds$$

and

$$b_n^k(t) := \int_{k-1}^{k} \left\|\left[A(t+s_n-s)U(t+s_n,t+s_n-s)P(t+s_n-s)-\theta(t,t-s)\right]h(t-s)\right\|_\alpha ds$$

Using Eq. (13) and the Hölder's inequality it follows that

$$a_n^k(t) \le C_q(\alpha,\beta,\delta)\|\phi(t+s_n-s)-h(t-s)\|_{\mathbb{S}^p_\gamma(\mathbb{X}_\beta)}$$

Then, by (17), $\lim_{n\to\infty} a_n^k(t)=0$ and by using the Lebesgue dominated convergence theorem and (15), one can get $\lim_{n\to\infty} b_n^k(t)=0$. Thus,

$$\lim_{n\to\infty}\Phi_k(t+s_n) = \int_{k-1}^{k}\theta(t,t-\sigma)h(t-\sigma)d\sigma, \quad \text{for each} \quad t\in\mathbb{R}.$$

Analogously, one can prove that

$$\lim_{n\to\infty}\int_{k-1}^{k}\theta(t-s_n,t-s_n-s)h(t-s_n-s)ds = \Phi_k(t), \quad \text{for each} \quad t\in\mathbb{R}.$$

Therefore, $\Phi_k\in AA(\mathbb{X}_\alpha)$. Applying Theorem (2.7), we deduce that the uniform limit

$$\Phi(\cdot) = \sum_{k=1}^{\infty}\Phi_k(\cdot)\in AA(\mathbb{X}_\alpha).$$

Now, we prove that $\Psi\in PAP_0(\mathbb{X}_\alpha)$. For this; for each $t\in\mathbb{R}$ and $k\in\mathbb{N}$; we set

$$\Psi_k(t) := \int_{k-1}^{k} A(t-s)U(t,t-s)P(t-s)\psi(t-s)ds = \int_{t-k}^{t-k+1} A(s)U(t,s)P(s)\psi(s)ds,$$

By carrying similar arguments as above, we deduce that $\Psi_k(t) \in BC(\mathbb{R}, \mathbb{X}_\alpha)$, $\sum_{k=1}^{\infty} \Psi_k(t)$ is uniformly convergent on $\mathbb{R}$ and

$$\Psi(t) = \sum_{k=1}^{\infty} \Psi_k(t) = \int_{-\infty}^{t} A(s)U(t,s)P(s)\psi(s)ds \in BC(\mathbb{R}, \mathbb{X}_\alpha)$$

To complete the proof, it remains to show that

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \|\Psi(t)\|_\alpha dt = 0$$

In fact, the estimate in Eq. (13) yields

$$\begin{aligned}
\|\Psi_k(t)\|_\alpha &\leq c_2(\alpha)r(\alpha,\beta) \left(\int_{t-k}^{t-k+1} \gamma^{-1/p}(s-t+k) e^{\frac{-\delta}{4}(t-s)} (t-s)^{-\beta} \|\psi(s)\|_\beta \gamma^{1/p}(s-t+k) ds \right) \\
&\leq c_0^{-1/p} c_2(\alpha) r(\alpha,\beta) \sqrt[q]{\frac{1+e^{\frac{q\delta}{4}}}{\frac{q\delta}{4}}} (k-1)^{-\beta} e^{\frac{-\delta}{4}k} \left(\int_{t-k}^{t-k+1} \gamma(s-t+k) \|\psi(s)\|_\beta^p ds \right)^{1/p} \\
&= C_q(\alpha,\beta,\delta) \left(\int_{t-k}^{t-k+1} \gamma(s-t+k) \|\psi(s)\|_\beta^p ds \right)^{1/p}
\end{aligned}$$

Then, one has

$$\begin{aligned}
\frac{1}{2T}\int_{-T}^{T} \|\Psi_k(t)\|_\alpha dt &\leq \frac{C_q(\alpha,\beta,\delta)}{2T} \int_{-T}^{T} \left(\int_{t-k}^{t-k+1} \gamma(s-t+k) \|\psi(s)\|_\beta^p ds \right)^{1/p} dt \\
&\leq \frac{C_q(\alpha,\beta,\delta)}{2T} \int_{-T}^{T} \left(\int_{0}^{1} \gamma(s) \|\psi(s+t-k)\|_\beta^p ds \right)^{\frac{1}{p}} dt
\end{aligned}$$

Since $\psi^b \in PAP_0(L^p((0,1), \gamma(s)ds))$, the above inequality leads to $\Psi_k \in PAP_0(\mathbb{X}_\alpha)$. Then by the following inequality

$$\frac{1}{2T}\int_{-T}^{T} \|\Psi(t)\|_\alpha dt \leq \frac{1}{2T}\int_{-T}^{T} \|\Psi(t) - \sum_{k=1}^{\infty}\Psi_k(t)\|_\alpha dt + \sum_{k=1}^{\infty} \frac{1}{2T}\int_{-T}^{T} \|\Psi_k(t)\|_\alpha dt$$

we deduce that the uniform limit $\Psi(\cdot) = \sum_{k=1}^{\infty} \Psi_k(\cdot) \in PAP_0(\mathbb{X}_\alpha)$, which ends the proof.

Of course, the proof for $(\Gamma_2 u)(\cdot)$ is similar to that for $(\Gamma_1 u)(\cdot)$. However, one makes use of Eq. (14) rather than Eq. (13).

Lemma 3.5. *Under assumptions* (H1)–(H6), *the integral operators* Γ_3 *and* Γ_4 *defined by*

$$(\Gamma_3 u)(t) := \int_{-\infty}^{t} U(t,s)P(s)g(s, C(s)u(s))ds$$

and

$$(\Gamma_4 u)(t) := \int_t^{\infty} U_Q(t,s)Q(s)g(s,C(s)u(s))ds$$

map $PAA(\mathbb{X}_\alpha)$ into itself.

Proof. Let $u \in PAA(\mathbb{X}_\alpha) \subset PAA^p_\gamma(\mathbb{X}_\alpha)$, since $C(\cdot) \in AS^p_\gamma(B(\mathbb{X}_\alpha,\mathbb{X}))$ it follows that $C(\cdot)u(\cdot) \in PAA^p_\gamma(\mathbb{X})$. Using the theorem of composition of generalized Stepanov-like pseudo almost automorphic functions (Theorem (2.24)), we deduce that $G(t) := g(t, C(t)u(t)) \in PAA^p_\gamma(\mathbb{X})$. Now, write $G = \phi + \psi$, where $\phi^b \in AA(L^p((0,1),\gamma(s)ds))$ and $\psi^b \in PAP_0(L^p((0,1),\gamma(s)ds))$. Thus Γ_3 can be rewritten as

$$(\Gamma_3 u)(t) = \Phi(t) + \Psi(t)$$

where

$$\Phi(t) = \int_{-\infty}^{t} U(t,s)P(s)\phi(s)ds \quad \text{and} \quad \Psi(t) = \int_{-\infty}^{t} U(t,s)P(s)\psi(s)ds,$$

Next we show that $\Phi \in AA(\mathbb{X}_\alpha)$. For that; for each $t \in \mathbb{R}$ and $k \in \mathbb{N}$; we set

$$\Phi_k(t) := \int_{k-1}^{k} U(t,t-s)P(t-s)\phi(t-s)ds = \int_{t-k}^{t-k+1} U(t,s)P(s)\phi(s)ds,$$

Let $q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$, where $p > 1$. Using Eq. (6) and the Hölder's inequality, it follows that

$$\begin{aligned}
\|\Phi_k(t)\|_\alpha &\leq \int_{t-k}^{t-k+1} \|U(t,s)P(s)\phi(s)\|_\alpha ds \\
&\leq n(\alpha)\int_{t-k}^{t-k+1} e^{\frac{-\delta}{2}(t-s)}(t-s)^{-\alpha}\|\phi(s)\|ds \\
&= n(\alpha)\int_{t-k}^{t-k+1} \gamma^{-1/p}(s-t+k)e^{\frac{-\delta}{2}(t-s)}(t-s)^{-\alpha}\|\phi(s)\|\gamma^{1/p}(s-t+k)ds \\
&\leq n(\alpha)\Big[\int_{t-k}^{t-k+1} \gamma^{-q/p}(s-t+k)e^{\frac{-q\delta}{2}(t-s)}(t-s)^{-q\alpha}ds\Big]^{1/q} \\
&\quad \times \Big[\int_{t-k}^{t-k+1} \gamma(s-t+k)\|\phi(s)\|^p ds\Big]^{1/p} \\
&\leq c_0^{-1/p} n(\alpha)\Big[\int_{k-1}^{k} e^{\frac{-q\delta}{2}s}s^{-q\alpha}ds\Big]^{1/q}\|\phi\|_{\mathbb{S}^p_\gamma(\mathbb{X})}
\end{aligned} \tag{19}$$

$$\leq c_0^{-1/p} n(\alpha) \sqrt[q]{\frac{1+e^{\frac{q\delta}{2}}}{\frac{q\delta}{2}}} (k-1)^{-\alpha} e^{\frac{-\delta}{2}k} \|\phi\|_{\mathbb{S}^p_\gamma(\mathbb{X})}$$

$$:= C_q(\alpha,\delta)\|\phi\|_{\mathbb{S}^p_\gamma(\mathbb{X})}.$$

Since the series $\sum_{k=1}^{\infty}\left((k-1)^{-\alpha}e^{\frac{-\delta}{2}k}\right)$ is convergent, we deduce from the well-known Weierstrass test that the series $\sum_{k=1}^{\infty}\Phi_k(t)$ is uniformly convergent on $\mathbb{R}$. Furthermore

$$\Phi(t) = \int_{-\infty}^{t} U(t,s)P(s)\phi(s)ds = \sum_{k=1}^{\infty}\Phi_k(t),$$

$\Phi \in C(\mathbb{R},\mathbb{X}_\alpha)$ and

$$\|\Phi(t)\|_\alpha \leq \sum_{k=1}^{\infty}\|\Phi_k(t)\| \leq K_2\|\phi\|_{\mathbb{S}^p_\gamma(\mathbb{X})}.$$

Fix $k\in\mathbb{N}$, let us take a sequence $(s'_n)_n$ of real numbers. Since $\phi \in AS^p_\gamma(\mathbb{X})$ and $U(t,s)y \in bAA(\mathbb{T},\mathbb{X}_\alpha)$ uniformly for $y\in\mathbb{X}$, then for every sequence $(s'_n)_n$ there exists a subsequence $(s_n)_n$ and functions θ, h such that

$$\lim_{n\to\infty} U(t+s_n, s+s_n)P(s+s_n)x = \theta(t,s)x \quad \text{for each} \quad t,s\in\mathbb{R}, x\in\mathbb{X}. \tag{20}$$

$$\lim_{n\to\infty} \theta(t-s_n, s-s_n)x = U(t,s)P(s)x \quad \text{for each} \quad t,s\in\mathbb{R}, x\in\mathbb{X}. \tag{21}$$

$$\lim_{n\to\infty} \|\phi(t+s_n+\cdot) - h(t+\cdot)\|_{S^p_\gamma(\mathbb{X})} = 0, \quad \text{for each} \quad t\in\mathbb{R}. \tag{22}$$

$$\lim_{n\to\infty} \|h(t-s_n+\cdot) - \phi(t+\cdot)\|_{S^p_\gamma(\mathbb{X})} = 0 \quad \text{for each} \quad t\in\mathbb{R}. \tag{23}$$

We set

$$H_k(t) := \int_{k-1}^{k}\theta(t,t-s)h(t-s)ds$$

Using triangle inequality, Eq. (6) and the Hölder's inequality, we obtain that

$$\|\Phi_k(t+s_n) - H_k(t)\|_\alpha \leq c_n^k(t) + d_n^k(t)$$

where

$$\begin{aligned} c_n^k(t) &:= \Big\| \int_{k-1}^{k} U(t+s_n, t+s_n-s)P(t+s_n-s)\left(\phi(t+s_n-s) - h(t-s)\right)ds \Big\|_\alpha \\ &\leq n(\alpha)\left(\int_{k-1}^{k}\gamma^{-1/p}(k-s)e^{\frac{-\delta}{2}s}s^{-\alpha}\|\phi(t+s_n-s)-h(t-s)\|\gamma^{1/p}(k-s)ds\right) \\ &\leq C_q(\alpha,\delta)\|\phi(t+s_n-s)-h(t-s)\|_{\mathbb{S}^p_\gamma(\mathbb{X})} \end{aligned}$$

and

$$
\begin{aligned}
d_n^k(t) &:= \Big\| \int_{k-1}^{k} \left[U(t+s_n, t+s_n-s)P(t+s_n-s) - \theta(t,t-s) \right] h(t-s)ds \Big\|_\alpha \\
&\leq \int_{k-1}^{k} \| U(t+s_n, t+s_n-s)P(t+s_n-s) - \theta(t,t-s)h(t-s)\|_\alpha ds
\end{aligned}
$$

By (22), $\lim_{n\to\infty} c_n^k(t) = 0$. By using the Lebesgue dominated convergence theorem and (20), one can get $\lim_{n\to\infty} c_n^k(t) = 0$. Thus,

$$
\lim_{n\to\infty} \Phi_k(t+s_n) = \int_{k-1}^{k} \theta(t,t-\sigma)h(t-\sigma)d\sigma, \quad \text{for each} \quad t \in \mathbb{R}.
$$

Analogously, one can prove that

$$
\lim_{n\to\infty} \int_{k-1}^{k} \theta(t-s_n, t-s_n-s)h(t-s_n-s)ds = \Phi_k(t), \quad \text{for each} \quad t \in \mathbb{R}.
$$

Therefore, $\Phi_k \in AA(\mathbb{X}_\alpha)$. Applying Theorem (2.7), we deduce that the uniform limit

$$
\Phi(\cdot) = \sum_{k=1}^{\infty} \Phi_k(\cdot) \in AA(\mathbb{X}_\alpha).
$$

Now, we prove that $\Psi \in PAP_0(\mathbb{X}_\alpha)$. For this; for each $t \in \mathbb{R}$ and $k \in \mathbb{N}$; we set

$$
\Psi_k(t) := \int_{k-1}^{k} U(t,t-s)P(t-s)\psi(t-s)ds = \int_{t-k}^{t-k+1} U(t,s)P(s)\psi(s)ds,
$$

By carrying similar arguments as above, we deduce that $\Psi_k(t) \in BC(\mathbb{R}, \mathbb{X}_\alpha)$, $\sum_{k=1}^{\infty} \Psi_k(t)$ is uniformly convergent on $\mathbb{R}$ and

$$
\Psi(t) = \sum_{k=1}^{\infty} \Psi_k(t) = \int_{-\infty}^{t} A(s)U(t,s)P(s)\psi(s)ds \in BC(\mathbb{R}, \mathbb{X}_\alpha)
$$

To complete the proof, it remains to show that

$$
\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \|\Psi(t)\|_\alpha dt = 0
$$

In fact, the estimate in Eq. (6) yields

$$
\begin{aligned}
\|\Psi_k(t)\|_\alpha &\leq n(\alpha) \left(\int_{t-k}^{t-k+1} \gamma^{-1/p}(s-t+k) e^{\frac{-\delta}{2}(t-s)} (t-s)^{-\alpha} \|\psi(s)\| \gamma^{1/p}(s-t+k) ds \right) \\
&\leq c_0^{-1/p} n(\alpha) \sqrt[q]{\frac{1+e^{\frac{q\delta}{2}}}{\frac{q\delta}{2}}} (k-1)^{-\alpha} e^{\frac{-\delta}{2}k} \left(\int_{t-k}^{t-k+1} \gamma(s-t+k) \|\psi(s)\|^p ds \right)^{1/p} \\
&= C_q(\alpha,\delta) \|\psi\|_{\mathbb{S}^p_\gamma(\mathbb{X})}
\end{aligned}
$$

Then, one has

$$\begin{aligned}\frac{1}{2T}\int_{-T}^{T}\|\Psi_k(t)\|_\alpha dt &\leq \frac{C_q(\alpha,\delta)}{2T}\int_{-T}^{T}\left(\int_{t-k}^{t-k+1}\gamma(s-t+k)\|\psi(s)\|^p ds\right)^{\frac{1}{p}} dt\\ &\leq \frac{C_q(\alpha,\delta)}{2T}\int_{-T}^{T}\left(\int_{0}^{1}\gamma(s)\|\psi(s+t-k)\|^p ds\right)^{\frac{1}{p}} dt\end{aligned}$$

Since $\psi^b \in PAP_0(L^p((0,1),\gamma(s)ds))$, the above inequality leads to $\Psi_k \in PAP_0(\mathbb{X}_\alpha)$. Then by the following inequality

$$\frac{1}{2T}\int_{-T}^{T}\|\Psi(t)\|_\alpha dt \leq \frac{1}{2T}\int_{-T}^{T}\|\Psi(t)-\sum_{k=1}^{\infty}\Psi_k(t)\|_\alpha dt+\sum_{k=1}^{\infty}\frac{1}{2T}\int_{-T}^{T}\|\Psi_k(t)\|_\alpha dt$$

we deduce that the uniform limit $\Psi(\cdot)=\sum_{k=1}^{\infty}\Psi_k(\cdot)\in PAP_0(\mathbb{X}_\alpha)$, which ends the proof.

Of course, the proof for $(\Gamma_4 u)(\cdot)$ is similar to that for $(\Gamma_3 u)(\cdot)$. However, one makes use of Eq. (7) rather than Eq. (6).

Theorem 3.6. *Under the assumptions* (H1)–(H6)*, the evolution equation (1) has a unique pseudo-almost automorphic mild solution whenever L is small enough.*

Proof. Consider the nonlinear operator Π defined on $PAA(\mathbb{X}_\alpha)$ by

$$\begin{aligned}&\Pi u(t) = -f(t,B(t)u(t)) - \int_{-\infty}^{t} A(s)U(t,s)P(s)f(s,B(s)u(s))ds\\ &+\int_{t}^{\infty} A(s)U_Q(t,s)Q(s)f(s,B(s)u(s))ds+\int_{-\infty}^{t}U(t,s)P(s)g(s,C(s)u(s))ds\\ &-\int_{t}^{\infty}U_Q(t,s)Q(s)g(s,C(s)u(s))ds\end{aligned}$$

for each $t\in\mathbb{R}$. As we have previously seen, for every $u\in PAA(\mathbb{X}_\alpha)$, $f(\cdot,Bu(\cdot))\in PAA(\mathbb{X}_\beta)\subset PAA(\mathbb{X}_\alpha)$. In view of Lemmas (3.4) and (3.5), it follows that Π maps $PAA(\mathbb{X}_\alpha)$ into its self. To complete the proof one has to show that Π has a unique fixed point.

Let $u, v \in PAA(\mathbb{X}_\alpha)$. For Γ_1 and Γ_2, we have the following approximations

$$\begin{aligned}
\|(\Gamma_1 u)(t) - (\Gamma_1 v)(t)\|_\alpha &\leq \int_{-\infty}^{t} \|A(s)U(t,s)P(s)\left[f(s,B(s)u(s)) - f(s,B(s)v(s))\right]\|_\alpha ds \\
&\leq c_2(\alpha)c_4 r(\alpha,\beta)\int_{-\infty}^{t}(t-s)^{-\alpha}e^{\frac{-\delta}{4}(t-s)}\|f(s,B(s)u(s)) - f(s,B(s)v(s))\|_\beta ds \\
&\leq Lc_2(\alpha)c_4 r(\alpha,\beta)\int_{-\infty}^{t}(t-s)^{-\alpha}e^{\frac{-\delta}{4}(t-s)}\|B(s)u(s) - B(s)v(s)\| ds \\
&\leq Lc_2(\alpha)c_4 c_5 r(\alpha,\beta)\int_{-\infty}^{t}(t-s)^{-\alpha}e^{\frac{-\delta}{4}(t-s)}\|u(s) - v(s)\|_\alpha ds \\
&\leq Lc_2(\alpha)c_4 c_5 r(\alpha,\beta)(4\delta^{-1})^{1-\alpha}\Gamma(1-\alpha)\|u-v\|_{\alpha,\infty}.
\end{aligned}$$

where Γ is the classical Γ function.

$$\begin{aligned}
\|(\Gamma_2 u)(t) - (\Gamma_2 v)(t)\|_\alpha &\leq \int_{t}^{\infty} \|A(s)U_Q(t,s)Q(s)\left[f(s,B(s)u(s)) - f(s,B(s)v(s))\right]\|_\alpha ds \\
&\leq c_2(\alpha)c_4 d(\beta)\int_{t}^{\infty} e^{-\delta(s-t)}\|f(s,B(s)u(s)) - f(s,B(s)v(s))\|_\beta ds \\
&\leq Lc_2(\alpha)c_4 d(\beta)\int_{t}^{\infty} e^{-\delta(s-t)}\|B(s)u(s) - B(s)v(s)\| ds \\
&\leq Lc_2(\alpha)c_4 c_5 d(\beta)\int_{t}^{\infty} e^{-\delta(s-t)}\|u(s) - v(s)\|_\alpha ds \\
&\leq Lc_2(\alpha)c_4 c_5 d(\beta)\delta^{-1}\|u-v\|_{\alpha,\infty}.
\end{aligned}$$

Similarly, For Γ_3 and Γ_4, we have the following approximations

$$\begin{aligned}
\|(\Gamma_3 u)(t) - (\Gamma_3 v)(t)\|_\alpha &\leq \int_{-\infty}^{t} \|U(t,s)P(s)\left[g(s,C(s)u(s)) - f(s,C(s)v(s))\right]\|_\alpha ds \\
&\leq n(\alpha)\int_{-\infty}^{t}(t-s)^{-\alpha}e^{\frac{-\delta}{2}(t-s)}\|g(s,C(s)u(s)) - g(s,C(s)v(s))\|_\beta ds \\
&\leq Ln(\alpha)\int_{-\infty}^{t}(t-s)^{-\alpha}e^{\frac{-\delta}{4}(t-s)}\|C(s)u(s) - C(s)v(s)\| ds \\
&\leq Ln(\alpha)c_5\int_{-\infty}^{t}(t-s)^{-\alpha}e^{\frac{-\delta}{2}(t-s)}\|u(s) - v(s)\|_\alpha ds \\
&\leq Lc(\alpha)c_5(2\delta^{-1})^{1-\alpha}\Gamma(1-\alpha)\|u-v\|_{\alpha,\infty}.
\end{aligned}$$

and

$$\begin{aligned}
\|(\Gamma_4 u)(t)-(\Gamma_4 v)(t)\|_\alpha &\leq \int_t^\infty \|U_Q(t,s)Q(s)\left[g(s,C(s)u(s))-g(s,C(s)v(s))\right]\|_\alpha ds \\
&\leq m(\alpha)\int_t^\infty e^{-\delta(s-t)}\|g(s,C(s)u(s))-g(s,C(s)v(s))\|ds \\
&\leq Lm(\alpha)\int_t^\infty e^{-\delta(s-t)}\|C(s)u(s)-C(s)v(s)\|ds \\
&\leq Lm(\alpha)c_5\int_t^\infty e^{-\delta(s-t)}\|u(s)-v(s)\|_\alpha ds \\
&\leq Lm(\alpha)c_5\delta^{-1}\|u-v\|_{\alpha,\infty}.
\end{aligned}$$

Consequently,

$$\|\Pi u-\Pi v\|_{\alpha,\infty} \leq L\Theta\|u-v\|_{\alpha,\infty}$$

where

$$\begin{aligned}
\Theta := {} & c_5(c_2(\alpha)+c_2(\alpha)c_4r(\alpha,\beta)(4\delta^{-1})^{1-\alpha}\Gamma(1-\alpha) \\
&+c_2(\alpha)c_4d(\beta)\delta^{-1}+c(\alpha)(2\delta^{-1})^{1-\alpha}\Gamma(1-\alpha)+m(\alpha)\delta^{-1}).
\end{aligned}$$

By taking L small enough, that is, $L<\Theta^{-1}$, the operator Π becomes a contraction on $PAA(\mathbb{X}_\alpha)$ and hence has a unique fixed point in $PAA(\mathbb{X}_\alpha)$, which obviously is the unique pseudo-almost automorphic mild solution to (1).

4. Application

Fix $\gamma\in\mathbb{U}$ and $p>1$. Let $\Omega\subset\mathbb{R}^N$ $(N\geq 1)$ be an open bounded subset with C^2 boundary $\Gamma=\partial\Omega$ and let $\mathbb{X}=L^2(\Omega)$ equipped with its natural topology $\|\cdot\|_2$.

Let $0<\alpha<\beta<1$ with $2\beta>\alpha+1$. In this section we study the existence and uniqueness of pseudo almost automorphic solution to perturbed nonautonomous equation

(24)
$$\frac{\partial}{\partial t}\Big[u(t,x)+F\left(t,b(t,x)u(t,x)\right)\Big]=a(t,x)\Delta^2u(t,x)+G\left(t,c(t,x)u(t,x)\right), \quad \text{in } \mathbb{R}\times\Omega$$

(25)
$$u=\Delta u=0, \quad \text{on } \mathbb{R}\times\Gamma$$

where $F : \mathbb{R} \times L^2(\Omega) \to \mathbb{X}_\beta$, $G : \mathbb{R} \times L^2(\Omega) \to L^2(\Omega)$ are S^p_γ-pseudo almost automorphic with

$$\mathbb{X}_\alpha = (L^2(\Omega), H_0^2(\Omega) \cap H^4(\Omega))_{\alpha,\infty}$$

The coefficients $a, b, c : \mathbb{R} \times \Omega \to \mathbb{R}$ are almost automorphic, $x \to a(t, x)$ is differentiable for all $t \in \mathbb{R}$, $t \to a(t, x)$ is ω-periodic ($\omega > 0$) in the sense that

$$a(t + \omega, x) = a(t, x), \quad \text{for all } t \in \mathbb{R} \quad \text{and } x \in \Omega$$

In addition to the above, we add the following assumptions:

(H7) $\inf_{t\in\mathbb{R},x\in\Omega} a(t, x) = m_0 > 0$, and

(H8) there exists $d > 0$ and $0 < \mu \leq 1$ such that $|a(t, x) - a(s, x)| \leq d|s - t|^\mu$ for all $t, s \in \mathbb{R}$ uniformly in $x \in \Omega$.

Define the linear operators $A(t), B(t)$ and $C(t)$ by

$$A(t)u = a(t, x)\Delta^2 u \quad \text{for all } u \in D(A(t)) = \mathbb{D} = H_0^2(\Omega) \cap H^4(\Omega),$$

$$B(t)u = b(t, x)u \quad \text{for all } L^2(\Omega),$$

$$C(t)u = c(t, x)u \quad \text{for all } L^2(\Omega).$$

Under previous assumptions, it is clear that the operators $A(t)$ defined above are invertible and satisfy Acquistapace-Terreni conditions. Clearly, the system

$$u'(t) = A(t)u(t), \quad t \geq s,$$
$$u(s) = \varphi \in L^2(\Omega),$$

has an associated evolution family $(U(t, s))_{t\geq s}$ on $L^2(\Omega)$, which satisfies: there exist $\omega_0 > 0$ and $M \geq 1$ such that

$$\|U(t, s)\|_{B(L^2(\Omega))} \leq Me^{-\omega_0(t-s)} \quad \text{for every } t \geq s.$$

Moreover, since $A(t + \omega) = A(t)$ for all $t \in \mathbb{R}$, it follows that

$$U(t+\omega, s+\omega) = U(t, s), \quad \Gamma(t+\omega, s+\omega) = \Gamma(t, s), \quad A(s+\omega)\Gamma(t+\omega, s+\omega) = A(s)\Gamma(t, s)$$

for all $t, s \in \mathbb{R}$ with $t \geq s$. Therefore, $(t,s) \mapsto \Gamma(t,s)w$ belongs to $bAA(\mathbb{T}, L^2(\Omega))$ uniformly in $w \in L^2(\Omega)$ and $(t,s) \mapsto A(s)\Gamma(t,s)w$ belongs to $bAA(\mathbb{T}, \mathbb{D})$ uniformly in $w \in \mathbb{D}$. It is also clear that (H2) holds.

We need the following additional assumption:

(H9) The functions $F \in PAA(\mathbb{R} \times L^2(\Omega), H_0^1(\Omega) \cap H^2(\Omega))$ and $G \in PAA_\gamma^p(\mathbb{R} \times L^2(\Omega), L^2(\Omega)) \cap C(\mathbb{R} \times L^2(\Omega), L^2(\Omega))$. Moreover, there exits $L > 0$ such that

$$\|F(t,u) - F(t,v)\|_\beta \leq L\|u - v\|_2$$

for all $u, v \in L^2(\Omega)$ and $t \in \mathbb{R}$, and

$$\|G(t,u) - G(t,v)\|_2 \leq L\|u - v\|_2$$

for all $u, v \in L^2(\Omega)$ and $t \in \mathbb{R}$.

Theorem 4.1. Under assumptions (H7)–(H9), the equation (24)-(25), has a unique solution $u \in PAA((L^2(\Omega), H_0^2(\Omega) \cap H^4(\Omega))_{\alpha,\infty})$ whenever L is small enough.

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