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LAGRANGE COMPUTATIONAL APPROACH FOR FRACTIONAL-ORDER DELAY SYSTEMS OF VOLTERRA INTEGRAL AND INTEGRO-DIFFERENTIAL EQUATIONS

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Abstract. In this paper, numerical solution depending on Lagrange cardinal operational collocation optimization method (LCOCOM) is introduced. The LCOCOM is developed to obtain the solutions of Volterra delay integral and integro-differential equations, as well fractional-order delay systems of Volterra integral and integro-differential equations. We present numerical results and comparisons of existing treatments to demonstrate the efficiency and applicability of the proposed method. The proposed method gives more accurate solution with minimum number of approximation nodes in linear as well as nonlinear cases.

Keywords: Lagrange cardinal function; Volterra delay integral and integro-differential equations; fractional-order delay system of Volterra integral and integro-differential equations.

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1. Introduction

The solutions of systems of Volterra integral and integro-differential equations have a significant role in science and engineering. There are numerical methods for solving systems of linear and nonlinear integro-differential equations such as Bernoulli matrix method [1] and Legendre wavelets [2]. Loh and Phang [3] introduced Genocchi polynomials to solve system of Volterra

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integro-differential equations numerically. Shidfar and Molabahrami [4] introduced homotopy analysis method (HAM) for solving a system of linear and nonlinear Volterra and Fredholm integral equations. Heydari [5] utilized Chebyshev cardinal wavelets to solve nonlinear fractional quadratic integral equations (QIEs) with variable-order (V-O). Wang et al. [6] presented Bernoulli wavelets and operational matrix to solve coupled systems of nonlinear fractional order integro-differential equations. Hesameddini and Riahi [7] utilized Bernoulli polynomials and the Galerkin method for solving system of Volterra-Fredholm integro-differential equations. El-Sayed et al. [8] presented the solutions of a delay functional integral equation of Volterra-Stieltjes type. Sahu and Ray [9] have developed Bernoulli wavelet method to solve nonlinear fuzzy Hammerstein-Volterra delay integral equations. Rahimkhani and Ordokhani [10] introduced another mathematical method dependent on the Chelyshkov wavelets and derived the integral and derivative operators of these wavelets. Abdi et al. [11] presented method for solving delay Volterra integro-differential equations depended on two utilizations of linear barycentric rational interpolation namely barycentric rational quadrature and barycentric rational finite differences. Nemati et al. [12] solved a class of nonlinear delay-type fractional integro-differential equations by using Legendre wavelets combined with the Gauss-Jacobi quadrature technique. Zheng and Chen [13] used Chebyshev spectral-collocation method for solving Volterra integral equation with two kinds of delay which are proportional delay and nonproportional delay. Ghomanjani et al. [14] introduced Bezier curves method for solving Volterra delay-integro-differential equations. Saeed et al. [15] developed the Chebyshev wavelet method for solving the fractional delay differential and integro-differential equations. Collocation method is known as a highly accurate numerical procedure for solving integro-differential equations. In the last centuries, the collocation method was used to find approximate solutions of Volterra and Fredholm integral equations and systems of integro-differential equations. [17, 18, 19]. Wang et al. [20] utilized Lagrange collocation method to solve the Volterra-Fredholm integral equations. Hussien [21] used collocation method with generalized Laguerre polynomials basis for solving two common of delay fractional-order differential equations.

In this paper, we consider two kinds of fractional-order delay system of Volterra integral equations, for $\tau > 0$:

$$\begin{cases} u_\ell(t) = g_\ell(t) + \dfrac{1}{\Gamma(\mu)} \displaystyle\int_{\theta(t)}^{t} (t-s)^{\mu-1} K\big(s, u_1(s), u_2(s), ..., u_m(s)\big) ds, & t \in [0, X], \\ u_\ell(t) = \phi_\ell \quad \ell = 1,2,...,m, \quad m \in \mathbb{N}, & t \in [-\tau, 0], \end{cases} \tag{1}$$

$$\begin{cases} u_\ell(t) = z_\ell(t) + \dfrac{1}{\Gamma(\mu)} \displaystyle\int_{0}^{t} (t-s)^{\mu-1} K\big(s, u_1(s-\tau_1), u_2(s-\tau_2), ..., u_m(s-\tau_m)\big) ds, & t \in [0, X], \\ u_\ell(t) = \eta_\ell \quad \ell = 1,2,...,m, \quad m \in \mathbb{N}, & t \in [-\tau_{max}, 0], \end{cases} \tag{2}$$

with $\tau_{max} = max_{i=1}(\tau_i)$, $i = 1,2,...,m$.

The third problem is the fractional-order delay system of Volterra integro-differential equations:

$$\begin{cases} D^\mu u_\ell(t) = h_\ell(t) + \displaystyle\int_{\theta(t)}^{t} K\big(s, u_1(s), u_2(s), ..., u_m(s)\big) ds, & t \in [0, X], \\ u_\ell^{(r)}(0) = \xi_{r\ell} \quad r = 0,1,2,...(n-1), \quad \ell = 1,2,...,m, \quad m \in \mathbb{N}, & t \in [-\tau, 0], \end{cases} \tag{3}$$

where u_ℓ are unknown functions, g_ℓ, z_ℓ, h_ℓ are known functions, $\theta(t)$ and t are integration limits, and D^μ is the Caputo's fractional derivative order of μ.

The main purpose of this paper is to present numerical approximation namely LCOCOM in order to solve Volterra delay integral and integro-differential equations in addition to fractional-order delay systems of Volterra integral and integro-differential equations.

This paper is organized as follows. In Section 2, we introduce some definitions and properties of Lagrange cardinal functions and the derivatives and integrals of fractional-order. In Section 3, we construct the fractional-order derivative and integral operators of the Lagrange cardinal functions. In Section 4, we present LCOCOM. In Section 5, we show some numerical examples. Finally, our conclusion is presented in Section 6.

2. PRELIMINARIES

In this section, we present some definitions of the Lagrange cardinal function and Caputo's derivative and Riemann– Liouville's integral which are utilized in this paper:

Definition 2.1. [20]: The Lagrange cardinal functions for points $x_0, x_1, ..., x_n$ is defined as:

$$L_j(x)=\prod_{\substack{i=0\\i\neq j}}^{n}\frac{(x-x_i)}{(x_j-x_i)}=\frac{p_j(x)}{p_j(x_j)},$$

where

$$p_j(x)=\prod_{\substack{i=0\\i\neq n}}^{n}(x-x_i)=x^n-(\sum_{\substack{i=0\\i\neq n}}^{n}x_i)x^{n-1}+(\sum_{\substack{i=0\\i\neq n}}^{n}\sum_{\substack{j=0\\j\neq n}}^{n}x_ix_j)x^{n-2}$$
$$-(\sum_{\substack{i=0\\i\neq n}}^{n}\sum_{\substack{j=0\\j\neq n}}^{n}\sum_{\substack{k=0\\k\neq n}}^{n}x_ix_jx_k)x^{n-3}+...+(-1)^n(x_1x_2...x_n).$$

The polynomials $\{L_j(x)\}_{j=0}^{n}$ have the property that:

$$L_j(x_i)=\begin{cases}0 & \text{if } i\neq j,\\ 1 & \text{if } i=j.\end{cases}$$

Now any function $y(x)$ on $[0,X]$ can be approximated as:

$$y(x)\approx y_n(x)=\sum_{j=0}^{n}y(x_j)L_j(x). \tag{4}$$

Definition 2.2. [22]: The Riemann-Liouville fractional integral I^{μ} of order μ, is defined as:

$$I^{\mu}u(t)=\begin{cases}\frac{1}{\Gamma(\mu)}\int_0^t(t-s)^{\mu-1}u(s)ds, & t>0,\quad \mu>0\\ u(t), & \mu=0.\end{cases}$$

For the Riemann-Liouville fractional integral, we have the following properties [23, 22]:

$$I^{\mu}\Big(\lambda_1u_1(t)+\lambda_2u_2(t)+...+\lambda_nu_n(t)\Big)=\Big(\lambda_1I^{\mu}u_1(t)+\lambda_2I^{\mu}u_2(t)+...+\lambda_nI^{\mu}u_n(t)\Big),$$

where $\lambda_1,\lambda_2,....,\lambda_n$ are constants.

$$I^{\mu}t^{w}=\frac{\Gamma(w+1)}{\Gamma(w+1+\mu)}t^{w+\mu},\quad w>-1.$$

Definition 2.3. [23]: Caputo's fractional derivative of order μ is defined as:

$$D^{\mu}u(t)=\frac{1}{\Gamma(n-\mu)}\int_0^t(t-s)^{n-\mu-1}u^{(n)}(s)ds,\quad t>0,$$

where $n-1<\mu\leq n$, $n\in\mathbb{N}$.

For the Caputo derivative, we have [23]:

$$D^{\mu}I^{\mu}u(t)=u(t),$$

where I^{μ} is the Riemann-Liouville fractional integral, that defined in the previous definition.

$$I^{\mu}D^{\mu}u(t)=u(t)-\sum_{k=0}^{n-1}u^{(k)}(0^{+})\frac{t^{k}}{k!};\quad t>0.$$

$$D^{\mu}t^{w}=\begin{cases}\frac{\Gamma(w+1)}{\Gamma(w+1-\mu)}t^{w-\mu}, & \text{for } w\in\mathbb{N}_0,\quad w\geq\lceil\mu\rceil,\\ 0, & \text{for } w\in\mathbb{N}_0,\quad w<\lceil\mu\rceil.\end{cases}$$

3. Lagrange Cardinal Function Operators

In this section, we derive the derivative and integral operators of a function $y(x)$ represented in Eq. (4) which are used to find approximate solutions to the problems (1), (2) and (3).

3.1. Fractional-order derivatives of Lagrange cardinal function:

In this subsection, we are going to construct the derivative of a function $y(x)$ represented in Eq. (4).

Lemma 3.1.1. The fractional-order derivative of function $y(x)$ represented in Eq.(4) can be approximated as:

$$D^{\mu}y(x)=\sum_{j=0}^{n}\frac{y_j}{p_j(x_j)}\left(B_0+\sum_{i=1}^{n}B_i\frac{\Gamma(i+1)}{\Gamma(i+1-\mu)}x^{i-\mu}\right).\tag{5}$$

Proof. By differentiation of Eq. (4), we get

$$D^{\mu}y(x)=\sum_{j=0}^{n}y(x_j)D^{\mu}L_j(x),\tag{6}$$

where

$$D^{\mu}L_j(x)=\frac{D^{\mu}p_j(x)}{p_j(x_j)}.\tag{7}$$

Therefore, we get

(8)

$$D^{\mu}p_j(x)=\frac{\Gamma(n+1)}{\Gamma(n+1-\mu)}x^{n-\mu}-\frac{\Gamma(n)}{\Gamma(n-\mu)}(\sum_{\substack{i=0\\i\neq n}}^{n}x_i)x^{n-1-\mu}+\frac{\Gamma(n-1)}{\Gamma(n-1-\mu)}(\sum_{\substack{i=0\\i\neq n}}^{n}\sum_{\substack{j=0\\j\neq n}}^{n}x_ix_j)x^{n-2-\mu}$$

$$-\frac{\Gamma(n-2)}{\Gamma(n-2-\mu)}(\sum_{\substack{i=0\\i\neq n}}^{n}\sum_{\substack{j=0\\j\neq n}}^{n}\sum_{\substack{k=0\\k\neq n}}^{n}x_ix_jx_k)x^{n-3-\mu}+...+0=B_0+\sum_{i=1}^{n}B_i\frac{\Gamma(i+1)}{\Gamma(i+1-\mu)}x^{i-\mu},$$

where

$$B_n=1,B_{n-1}=-(\sum_{\substack{i=0\\i\neq n}}^{n}x_i),B_{n-2}=(\sum_{\substack{i=0\\i\neq n}}^{n}\sum_{\substack{j=0\\j\neq n}}^{n}x_ix_j),...,B_0=0.$$

Substituting from Eq.(8) into (7), we get

$$D^{\mu}L_j(x)=\frac{1}{p_j(x_j)}\left(B_0+\sum_{i=1}^{n}B_i\frac{\Gamma(i+1)}{\Gamma(i+1-\mu)}x^{i-\mu}\right). \tag{9}$$

Thus loading (9) in (6), yields to the conclusion (5).

3.2. Fractional-order integrals of Lagrange cardinal functions:

Here, we construct the fractional-order integral of a function $y(x)$ represented in Eq. (4).

Lemma 3.2.2. The fractional integral of function $y(x)$ represented in Eq. (4)

$$I^{\mu}y(x)=\sum_{j=0}^{n}\sum_{i=0}^{n}\frac{y_j}{p_j(x_j)}\frac{C_i\Gamma(i+1)}{\Gamma(i+1+\mu)}x^{i+\mu}. \tag{10}$$

Proof. By integration of Eq.(4), we get

$$I^{\mu}y(x)=\sum_{j=0}^{n}y(x_j)I^{\mu}L_j(x), \tag{11}$$

where

$$I^{\mu}L_j(x)=\frac{I^{\mu}p_j}{p_j(x_j)}. \tag{12}$$

Therefore, we get

(13)

$$I^{\mu}p_j(x)=\frac{\Gamma(n+1)}{\Gamma(n+1+\mu)}x^{n+\mu}-\frac{\Gamma(n)}{\Gamma(n+\mu)}(\sum_{\substack{i=0\\i\neq n}}^{n}x_i)x^{n-1+\mu}+\frac{\Gamma(n-1)}{\Gamma(n-1+\mu)}(\sum_{\substack{i=0\\i\neq n}}^{n}\sum_{\substack{j=0\\j\neq n}}^{n}x_ix_j)x^{n-2+\mu}$$

$$-\frac{\Gamma(n-3)}{\Gamma(n-2+\mu)}(\sum_{\substack{i=0\\i\neq n}}^{n}\sum_{\substack{j=0\\j\neq n}}^{n}\sum_{\substack{k=0\\k\neq n}}^{n}x_ix_jx_k)x^{n-2+\mu}+...+(-1)^n\frac{x^{\mu}}{\Gamma(1+\mu)}(x_1x_2...x_n)=\sum_{i=0}^{n}C_i\frac{\Gamma(i+1)}{\Gamma(i+1+\mu)}x^{i+\mu},$$

where

$$C_n = 1, C_{n-1} = -(\sum_{\substack{i=0\\ i\neq n}}^{n} x_i), C_{n-2} = (\sum_{\substack{i=0\\ i\neq n}}^{n}\sum_{\substack{j=0\\ j\neq n}}^{n} x_i x_j), \ldots, C_0 = (-1)^n (x_1 x_2 \ldots x_n).$$

Substituting from (13) into (12), we have

$$I^{\mu} L_j(x) = \frac{1}{p_j(x_j)} \sum_{i=0}^{n} C_i \frac{\Gamma(i+1)}{\Gamma(i+1+\mu)} x^{i+\mu} \tag{14}$$

Thus loading (14) in (11), yields to the conclusion (10).

4. Description of Proposed Method

In this section, we are going to construct LCOCOM for solving Volterra delay integral and integro-differential equations and fractional-order delay systems of Volterra integral and integro-differential equations given in Eqs. (1), (2) and (3), respectively.

4.1. Systems of fractional Volterra delay integal equations:

Here, the fractional-order delay system of Volterra integral equations given by Eq. (1) becomes:

$$\begin{cases} u_{\ell}(t) = G_{\ell}(t) + \dfrac{1}{\Gamma(\mu)} \displaystyle\int_0^t (t-s)^{\mu-1} K\big(s, u_1(s), u_2(s), \ldots, u_m(s)\big) ds, & t \in [0, X], \\ u_{\ell}(t) = \phi_{\ell} \quad \ell = 1, 2, \ldots, m, \quad m \in \mathbb{N}, & t \in [-\tau, 0], \end{cases} \tag{15}$$

where $G_{\ell}(t) = g_{\ell}(t) + \dfrac{1}{\Gamma(\mu)} \displaystyle\int_{\theta(t)}^{0} (t-s)^{\mu-1} K\big(s, u_1(s), u_2(s), \ldots, u_m(s)\big) ds.$
We can write Eqs. (15) and (2) as:

$$u_{\ell}(t) = G_{\ell}(t) + I^{\mu} K\big(t, u_1(t), u_2(t), \ldots, u_m(t)\big). \tag{16}$$

$$u_{\ell}(t) = z_{\ell}(t) + I^{\mu} K\big(t, u_1(t-\tau_1), u_2(t-\tau_2), \ldots, u_m(t-\tau_m)\big). \tag{17}$$

Applying **Lemma 3.2.2** to approximate the solutions of problems (16) and (17), we get:

$$u_{\ell}(t_i) = G_{\ell}(t_i) + \sum_{j=0}^{n} K\big(t_j, u_1(t_j), u_2(t_j), \ldots, u_m(t_j)\big) I^{\mu} L_j(t_i). \tag{18}$$

$$u_{\ell}(t_0) = \phi_{\ell}. \tag{19}$$

$$u_{\ell}(t_i) = z_{\ell}(t_i) + \sum_{j=0}^{n} K\big(t_j, u_1(t_j-\tau_1), u_2(t_j-\tau_2), \ldots, u_m(t_j-\tau_m)\big) I^{\mu} L_j(t_i). \tag{20}$$

$$u_\ell(t_0) = \eta_\ell. \tag{21}$$

By using a collocation method with least squares approximation of Eqs. (18)-(21) lead to :

$$R_1(t_i){=}u_\ell(t_i) - G_\ell(t_i) - \sum_{j=0}^{n} K\big(t_j, u_1(t_j), u_2(t_j), ..., u_m(t_j)\big) I^\mu L_j(t_i). \tag{22}$$

$$R_1(t_0) = \Big(u_\ell(t_0) - \phi_\ell\Big) \tag{23}$$

$$R_2(t_i) = u_\ell(t_i) - z_\ell(t_i) - \sum_{j=0}^{n} K\big(t_j, u_1(t_j - \tau_1), u_2(t_j - \tau_2), ..., u_m(t_j - \tau_m\big) I^\mu L_j(t_i) \tag{24}$$

$$R_2(t_0) = \Big(u_\ell(t_0) - \eta_\ell\Big). \tag{25}$$

So we can construct an unconstrained optimization problem with objective function as:

$$\begin{aligned}\bar{R_1} = \big(R_1(t_0)\big)^2 + \sum_{i=1}^{n} (R_1(t_i))^2 = \sum_{i=1}^{n} \Big(u_\ell(t_i) - G_\ell(t_i) - \\ \sum_{j=0}^{n} K\big(t_j, u_1(t_j), u_2(t_j), ..., u_m(t_j)\big) I^\mu L_j(t_i)\Big)^2 + \Big(u_\ell(t_0) - \phi_\ell\Big)^2.\end{aligned} \tag{26}$$

$$\begin{aligned}\bar{R_2} = \big(R_2(t_0)\big)^2 + \sum_{i=1}^{n} (R_2(t_i))^2 = \sum_{i=1}^{n} \Big(u_\ell(t_i) - z_\ell(t_i) - \\ \sum_{j=0}^{n} K\big(t_j, u_1(t_j - \tau_1), u_2(t_j - \tau_2), ..., u_m(t_j - \tau_m\big) I^\mu L_j(t_i)\Big)^2 + \Big(u_\ell(t_0) - \eta_\ell\Big)^2.\end{aligned} \tag{27}$$

The unknown values $u_\ell = \{u_1, u_2, ...u_m\}$ can be obtained by applying Leap-Frog algorithm (LFOPC) method [24] for the unconstrained optimization problem with the cost function (26) and (27).

4.2. System of fractional Volterra delay integro-differential equations:

The fractional-order delay system of Volterra integro-differential equations given Eq.(3) becomes:

$$\begin{cases} D^\mu u_\ell(t) = H_\ell(t) + \int_0^t K\big(s, u_1(s), u_2(s), ..., u_m(s)\big) ds, & t \in [0, X], \\ u_\ell^{(r)}(0) = \xi_{r\ell} \quad r = 0, 1, 2, ...(n-1), \quad \ell = 1, 2, ..., m, \quad m \in \mathbb{N}, & t \in [-\tau, 0]. \end{cases} \tag{28}$$

where $H_\ell(t) = h_\ell(t) + \int_{\theta(t)}^{0} K\big(s, u_1(s), u_2(s), \ldots, u_m(s)\big)ds$.
we can write Eq. (28) as:

$$D^\mu u_\ell(t) = H_\ell(t) + I\Big(K\big(t, u_1(t), u_2(t), \ldots, u_m(t)\big)\Big), \tag{29}$$

where

$$I\Big(K\big(t, u_1(t), u_2(t), \ldots, u_m(t)\big)\Big) = \int_0^t K\big(s, u_1(s), u_2(s), \ldots, u_m(s)\big)ds. \tag{30}$$

Applying **Lemma 3.2.1** and **Lemma 3.2.2** to approximate the solutions of problem (29), we obtain:

$$\sum_{j=0}^{n} u_\ell(t_j) D^\mu L_j(t_i) = H_\ell(t_i) + \sum_{j=0}^{n} K\big(t_j, u_1(t_j), u_2(t_j), \ldots, u_m(t_j)\big) I\big(L_j(t_i)\big) \tag{31}$$

$$\sum_{r=0}^{n-1} u_\ell(0) D^{(r)} L_j(t_0) = \xi_{r\ell}. \tag{32}$$

By using a collocation method with least squares approximation of Eqs. (31) and (32) lead to:

$$R_3(t_i) = \sum_{j=0}^{n} u_\ell(t_j) D^\mu L_j(t_i) - H_\ell(t_i) - \sum_{j=0}^{n} K\big(t_j, u_1(t_j), u_2(t_j), \ldots, u_m(t_j)\big) I\big(L_j(t_i)\big) \tag{33}$$

$$R_3(t_0) = \left(\sum_{r=0}^{n-1} u_\ell(0) D^{(r)} L_j(t_0) - \xi_{r\ell}\right). \tag{34}$$

So we can construct an unconstrained optimization problem with objective function as:

$$\begin{aligned} \bar{R_3} = \big(R_3(t_0)\big)^2 + \sum_{i=1}^{n} (R_3(t_i))^2 = \sum_{i=1}^{n} \Bigg(\sum_{j=0}^{n} u_\ell(t_j) D^\mu L_j(t_i) - H_\ell(t_i) - \\ \sum_{j=0}^{n} K\big(t_j, u_1(t_j), u_2(t_j), \ldots, u_m(t_j)\big) I\big(L_j(t_i)\big)\Bigg)^2 + \left(\sum_{r=0}^{n-1} u_\ell(0) D^{(r)} L_j(t_0) - \xi_{r\ell}\right)^2. \end{aligned} \tag{35}$$

The unknown values $u_\ell = \{u_1, u_2, \ldots u_m\}$ can be obtained by applying LFOPC to the unconstrained optimization problem with the cost function (35).

5. Numerical Experiments and Discussion

Here, we present some examples to show the applicability and efficiency of the proposed method. The examples considered here are taken from recent references.

Example 1: Consider the Volterra– Hammerstein delay integral equation [9]:

$$\begin{cases} y(t) = e^{t-\tau} + \int_{t-\tau}^{t} y(s)ds, & t \in [0, X], \\ y(t) = \phi(t), & t \in [-\tau, 0]. \end{cases} \tag{36}$$

The equation (36) can be written as:

$$\begin{cases} y(t) = 1 + \int_{0}^{t} y(s)ds, & t \in [0, X], \\ y(t) = \phi(t), & t \in [-\tau, 0]. \end{cases} \tag{37}$$

where $\tau = 0.5$, $X = 0.5$, $\phi(t) = e^t$. The exact solution is $y(t) = e^t$, $t \in [-0.5, 0.5]$. The above problem has been solved by LCOCOM and then the absolute error and computational time (in seconds) have been compared with that of obtained by Bernoulli wavelets method [9]. In Table 1, this comparison has been presented . In Fig. 1, we show the comparison between the LCOCOM for $n = 8$ with B-spline wavelets method [9].

		LCOCOM		
t	Bernoulli wavelets [9]	$n=4$	$n=6$	$n=8$
0	–	1.9984e-15	8.7708e-15	1.1102e-16
1.0000e-01	1.05e-5	1.9471e-09	5.5378e-13	3.9968e-15
2.0000e-01	4.15e-6	1.5430e-09	6.7479e-13	3.5083e-14
3.0000e-01	5.17e-6	2.1330e-09	6.4593e-13	2.2582e-13
4.0000e-01	9.76e-6	1.7526e-09	8.0136e-13	2.2649e-14
5.0000e-01	9.22e-5	3.8543e-09	1.6829e-12	7.2453e-13
CPU	1.43	2.92	3.27	2.81

Table 1: Absolute error of Example 1

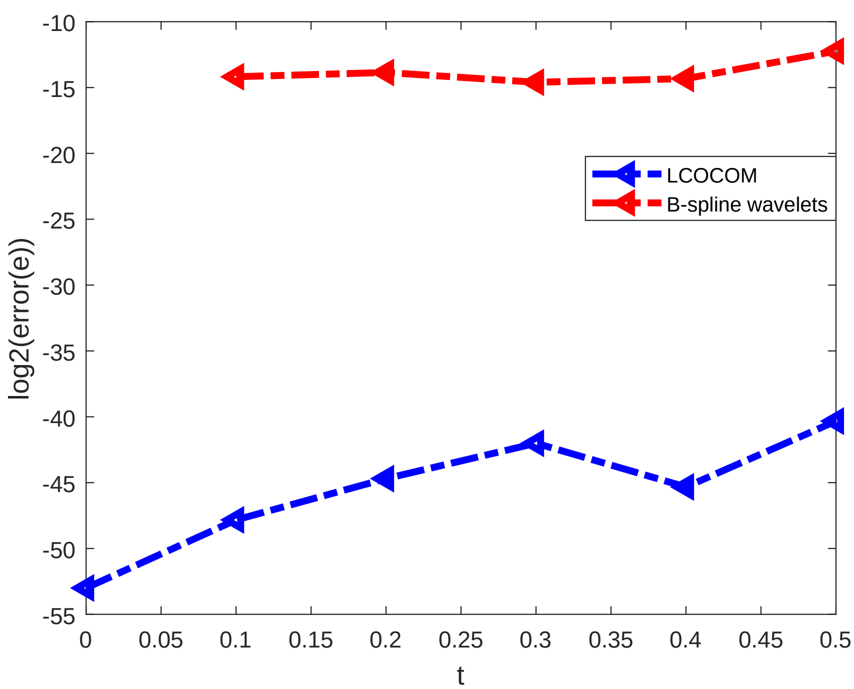

Figure 1: Absolute error of Example 1.

Example 2: Consider the Volterra– Hammerstein delay integral equation [9]:

$$\begin{cases} y(t) = e^{t+1} - \tau(1 + t - \frac{\tau}{2}) + \int_{t-\tau}^{t} \ln(y(s))ds, & t \in [0, X], \\ y(t) = \phi(t), & t \in [-\tau, 0]. \end{cases} \tag{38}$$

The nonlinear integral equation (38) can be written as:

$$\begin{cases} y(t) = e^{t+1} - t(1 + \frac{t}{2}) + \int_{0}^{t} \ln(y(s))ds, & t \in [0, X], \\ y(t) = \phi(t), & t \in [-\tau, 0]. \end{cases} \tag{39}$$

where $\tau = 0.5$, $X = 1$, $\phi(t) = e^{t+1}$. The exact solution is $y(t) = e^{t+1}$, $t \in [-0.5, 1]$. The above problem has been solved by LCOCOM and then the absolute error and computational time (in seconds) have been compared with that of obtained by B-spline wavelets method [9]. In Table 2, this comparison has been presented. In Fig. 2, we show the comparison between the LCOCOM for $n = 8$ with Bernouli wavelets method [9].

		LCOCOM		
t	B-spline wavelets [9]	$n = 4$	$n = 6$	$n = 8$
0	–	4.2588e-12	2.9714e-12	2.2413e-12
2.0000e-01	2.01e-6	2.2611e-11	5.2240e-11	3.1672e-11
4.0000e-01	8.56e-6	1.0954e-10	8.4359e-12	5.8894e-11
6.0000e-01	2.98e-5	1.4048e-11	3.0703e-11	3.2133e-11
8.0000e-01	7.41e-5	1.6966e-11	1.3294e-11	8.7095e-12
1.0000e+00	1.40e-4	1.6857e-10	1.2284e-11	2.2826e-11
CPU	462.41	4.05	2.56	4.23

Table 2: Absolute error of Example 2

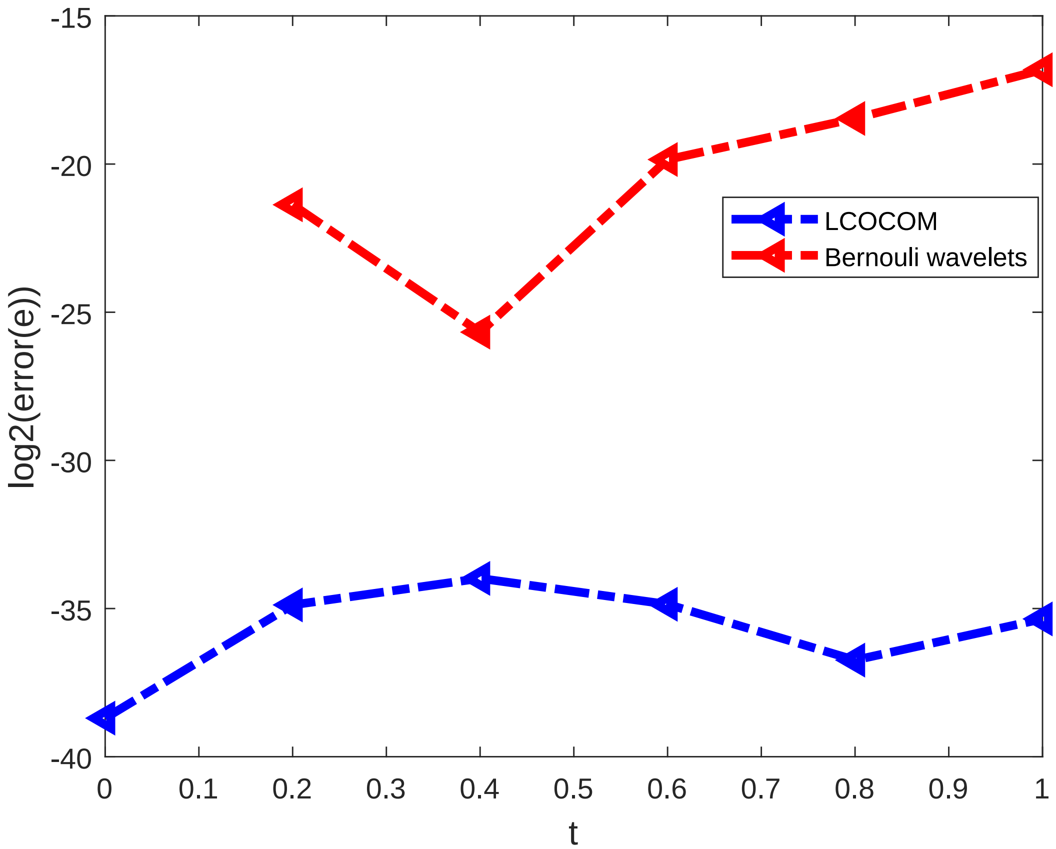

Figure 2: Absolute error of Example 2.

Example 3: Consider Volterra delay integro-differential equation [15]:

$$\begin{cases} y'(t) = y(t-1) + \int_{t-1}^{t} y(s)ds, & t \geq 0, \\ y(t) = e^{t}, & t \leq 0. \end{cases} \tag{40}$$

The equation (40) can be written as:

$$\begin{cases} y'(t) = 1 + \int_{0}^{t} y(s)ds, & t \geq 0, \\ y(t) = e^{t}, & t \leq 0. \end{cases} \tag{41}$$

The exact solution of this problem is $y(t) = e^t$. The above problem has been solved by LCOCOM. We presente the absolute error and computational time (in seconds) of LCOCOM for some values of n. In Table 3, this comparison has been presented.

	LCOCOM		
t	$n=4$	$n=6$	$n=8$
0	1.0236e-13	4.4209e-13	3.1826e-11
5.0000e-01	5.8123e-04	2.0189e-06	4.4924e-09
1.0000e+00	7.6386e-04	2.6416e-06	6.0108e-09
CPU	0.91	3.02	15.00

Table 3: Absolute error of Example 3

Example 4: Consider the following system of fractional system of delay integral equations:

(42)
$$\begin{cases} u(t) = 1 + H_1(t) + \dfrac{1}{\Gamma(\mu)} \displaystyle\int_0^t (t-s)^{\mu-1} u(s - \frac{1}{3}) ds + \dfrac{2}{\Gamma(\mu)} \displaystyle\int_0^t (t-s)^{\mu-1} v(s - \frac{2}{3}) ds, & t \in [0, X], \\ v(t) = H_2(t) + \dfrac{2}{\Gamma(\mu)} \displaystyle\int_0^t t(t-s)^{\mu-1} u(s - \frac{2}{3}) ds + \dfrac{1}{\Gamma(\mu)} \displaystyle\int_0^t t(t-s)^{\mu-1} v(s - \frac{1}{3}) ds, & \\ u(t) = \phi_1(t), \quad v(t) = \phi_2(t), & t \in [-\tau, 0]. \end{cases}$$

with the conditions $u(0) = 1$, $v(0) = 0$, $t \in [-\frac{2}{3}, 0]$, $\phi_1(t) = t^3 + 1$ and $\phi_2(t) = t^2$.
$H_1(t)$ and $H_2(t)$ are choosing such that the exact solution of this system is $u(t) = t^3 + 1$ and $v(t) = t^2$. This problem has been taken up similarly as a differential equation in [21]. The above problem has been solved by LCOCOM at $n = 3$. The maximum error obtained by present method is $O(10^{-14})$ compared to $O(10^{-11})$ obtained in [21]. We obtain the approximate solution plotted in Fig. 3, at some selected values of μ. In Table 4, we present the absolute error of u and v.

	Absolute error	
t	$u(t)$	$v(t)$
0	7.5495e-15	1.3679e-14
3.3333e-01	1.6431e-14	1.3559e-14
6.6667e-01	4.6629e-15	7.1054e-15
1.0000e+00	2.8866e-15	3.2752e-14

Table 4: Absolute error at $\mu = 1$ and $n = 3$ of Example 4

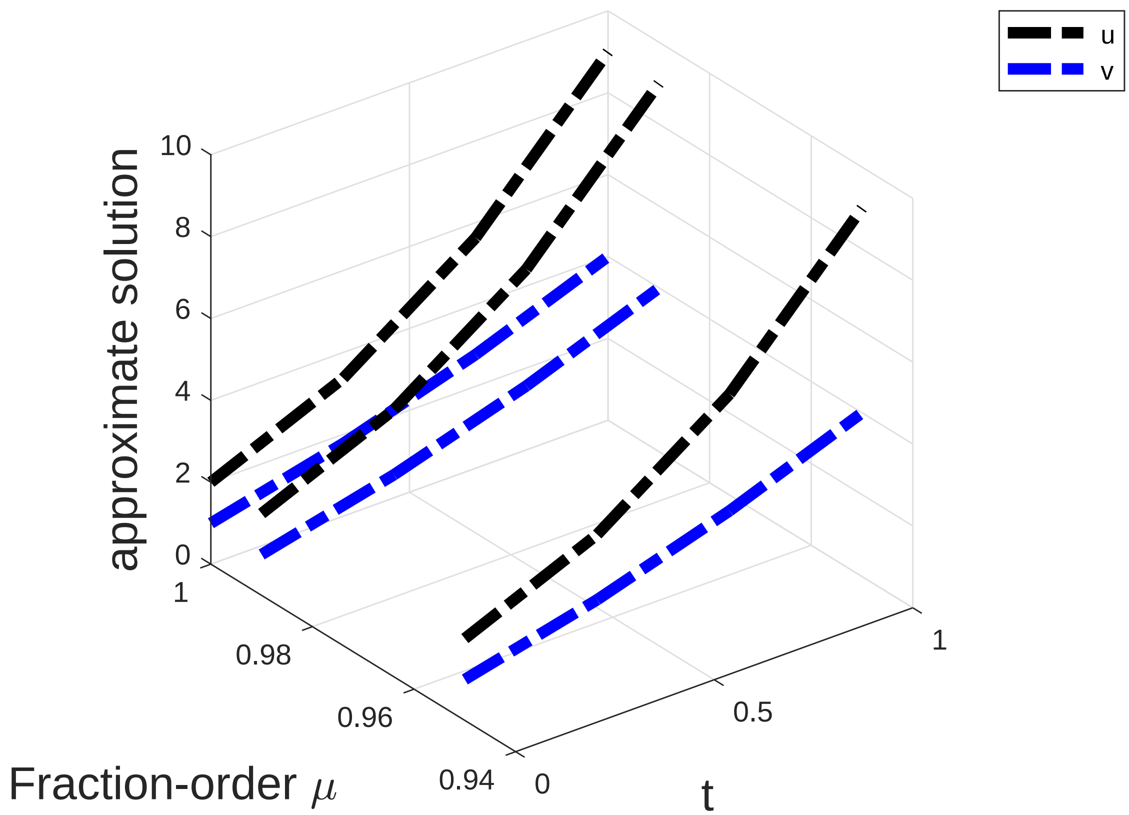

Figure 3: Approximate solution at different values for μ of Example 4.

Example 5: Consider the fractional system of delay Volterra integro differential equations, $0<\mu\leq 2$:

(43)
$$\begin{cases} D^{\mu}u(t)=-cos(t-\tau)+sin(t-\tau)-\tau(2t-\tau)-sin(t)+\int_{t-\tau}^{t}(u(s)+v(s))ds, & t\in[0,X], \\ D^{\mu}v(t)=cos(t-\tau)+sin(t-\tau)-2sin(t)-cos(t)+\int_{t-\tau}^{t}(u(s)-v(s))ds, & \\ u(t)=\phi_1(t), \quad v(t)=\phi_2(t), & t\in[-\tau,0]. \end{cases}$$

The system of equations (43) can be written as in [16]:

$$\begin{cases} D^{\mu}u(t)=-1-t^2-sin(t)+\int_{0}^{t}(u(s)+v(s))ds, & t\in[0,X] \\ D^{\mu}v(t)=1-2sin(t)-cos(t)+\int_{0}^{t}(u(s)-v(s))ds, & \\ u(t)=\phi_1(t), v(t)=\phi_2(t), & t\in[-\tau,0], \end{cases} \tag{44}$$

and the ICs $u(0)=1$, $u'(0)=1$, $v(0)=0$ and $v'(0)=2$, where $\tau=0.5$, $X=1$, $\phi_1(t)=t+cos(t)$ and $\phi_2(t)=t+sin(t)$. The exact solution of this system for $\mu=2$ is $u(t)=t+cos(t)$ and $v(t)=t+sin(t)$. The above problem has been solved by LCOCOM at $n=5$, we obtain the approximate solution plotted in Fig. 4, at some selected values of μ. In Table 5, we present an absolute error of u and v.

	Absolute error	
t	$u(t)$	$v(t)$
0	5.7732e-15	3.8908e-06
2.0000e-01	2.0243e-05	1.6292e-04
4.0000e-01	5.3877e-05	1.9941e-04
6.0000e-01	8.8248e-05	2.3180e-04
8.0000e-01	1.3097e-04	2.6268e-04
1.0000e+00	1.7953e-04	2.8775e-04

Table 5: Absolute error at $\mu=2$ and $n=5$ of Example 5

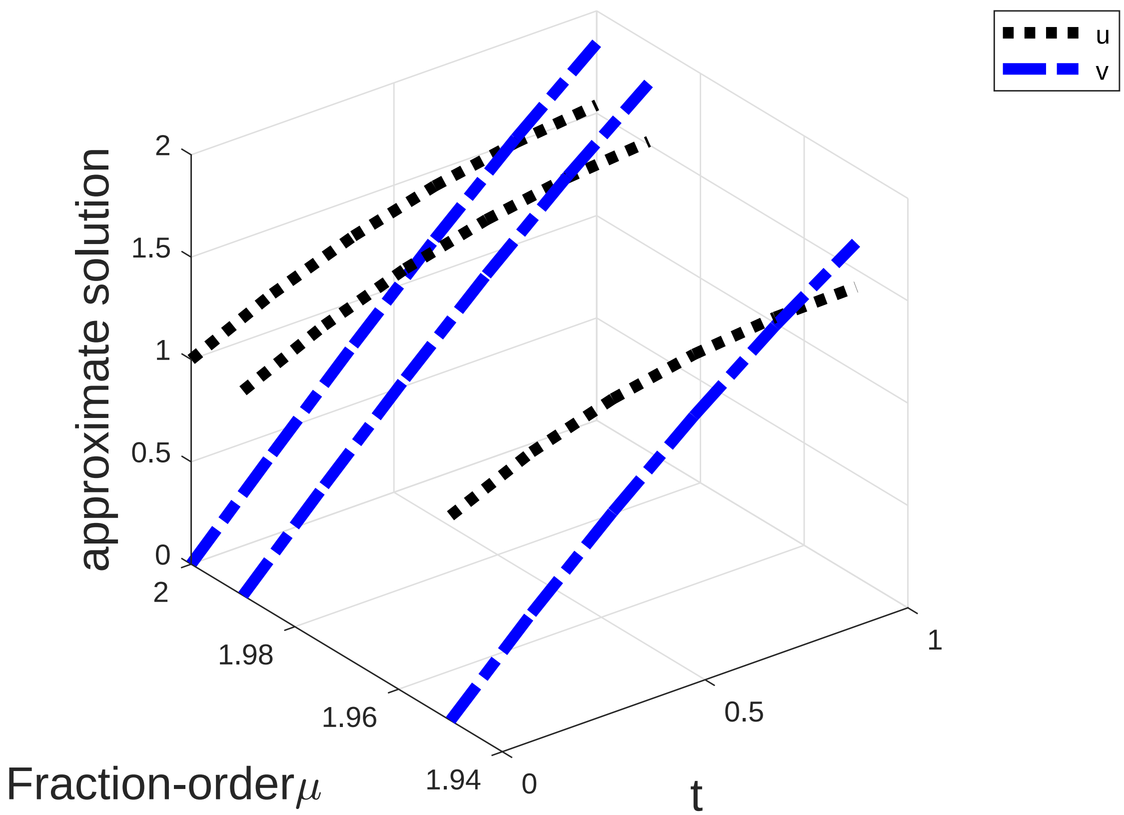

Figure 4: Approximate solution at different values for μ of Example 5.

Examples No.	CPU time(Sec.)
4	14.25
5	27.28

Table 6: CPU time in seconds for Examples 4 and 5.

6. Conclusion

In this work, we presented the LCOCOM to solve Volterra delay integral and integro-differential equations in addition to fractional-order delay systems of Volterra integral and integro-differential equations. We derived fractional-order derivatives and integrals of Lagrange cardinal functions. Then, we used these derivatives and integrals to approximate the solution of the problems. The proposed method is implemented by using MATLAB (R 2017b). The numerical examples presented in this paper show that the present method gives accurate results and has the following advantages:

- The present method can solve Volterra delay integral and integro-differential equations.
- The absolute error of present method is decreasing when n increased (as in Examples 1, 2 and 3)
- The present method can solve fractional-order system of Volterra integral and integro-differential equations delay problems.
- The present method is used to show the effects of fractional order on the results (as in Examples 4 and 5).
- The present method gives more accurate results than other method such as collocation Laguerre operational matrix (CLOM) method (as in Example 4).
- CPU time of the present method is too small (as in Table 6).

Conflict of Interests

The author(s) declare that there is no conflict of interests.

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