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EVALUATION OF FOUR CONVOLUTION SUMS AND REPRESENTATION OF INTEGERS BY CERTAIN QUADRATIC FORMS IN TWELVE VARIABLES

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Abstract. In this paper the convolution sums $\sum_{6i+j=n}\sigma(l)\sigma_3(m)$, $\sum_{2i+3j=n}\sigma(l)\sigma_3(m)$, $\sum_{i+6j=n}\sigma(l)\sigma_3(m)$ and $\sum_{3i+2j=n}\sigma(l)\sigma_3(m)$ are evaluated for all $n\in\mathbb{N}$, and then their evaluations are used to determine the representation number formulae $N(1,1,1,1,1,2;n), N(1,1,1,1,2,2;n)$ and $N(1,1,1,2,2,2;n)$ where $N(a_1,\dots,a_6;n)$ denote the representation numbers of n by the form $a_1(x_1^2+x_1x_2+x_2^2)+a_2(x_3^2+x_3x_4+x_4^2)+a_3(x_5^2+x_5x_6+x_6^2)+a_4(x_7^2+x_7x_8+x_8^2)+a_5(x_9^2+x_9x_{10}+x_{10}^2)+a_6(x_{11}^2+x_{11}x_{12}+x_{12}^2)$.

Keywords: convolution sum; divisor function; Eisenstein series; quadratic forms; representation numbers.

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1. INTRODUCTION

Let $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, and $\mathbb{C}$ denote the set of positive integers, integers, real numbers and complex numbers respectively and let $\mathbb{N}_0=\mathbb{N}\cup\{0\}$. Throughout this paper $q\in\mathbb{C}$ is taken to satisfy $|q|<1$. For $k,n\in\mathbb{N}$ we set

(1) $$\sigma_k(n)=\sum_{d\mid n}d^k,$$

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where d runs through the positive divisors of n. If $n \notin \mathbb{N}$ we set $\sigma_k(n) = 0$. We write $\sigma(n)$ for $\sigma_1(n)$. For $a,b,r,s,n \in \mathbb{N}$, we define the convolution sum $W_{a,b}^{r,s}(n)$ by

$$W_{a,b}^{r,s}(n) := \sum_{\substack{(l,m)\in\mathbb{N}^2 \\ al+bm=n}} \sigma_r(l)\sigma_s(m). \tag{2}$$

Ramanujan [1] evaluated $W_{1,1}^{1,3}(n)$ explicitly. He proved that

$$W_{1,1}^{1,3}(n) = \frac{7}{80}\sigma_5(n) + \frac{1-3n}{24}\sigma_3(n) - \frac{1}{240}\sigma(n). \tag{3}$$

The following two sums are given by Huard, Ou, Spearman and Williams [2].

$$W_{1,2}^{1,3}(n) = \frac{1}{240}\Big(\sigma_5(n) + 20\sigma_5(\frac{n}{2}) + (10-30n)\sigma_3(\frac{n}{2}) - \sigma(n)\Big) \tag{4}$$

and

$$W_{2,1}^{1,3}(n) = \frac{1}{240}\Big(5\sigma_5(n) + 16\sigma_5(\frac{n}{2}) + (10-15n)\sigma_3(n) - \sigma(\frac{n}{2})\Big). \tag{5}$$

The convolution sums $W_{1,4}^{1,3}(n)$, $W_{4,1}^{1,3}(n)$ have been evaluated by Cheng and Williams [3]. A linear equation for $W_{1,3}^{1,3}(n)$ and $W_{3,1}^{1,3}(n)$ was given before by Huard, Ou, Spearman and Williams [2] without individual determination. In a recent publication, Yao and Xia [4] completed the determination of $W_{1,3}^{1,3}(n)$ and $W_{3,1}^{1,3}(n)$. They proved that

$$W_{1,3}^{1,3}(n) = \frac{1}{1040}\sigma_5(n) + \frac{9}{104}\sigma_5(\frac{n}{3}) + \frac{1-3n}{24}\sigma_3(\frac{n}{3}) - \frac{1}{240}\sigma(n) + \frac{1}{312}a(n) \tag{6}$$

and

$$W_{3,1}^{1,3}(n) = \frac{1}{104}\sigma_5(n) + \frac{81}{1040}\sigma_5(\frac{n}{3}) + \frac{1-n}{24}\sigma_3(n) - \frac{1}{240}\sigma(\frac{n}{3}) - \frac{1}{104}a(n), \tag{7}$$

where $a(n)$ is defined by

$$\sum_{n=1}^{\infty} a(n)\,q^n = q\prod_{n=1}^{\infty}(1-q^n)^6(1-q^{3n})^6. \tag{8}$$

Convolution sum $W_{a,b}^{r,s}(n)$ involving the divisor function $\sigma(n)$ has been evaluated for certain values a,b,r,s (see, for example, [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17]).

In this paper, motivated from the work of Yao and Xia[4] and using the (p,k)-parametrization of Eisenstein series and theta functions given by Alaca, Alaca and Williams we determine the convolution sums $W_{6,1}^{1,3}(n)$, $W_{2,3}^{1,3}(n)$, $W_{1,6}^{1,3}(n)$ and $W_{3,2}^{1,3}(n)$. We've used 4 Eisenstein series $N(q),N(q^2),N(q^3),N(q^6)$ and 3 eta products which form a basis of the Modular space $M_6(\Gamma_0(6))$ (with dimension 7) and express the products $\left(6L(q^6)-L(q)\right)M(q)$, $\left(3L(q^3)-2L(q^2)\right)M\left(q^3\right)$, $\left(2L(q^2)-L(q)\right)M\left(q^6\right)$ and $\left(3L(q^2)-2L(q^2)\right)M\left(q^2\right)$ of Eisenstein Series (all of which clearly belongs to space) as a linear combination the mentioned Eisenstein series and eta products. One of the eta products which is used here appeared in literature[4]. Today, some other authors use the theory of quasimodular forms with softwares Magma or the database LMFDP of L functions and modular forms to obtain formulae for convolution sums.

As an application we use our evaluations to determine formulae for $N(1,1,1,1,1,2;n)$, $N(1,1,1,1,2,2;n)$ and $N(1,1,1,2,2,2;n)$ where $N(a_1,...,a_6;n)$ denote the representation number of n by the form $a_1(x_1^2+x_1x_2+x_2^2)+a_2(x_3^2+x_3x_4+x_4^2)+a_3(x_5^2+x_5x_6+x_6^2)+a_4(x_7^2+x_7x_8+x_8^2)+a_5(x_9^2+x_9x_{10}+x_{10}^2)+a_6(x_{11}^2+x_{11}x_{12}+x_{12}^2)$, that is

$$N(a_1,...,a_6;n):=\text{card}\left\{\begin{array}{c}(x_1,...,x_{12})\in\mathbb{Z}^{12}:n=a_1(x_1^2+x_1x_2+x_2^2)\\ +a_2(x_3^2+x_3x_4+x_4^2)+a_3(x_5^2+x_5x_6+x_6^2)\\ +a_4(x_7^2+x_7x_8+x_8^2)+a_5(x_9^2+x_9x_{10}+x_{10}^2)\\ +a_6(x_{11}^2+x_{11}x_{12}+x_{12}^2).\end{array}\right\}$$

Formulae for $N(1,1,1,1,1,1;n)$ is given recently by Yao and Xia [4]. Our main results are as follows.

Theorem 1. *For $n\in\mathbb{N}$, we have*

(i)

$$\begin{aligned} W_{6,1}^{1,3}(n) &= \frac{5}{2184}\sigma_5(n)+\frac{2}{273}\sigma_5\left(\frac{n}{2}\right)+\frac{27}{1456}\sigma_5\left(\frac{n}{3}\right)+\frac{27}{455}\sigma_5\left(\frac{n}{6}\right)\\ &\quad+\frac{2-n}{48}\sigma_3(n)-\frac{1}{240}\sigma\left(\frac{n}{6}\right)-\frac{1}{4368}u_1(n), \end{aligned} \tag{9}$$

(ii)

$$W_{2,3}^{1,3}(n) = \frac{1}{4368}\sigma_5(n) + \frac{1}{1365}\sigma_5\left(\frac{n}{2}\right) + \frac{15}{728}\sigma_5\left(\frac{n}{3}\right) + \frac{6}{91}\sigma_5\left(\frac{n}{6}\right) + \frac{2-3n}{48}\sigma_3\left(\frac{n}{3}\right) - \frac{1}{240}\sigma\left(\frac{n}{2}\right) + \frac{1}{8736}u_2(n), \tag{10}$$

(iii)

$$W_{1,6}^{1,3}(n) = \frac{1}{21840}\sigma_5(n) + \frac{1}{1092}\sigma_5\left(\frac{n}{2}\right) + \frac{3}{728}\sigma_5\left(\frac{n}{3}\right) + \frac{15}{182}\sigma_5\left(\frac{n}{6}\right) + \frac{1-3n}{24}\sigma_3\left(\frac{n}{6}\right) - \frac{1}{240}\sigma(n) + \frac{1}{156}a(\frac{n}{2}) + \frac{1}{4368}u_3(n) \tag{11}$$

and

(iv)

$$W_{3,2}^{1,3}(n) = \frac{1}{2184}\sigma_5(n) + \frac{5}{546}\sigma_5\left(\frac{n}{2}\right) + \frac{27}{7280}\sigma_5\left(\frac{n}{3}\right) + \frac{27}{364}\sigma_5\left(\frac{n}{6}\right) + \frac{1-n}{24}\sigma_3\left(\frac{n}{2}\right) - \frac{1}{240}\sigma\left(\frac{n}{3}\right) - \frac{1}{8736}u_4(n) \tag{12}$$

where $u_1(n), u_2(n), u_3(n)$ and $u_4(n)$ are respectively defined by

$$\sum_{n=1}^{\infty} u_1(n) q^n = -4q\prod_{n=1}^{\infty}(1-q^n)^5(1-q^{2n})^5(1-q^{3n})(1-q^{6n}) + 105q\prod_{n=1}^{\infty}(1-q^n)^6(1-q^{3n})^6 + 972q^2\prod_{n=1}^{\infty}(1-q^n)(1-q^{2n})(1-q^{3n})^5(1-q^{6n})^5, \tag{13}$$

$$\sum_{n=1}^{\infty} u_2(n) q^n = -72q\prod_{n=1}^{\infty}(1-q^n)^5(1-q^{2n})^5(1-q^{3n})(1-q^{6n}) + 70q\prod_{n=1}^{\infty}(1-q^n)^6(1-q^{3n})^6 + 24q^2\prod_{n=1}^{\infty}(1-q^n)(1-q^{2n})(1-q^{3n})^5(1-q^{6n})^5, \tag{14}$$

$$\begin{aligned}
&\sum_{n=1}^{\infty} u_3(n)\,q^n \\
&+4q\prod_{n=1}^{\infty}(1-q^n)^5(1-q^{2n})^5(1-q^{3n})(1-q^{6n}) \\
&+14q\prod_{n=1}^{\infty}(1-q^n)^6(1-q^{3n})^6 \\
&+120q^2\prod_{n=1}^{\infty}(1-q^n)(1-q^{2n})(1-q^{3n})^5(1-q^{6n})^5, \qquad (15)
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{n=1}^{\infty} u_4(n)\,q^n \\
&-101q\prod_{n=1}^{\infty}(1-q^n)^5(1-q^{2n})^5(1-q^{3n})(1-q^{6n}) \\
&+105q\prod_{n=1}^{\infty}(1-q^n)^6(1-q^{3n})^6 \\
&-27q^2\prod_{n=1}^{\infty}(1-q^n)(1-q^{2n})(1-q^{3n})^5(1-q^{6n})^5. \qquad (16)
\end{aligned}$$

Theorem 2. *For $n \in \mathbb{N}$,*

(i)

$$\begin{aligned}
N(1,1,1,1,1,2;n) &= \frac{66}{7}\sigma_5(n) - \frac{192}{7}\sigma_5\left(\frac{n}{2}\right) + \frac{1782}{7}\sigma_5\left(\frac{n}{3}\right) - \frac{5184}{7}\sigma_5\left(\frac{n}{6}\right) \\
&\quad + \frac{432}{13}a\left(\frac{n}{2}\right) + \frac{108}{91}u_3(n) - \frac{18}{91}u_4(n), \qquad (17)
\end{aligned}$$

(ii)

$$\begin{aligned}
N(1,1,1,1,2,2;n) &= \frac{60}{13}\sigma_5(n) + \frac{192}{13}\sigma_5\left(\frac{n}{2}\right) - \frac{1620}{13}\sigma_5\left(\frac{n}{3}\right) - \frac{5184}{13}\sigma_5\left(\frac{n}{6}\right) \\
&\quad + \frac{18}{91}u_1(n) + \frac{27}{91}u_2(n) \qquad (18)
\end{aligned}$$

and

(iii)

$$\begin{aligned}
N(1,1,1,2,2,2;n) &= \frac{18}{7}\sigma_5(n) - \frac{144}{7}\sigma_5\left(\frac{n}{2}\right) + \frac{486}{7}\sigma_5\left(\frac{n}{3}\right) - \frac{3888}{7}\sigma_5\left(\frac{n}{6}\right) \\
&\quad + \frac{324}{13}a\left(\frac{n}{2}\right) + \frac{81}{91}u_3(n) - \frac{27}{91}u_4(n). \qquad (19)
\end{aligned}$$

2. Proof of Theorem 1

In his second notebook [18] Ramanujan gives the definitions of Eisenstein series $L(q), M(q)$ and $N(q)$ by

$$L(q) := 1 - 24\sum_{n=1}^{\infty} \frac{nq^n}{1-q^n}, \tag{20}$$

$$M(q) := 1 + 240\sum_{n=1}^{\infty} \frac{n^3q^n}{1-q^n} \tag{21}$$

and

$$N(q) := 1 - 504\sum_{n=1}^{\infty} \frac{n^5q^n}{1-q^n}. \tag{22}$$

It can be easily seen that

$$L(q) := 1 - 24\sum_{n=1}^{\infty} \sigma(n)\, q^n, \tag{23}$$

$$M(q) := 1 + 240\sum_{n=1}^{\infty} \sigma_3(n)\, q^n \tag{24}$$

and

$$N(q) := 1 - 504\sum_{n=1}^{\infty} \sigma_5(n)\, q^n. \tag{25}$$

The Jacobi theta function $\varphi(q)$ is defined by

$$\varphi(q) := \sum_{n=-\infty}^{\infty} q^{n^2}, \tag{26}$$

see for example ([20], p.92).

Alaca, Alaca, and Williams [19] defined p and k respectively by

$$p = p(q) := \frac{\varphi^2(q) - \varphi^2(q^3)}{2\varphi^2(q^3)} \tag{27}$$

and

$$k = k(q) := \frac{\varphi^3(q^3)}{\varphi(q)}. \tag{28}$$

The $(p-k)$-parametrization of $2L(q^2)-L(q)$, $6L(q^6)-L(q)$ and $3L(q^3)-2L(q^2)$ are given by the equations (3.84), (3.89) and (3.90) in [9] respectively as follows:

$$2L(q^2)-L(q)=(1+14p+24p^2+14p^3+p^4)k^2, \tag{29}$$

$$6L(q^6)-L(q)=(5+22p+36p^2+22p^3+5p^4)k^2 \tag{30}$$

and

$$3L(q^3)-2L(q^2)=(1+2p+12p^2+2p^3+p^4)k^2. \tag{31}$$

Formulae for the series $M(q),M(q^2),M(q^3),M(q^6),N(q),N(q^2),N(q^3),N(q^6)$ in terms of p and k are determined by Alaca, Alaca, and Williams [5]. Equations (3.14)-(3.16), (3.18), (3.22)-(3.24) and (3.26) are as follows

$$\begin{aligned} M(q) &= (1+124p+964p^2+2788p^3+3910p^4 \\ &\quad +2788p^5+964p^6+124p^7+p^8)k^4, \end{aligned} \tag{32}$$

$$\begin{aligned} M(q^2) &= (1+4p+64p^2+178p^3+235p^4 \\ &\quad +178p^5+64p^6+4p^7+p^8)k^4, \end{aligned} \tag{33}$$

$$\begin{aligned} M(q^3) &= (1+4p+4p^2+28p^3+70p^4 \\ &\quad +28p^5+4p^6+4p^7+p^8)k^4, \end{aligned} \tag{34}$$

$$\begin{aligned} M(q^6) &= (1+4p+4p^2-2p^3-5p^4 \\ &\quad -2p^5+4p^6+4p^7+p^8)k^4, \end{aligned} \tag{35}$$

$$\begin{aligned} N(q) &= (1-246p-5532p^2-38614p^3-135369p^4 \\ &\quad -276084p^5-348024p^6-276084p^7-135369p^8 \\ &\quad -38614p^9-5532p^{10}-246p^{11}+p^{12})k^6, \end{aligned} \tag{36}$$

$$\begin{aligned} N(q^2) &= (1+6p-114p^2-625p^3-\frac{4059}{2}p^4 \\ &\quad -4302p^5-5556p^6-4302p^7-\frac{4059}{2}p^8 \\ &\quad -625p^9-114p^{10}+6p^{11}+p^{12})k^6, \end{aligned} \tag{37}$$

$$\begin{aligned} N(q^3) &= (1+6p+12p^2-58p^3-297p^4-396p^5-264p^6 \\ &\quad -396p^7-297p^8-58p^9+12p^{10}+6p^{11}+p^{12})k^6, \end{aligned} \tag{38}$$

and

$$\begin{aligned} N(q^6) &= (1+6p+12p^2+5p^3-\frac{27}{2}p^4-18p^5-12p^6 \\ &\quad -18p^7-\frac{27}{2}p^8+5p^9+12p^{10}+6p^{11}+p^{12})k^6. \end{aligned} \tag{39}$$

In that study, Alaca, Alaca and Williams also derived formulae for $\prod_{n=1}^{\infty}(1-q^n)$, $\prod_{n=1}^{\infty}(1-q^{2n})$, $\prod_{n=1}^{\infty}(1-q^{3n})$ and $\prod_{n=1}^{\infty}(1-q^{6n})$ in terms of p and k. Equations (3.28)-(3.30) and (3.32) in [5] are

$$\prod_{n=1}^{\infty}(1-q^n) = q^{-\frac{1}{24}}2^{-\frac{1}{6}}p^{\frac{1}{24}}(1-p)^{\frac{1}{2}}(1+p)^{\frac{1}{6}}(1+2p)^{\frac{1}{8}}(2+p)^{\frac{1}{8}}k^{\frac{1}{2}}, \tag{40}$$

$$\prod_{n=1}^{\infty}(1-q^{2n}) = q^{-\frac{1}{12}}2^{-\frac{1}{3}}p^{\frac{1}{12}}(1-p)^{\frac{1}{4}}(1+p)^{\frac{1}{12}}(1+2p)^{\frac{1}{4}}(2+p)^{\frac{1}{4}}k^{\frac{1}{2}}, \tag{41}$$

$$\prod_{n=1}^{\infty}(1-q^{3n}) = q^{-\frac{1}{8}}2^{-\frac{1}{6}}p^{\frac{1}{8}}(1-p)^{\frac{1}{6}}(1+p)^{\frac{1}{2}}(1+2p)^{\frac{1}{24}}(2+p)^{\frac{1}{24}}k^{\frac{1}{2}} \tag{42}$$

and

$$\prod_{n=1}^{\infty}(1-q^{6n}) = q^{-\frac{1}{4}}2^{-\frac{1}{3}}p^{\frac{1}{4}}(1-p)^{\frac{1}{12}}(1+p)^{\frac{1}{4}}(1+2p)^{\frac{1}{12}}(2+p)^{\frac{1}{12}}k^{\frac{1}{2}}. \tag{43}$$

Using (40), (41), (42) and (43) we obtain

$$q\prod_{n=1}^{\infty}(1-q^n)^5(1-q^{2n})^5(1-q^{3n})(1-q^{6n}) = \frac{1}{8}p(1-p)^4(1+p)^2(1+2p)^2(2+p)^2k^6, \tag{44}$$

$$q\prod_{n=1}^{\infty}(1-q^n)^6(1-q^{3n})^6 = \frac{1}{4}p(1-p)^4(1+p)^4(1+2p)(2+p)k^6, \tag{45}$$

$$q^2\prod_{n=1}^{\infty}(1-q^n)(1-q^{2n})(1-q^{3n})^5(1-q^{6n})^5=\frac{1}{8}p^2(1-p)^2(1+p)^4(1+2p)(2+p)k^6. \tag{46}$$

From (8) and (45) it is clear that the eta products in equations are same.

Proof. (i) From (30), (32), (36)-(39), (44), (45), and (46) we deduce that

$$\begin{aligned}\Big(6L(q^6)-L(q)\Big)M(q) &= \frac{-537}{637}N(q)+\frac{320}{637}N\left(q^2\right)+\frac{810}{637}N\left(q^3\right)+\frac{2592}{637}N\left(q^6\right)\\ &\quad-\frac{2880}{91}q\prod_{n=1}^{\infty}(1-q^n)^5(1-q^{2n})^5(1-q^{3n})(1-q^{6n})\\ &\quad+\frac{10800}{13}q\prod_{n=1}^{\infty}(1-q^n)^6(1-q^{3n})^6\\ &\quad+\frac{699840}{91}q^2\prod_{n=1}^{\infty}(1-q^n)(1-q^{2n})(1-q^{3n})^5(1-q^{6n})^5.\end{aligned} \tag{47}$$

By (23) and (24), we see that

$$\begin{aligned}&\Big(6L(q^6)-L(q)\Big)M(q)\\ &= (5-144\sum_{n=1}^{\infty}\sigma(n)q^{6n}+24\sum_{n=1}^{\infty}\sigma(n)q^n)(1+240\sum_{n=1}^{\infty}\sigma_3(n)q^n)\\ &= 5-144\sum_{n=1}^{\infty}\sigma(n)q^{6n}+24\sum_{n=1}^{\infty}\sigma(n)q^n+1200\sum_{n=1}^{\infty}\sigma_3(n)q^n\\ &\quad-34560\sum_{n=1}^{\infty}\sigma(n)q^{6n}\sum_{n=1}^{\infty}\sigma_3(n)q^n+5760\sum_{n=1}^{\infty}\sigma(n)q^n\sum_{n=1}^{\infty}\sigma_3(n)q^n.\end{aligned} \tag{48}$$

Appealing to (13) and (25), we obtain

$$\begin{aligned}&\frac{-537}{637}N(q)+\frac{320}{637}N\left(q^2\right)+\frac{810}{637}N\left(q^3\right)+\frac{2592}{637}N\left(q^6\right)\\ &-\frac{2880}{91}q\prod_{n=1}^{\infty}(1-q^n)^5(1-q^{2n})^5(1-q^{3n})(1-q^{6n})\\ &+\frac{10800}{13}q\prod_{n=1}^{\infty}(1-q^n)^6(1-q^{3n})^6\\ &+\frac{699840}{91}q^2\prod_{n=1}^{\infty}(1-q^n)(1-q^{2n})(1-q^{3n})^5(1-q^{6n})^5\\ &= \frac{-537}{637}\left(1-504\sum_{n=1}^{\infty}\sigma_5(n)q^n\right)+\frac{320}{637}\left(1-504\sum_{n=1}^{\infty}\sigma_5(n)q^{2n}\right)\end{aligned}$$

$$
\begin{aligned}
& +\frac{810}{637}\left(1-504\sum_{n=1}^{\infty}\sigma_5(n)q^{3n}\right)+\frac{2592}{637}\left(1-504\sum_{n=1}^{\infty}\sigma_5(n)q^{6n}\right) \\
& +\frac{720}{91}\sum_{n=1}^{\infty}u_1(n)q^n. \\
= \ & 5+\frac{38664}{91}\sum_{n=1}^{\infty}\sigma_5(n)q^n-\frac{23040}{91}\sum_{n=1}^{\infty}\sigma_5(n)q^{2n} \\
& -\frac{58320}{91}\sum_{n=1}^{\infty}\sigma_5(n)q^{3n}-\frac{186624}{91}\sum_{n=1}^{\infty}\sigma_5(n)q^{6n}+\frac{720}{91}\sum_{n=1}^{\infty}u_1(n)q^n. \qquad (49)
\end{aligned}
$$

Combining (47), (48) and (49) we have

$$
\begin{aligned}
& \sum_{n=1}^{\infty}\sigma(n)q^{6n}\sum_{n=1}^{\infty}\sigma_3(n)q^n \\
= \ & -\frac{179}{14560}\sum_{n=1}^{\infty}\sigma_5(n)q^n+\frac{2}{273}\sum_{n=1}^{\infty}\sigma_5(n)q^{2n}+\frac{27}{1456}\sum_{n=1}^{\infty}\sigma_5(n)q^{3n} \\
& +\frac{27}{455}\sum_{n=1}^{\infty}\sigma_5(n)q^{6n}+\frac{5}{144}\sum_{n=1}^{\infty}\sigma_3(n)q^n+\frac{1}{1440}\sum_{n=1}^{\infty}\sigma(n)q^n \\
& -\frac{1}{240}\sum_{n=1}^{\infty}\sigma(n)q^{6n}+\frac{1}{6}\sum_{n=1}^{\infty}\sigma(n)q^n\sum_{n=1}^{\infty}\sigma_3(n)q^n \\
& -\frac{1}{4368}\sum_{n=1}^{\infty}u_1(n)q^n. \qquad (50)
\end{aligned}
$$

For $n\in\mathbb{N}$, equating the coefficients of q^n on both sides of (50) and using (2) we have

$$
\begin{aligned}
W_{6,1}^{1,3}(n) = \ & -\frac{179}{14560}\sigma_5(n)+\frac{2}{273}\sigma_5\left(\frac{n}{2}\right)+\frac{27}{1456}\sigma_5\left(\frac{n}{3}\right)+\frac{27}{455}\sigma_5\left(\frac{n}{6}\right)+\frac{5}{144}\sigma_3(n) \\
& +\frac{1}{1440}\sigma(n)-\frac{1}{240}\sigma\left(\frac{n}{6}\right)+\frac{1}{6}W_{1,1}^{1,3}(n)-\frac{1}{4368}u_1(n). \qquad (51)
\end{aligned}
$$

Identity (9) follows from (3) and (51).

(ii) From (31), (34), (36)-(39), (44), (45), and (46) we obtain

$$
\begin{aligned}
\left(3L(q^3)-2L(q^2)\right)M\left(q^3\right) &= \frac{-10}{1911}N(q)-\frac{32}{1911}N\left(q^2\right)+\frac{1611}{637}N\left(q^3\right)-\frac{960}{637}N\left(q^6\right)\\
&\quad -\frac{8640}{91}q\prod_{n=1}^{\infty}(1-q^n)^5(1-q^{2n})^5(1-q^{3n})(1-q^{6n})\\
&\quad +\frac{1200}{13}q\prod_{n=1}^{\infty}(1-q^n)^6(1-q^{3n})^6\\
&\quad +\frac{2880}{91}q^2\prod_{n=1}^{\infty}(1-q^n)(1-q^{2n})(1-q^{3n})^5(1-q^{6n})^5.
\end{aligned}\tag{52}
$$

From (23) and (24) we have

$$
\begin{aligned}
&\left(3L(q^3)-2L(q^2)\right)M\left(q^3\right)\\
&= (1-72\sum_{n=1}^{\infty}\sigma(n)q^{3n}+48\sum_{n=1}^{\infty}\sigma(n)q^{2n})(1+240\sum_{n=1}^{\infty}\sigma_3(n)q^{3n})\\
&= 1-72\sum_{n=1}^{\infty}\sigma(n)q^{3n}+48\sum_{n=1}^{\infty}\sigma(n)q^{2n}+240\sum_{n=1}^{\infty}\sigma_3(n)q^{3n}\\
&\quad -17280\sum_{n=1}^{\infty}\sigma(n)q^{3n}\sum_{n=1}^{\infty}\sigma_3(n)q^{3n}\\
&\quad +11520\sum_{n=1}^{\infty}\sigma(n)q^{2n}\sum_{n=1}^{\infty}\sigma_3(n)q^{3n}
\end{aligned}\tag{53}
$$

It follows from (14) and (25) that

$$
\begin{aligned}
&\frac{-10}{1911}N(q)-\frac{32}{1911}N\left(q^2\right)+\frac{1611}{637}N\left(q^3\right)-\frac{960}{637}N\left(q^6\right)\\
&-\frac{8640}{91}q\prod_{n=1}^{\infty}(1-q^n)^5(1-q^{2n})^5(1-q^{3n})(1-q^{6n})\\
&+\frac{1200}{13}q\prod_{n=1}^{\infty}(1-q^n)^6(1-q^{3n})^6\\
&+\frac{2880}{91}q^2\prod_{n=1}^{\infty}(1-q^n)(1-q^{2n})(1-q^{3n})^5(1-q^{6n})^5\\
&= \frac{-10}{1911}\left(1-504\sum_{n=1}^{\infty}\sigma_5(n)q^n\right)-\frac{32}{1911}\left(1-504\sum_{n=1}^{\infty}\sigma_5(n)q^{2n}\right)
\end{aligned}
$$

$$
\begin{aligned}
& +\frac{1611}{637}\left(1-504\sum_{n=1}^{\infty}\sigma_5(n)q^{3n}\right)-\frac{960}{637}\left(1-504\sum_{n=1}^{\infty}\sigma_5(n)q^{6n}\right)\\
& +\frac{120}{91}\sum_{n=1}^{\infty}u_2(n)q^n\\
= \; & 1+\frac{240}{91}\sum_{n=1}^{\infty}\sigma_5(n)q^n+\frac{768}{91}\sum_{n=1}^{\infty}\sigma_5(n)q^{2n}\\
& -\frac{115992}{91}\sum_{n=1}^{\infty}\sigma_5(n)q^{3n}+\frac{69120}{91}\sum_{n=1}^{\infty}\sigma_5(n)q^{6n}+\frac{120}{91}\sum_{n=1}^{\infty}u_2(n)q^n. \qquad (54)
\end{aligned}
$$

Combining (52), (53) and (54) we have

$$
\begin{aligned}
& \sum_{n=1}^{\infty}\sigma(n)q^{2n}\sum_{n=1}^{\infty}\sigma_3(n)q^{3n}\\
= \; & \frac{1}{4368}\sum_{n=1}^{\infty}\sigma_5(n)q^n+\frac{1}{1365}\sum_{n=1}^{\infty}\sigma_5(n)q^{2n}-\frac{1611}{14560}\sum_{n=1}^{\infty}\sigma_5(n)q^{3n}\\
& +\frac{6}{91}\sum_{n=1}^{\infty}\sigma_5(n)q^{6n}-\frac{1}{48}\sum_{n=1}^{\infty}\sigma_3(n)q^{3n}-\frac{1}{240}\sum_{n=1}^{\infty}\sigma(n)q^{2n}\\
& +\frac{1}{160}\sum_{n=1}^{\infty}\sigma(n)q^{3n}+\frac{3}{2}\sum_{n=1}^{\infty}\sigma(n)q^{3n}\sum_{n=1}^{\infty}\sigma_3(n)q^{3n}\\
& +\frac{1}{8736}\sum_{n=1}^{\infty}u_2(n)q^n. \qquad (55)
\end{aligned}
$$

For $n\in\mathbb{N}$, equating the coefficients of q^n on both sides of (55) and using (2) we have

$$
\begin{aligned}
W_{2,3}^{1,3}(n) = \; & \frac{1}{4368}\sigma_5(n)+\frac{1}{1365}\sigma_5\left(\frac{n}{2}\right)-\frac{1611}{14560}\sigma_5\left(\frac{n}{3}\right)+\frac{6}{91}\sigma_5\left(\frac{n}{6}\right)-\frac{1}{48}\sigma_3\left(\frac{n}{3}\right)\\
& -\frac{1}{240}\sigma\left(\frac{n}{2}\right)+\frac{1}{160}\sigma\left(\frac{n}{3}\right)+\frac{3}{2}W_{1,1}^{1,3}\left(\frac{n}{3}\right)+\frac{1}{8736}u_2(n). \qquad (56)
\end{aligned}
$$

Identity (10) follows from (3) and (56).

(iii) From (29), (35)-(39), (44), (45), and (46) we deduce that

$$\begin{aligned}\left(2L(q^2)-L(q)\right)M\left(q^6\right) &= \frac{-1}{1911}N(q)+\frac{22}{1911}N\left(q^2\right)-\frac{30}{637}N\left(q^3\right)+\frac{660}{637}N\left(q^6\right)\\ &\quad+\frac{480}{91}q\prod_{n=1}^{\infty}(1-q^n)^5(1-q^{2n})^5(1-q^{3n})(1-q^{6n})\\ &\quad+\frac{240}{13}q\prod_{n=1}^{\infty}(1-q^n)^6(1-q^{3n})^6\\ &\quad+\frac{14400}{91}q^2\prod_{n=1}^{\infty}(1-q^n)(1-q^{2n})(1-q^{3n})^5(1-q^{6n})^5. \end{aligned} \tag{57}$$

By (23) and (24), we see that

$$\begin{aligned} &\left(2L(q^2)-L(q)\right)M\left(q^6\right)\\ = {}&(1-48\sum_{n=1}^{\infty}\sigma(n)q^{2n}+24\sum_{n=1}^{\infty}\sigma(n)q^n)(1+240\sum_{n=1}^{\infty}\sigma_3(n)q^{6n})\\ = {}&1-48\sum_{n=1}^{\infty}\sigma(n)q^{2n}+24\sum_{n=1}^{\infty}\sigma(n)q^n+240\sum_{n=1}^{\infty}\sigma_3(n)q^{6n}\\ &-11520\sum_{n=1}^{\infty}\sigma(n)q^{2n}\sum_{n=1}^{\infty}\sigma_3(n)q^{6n}+5760\sum_{n=1}^{\infty}\sigma(n)q^n\sum_{n=1}^{\infty}\sigma_3(n)q^{6n}. \end{aligned} \tag{58}$$

Appealing to (15) and (25), we obtain

$$\begin{aligned} &\frac{-1}{1911}N(q)+\frac{22}{1911}N\left(q^2\right)-\frac{30}{637}N\left(q^3\right)+\frac{660}{637}N\left(q^6\right)\\ &+\frac{480}{91}q\prod_{n=1}^{\infty}(1-q^n)^5(1-q^{2n})^5(1-q^{3n})(1-q^{6n})\\ &+\frac{240}{13}q\prod_{n=1}^{\infty}(1-q^n)^6(1-q^{3n})^6\\ &+\frac{14400}{91}q^2\prod_{n=1}^{\infty}(1-q^n)(1-q^{2n})(1-q^{3n})^5(1-q^{6n})^5\\ = {}&\frac{-1}{1911}\left(1-504\sum_{n=1}^{\infty}\sigma_5(n)q^n\right)+\frac{22}{1911}\left(1-504\sum_{n=1}^{\infty}\sigma_5(n)q^{2n}\right)\\ &-\frac{30}{637}\left(1-504\sum_{n=1}^{\infty}\sigma_5(n)q^{3n}\right)+\frac{660}{637}\left(1-504\sum_{n=1}^{\infty}\sigma_5(n)q^{6n}\right) \end{aligned}$$

$$
\begin{aligned}
& +\frac{120}{91}\sum_{n=1}^{\infty}u_3(n)q^n. \\
= \quad & 1+\frac{24}{91}\sum_{n=1}^{\infty}\sigma_5(n)q^n-\frac{528}{91}\sum_{n=1}^{\infty}\sigma_5(n)q^{2n} \\
& +\frac{2160}{91}\sum_{n=1}^{\infty}\sigma_5(n)q^{3n}-\frac{47520}{91}\sum_{n=1}^{\infty}\sigma_5(n)q^{6n}+\frac{120}{91}\sum_{n=1}^{\infty}u_3(n)q^n. \qquad (59)
\end{aligned}
$$

Combining (57), (58) and (59) we have

$$
\begin{aligned}
& \sum_{n=1}^{\infty}\sigma(n)q^n\sum_{n=1}^{\infty}\sigma_3(n)q^{6n} \\
= \quad & \frac{1}{21840}\sum_{n=1}^{\infty}\sigma_5(n)q^n-\frac{11}{10920}\sum_{n=1}^{\infty}\sigma_5(n)q^{2n}+\frac{3}{728}\sum_{n=1}^{\infty}\sigma_5(n)q^{3n} \\
& -\frac{33}{364}\sum_{n=1}^{\infty}\sigma_5(n)q^{6n}-\frac{1}{24}\sum_{n=1}^{\infty}\sigma_3(n)q^{6n}-\frac{1}{240}\sum_{n=1}^{\infty}\sigma(n)q^n \\
& +\frac{1}{120}\sum_{n=1}^{\infty}\sigma(n)q^{2n}+2\sum_{n=1}^{\infty}\sigma(n)q^{2n}\sum_{n=1}^{\infty}\sigma_3(n)q^{6n} \\
& +\frac{1}{4368}\sum_{n=1}^{\infty}u_3(n)q^n. \qquad (60)
\end{aligned}
$$

For $n\in\mathbb{N}$, equating the coefficients of q^n on both sides of (60) and using (2) we have

$$
\begin{aligned}
W_{1,6}^{1,3}(n) \quad = \quad & \frac{1}{21840}\sigma_5(n)-\frac{11}{10920}\sigma_5\left(\frac{n}{2}\right)+\frac{3}{728}\sigma_5\left(\frac{n}{3}\right)-\frac{33}{364}\sigma_5\left(\frac{n}{6}\right)-\frac{1}{24}\sigma_3\left(\frac{n}{6}\right) \\
& -\frac{1}{240}\sigma(n)+\frac{1}{120}\sigma\left(\frac{n}{2}\right)+2W_{1,3}^{1,3}(\frac{n}{2})+\frac{1}{4368}u_3(n) \qquad (61)
\end{aligned}
$$

Identity (11) follows from (6) and (61).

(iv) From (31), (33), (36)-(39), (44), (45), and (46) we deduce that

$$
\begin{aligned}
\left(3L(q^2)-2L(q^2)\right)M\left(q^2\right) \quad = \quad & \frac{10}{637}N(q)-\frac{1074}{637}N\left(q^2\right)+\frac{81}{637}N\left(q^3\right)+\frac{1620}{637}N\left(q^6\right) \\
& -\frac{18180}{91}q\prod_{n=1}^{\infty}(1-q^n)^5(1-q^{2n})^5(1-q^{3n})(1-q^{6n})
\end{aligned}
$$

$$+\frac{2700}{13}q\prod_{n=1}^{\infty}(1-q^n)^6(1-q^{3n})^6 \tag{62}$$
$$-\frac{4860}{91}q^2\prod_{n=1}^{\infty}(1-q^n)(1-q^{2n})(1-q^{3n})^5(1-q^{6n})^5.$$

By (23) and (24), we see that

$$\begin{aligned}
& \left(3L(q^3)-2L(q^2)\right)M\left(q^2\right) \\
& (1-72\sum_{n=1}^{\infty}\sigma(n)q^{3n}+48\sum_{n=1}^{\infty}\sigma(n)q^{2n})(1+240\sum_{n=1}^{\infty}\sigma_3(n)q^{2n}) \\
= \; & 1-72\sum_{n=1}^{\infty}\sigma(n)q^{3n}+48\sum_{n=1}^{\infty}\sigma(n)q^{2n}+240\sum_{n=1}^{\infty}\sigma_3(n)q^{2n} \\
& -17280\sum_{n=1}^{\infty}\sigma(n)q^{3n}\sum_{n=1}^{\infty}\sigma_3(n)q^{2n}+11520\sum_{n=1}^{\infty}\sigma(n)q^{2n}\sum_{n=1}^{\infty}\sigma_3(n)q^{2n}.
\end{aligned} \tag{63}$$

Appealing to (16) and (25), we obtain

$$\begin{aligned}
& \frac{10}{637}N(q)-\frac{1074}{637}N\left(q^2\right)+\frac{81}{637}N\left(q^3\right)+\frac{1620}{637}N\left(q^6\right) \\
& -\frac{18180}{91}q\prod_{n=1}^{\infty}(1-q^n)^5(1-q^{2n})^5(1-q^{3n})(1-q^{6n}) \\
& +\frac{2700}{13}q\prod_{n=1}^{\infty}(1-q^n)^6(1-q^{3n})^6 \\
& -\frac{4860}{91}q^2\prod_{n=1}^{\infty}(1-q^n)(1-q^{2n})(1-q^{3n})^5(1-q^{6n})^5 \\
= \; & \frac{10}{637}\left(1-504\sum_{n=1}^{\infty}\sigma_5(n)q^n\right)-\frac{1074}{637}\left(1-504\sum_{n=1}^{\infty}\sigma_5(n)q^{2n}\right) \\
& +\frac{81}{637}\left(1-504\sum_{n=1}^{\infty}\sigma_5(n)q^{3n}\right)+\frac{1620}{637}\left(1-504\sum_{n=1}^{\infty}\sigma_5(n)q^{6n}\right) \\
& +\frac{180}{91}\sum_{n=1}^{\infty}u_4(n)q^n. \\
= \; & 1-\frac{720}{91}\sum_{n=1}^{\infty}\sigma_5(n)q^n+\frac{77328}{91}\sum_{n=1}^{\infty}\sigma_5(n)q^{2n} \\
& -\frac{5832}{91}\sum_{n=1}^{\infty}\sigma_5(n)q^{3n}-\frac{116640}{91}\sum_{n=1}^{\infty}\sigma_5(n)q^{6n}+\frac{180}{91}\sum_{n=1}^{\infty}u_4(n)q^n.
\end{aligned} \tag{64}$$

Combining (62), (63) and (64) we have

$$\begin{aligned}
&\sum_{n=1}^{\infty}\sigma(n)q^{6n}\sum_{n=1}^{\infty}\sigma_3(n)q^n\\
=\;&\frac{1}{2184}\sum_{n=1}^{\infty}\sigma_5(n)q^n-\frac{179}{3640}\sum_{n=1}^{\infty}\sigma_5(n)q^{2n}+\frac{27}{7280}\sum_{n=1}^{\infty}\sigma_5(n)q^{3n}\\
&+\frac{27}{364}\sum_{n=1}^{\infty}\sigma_5(n)q^{6n}+\frac{1}{72}\sum_{n=1}^{\infty}\sigma_3(n)q^{2n}+\frac{1}{360}\sum_{n=1}^{\infty}\sigma(n)q^{2n}\\
&-\frac{1}{240}\sum_{n=1}^{\infty}\sigma(n)q^{3n}+\frac{2}{3}\sum_{n=1}^{\infty}\sigma(n)q^{2n}\sum_{n=1}^{\infty}\sigma_3(n)q^{2n}\\
&-\frac{1}{8736}\sum_{n=1}^{\infty}u_4(n)q^n. \qquad (65)
\end{aligned}$$

For $n\in\mathbb{N}$, equating the coefficients of q^n on both sides of (65) and using (2) we have

$$\begin{aligned}
W_{3,2}^{1,3}(n) =\;& \frac{1}{2184}\sigma_5(n)-\frac{179}{3640}\sigma_5\left(\frac{n}{2}\right)+\frac{27}{7280}\sigma_5\left(\frac{n}{3}\right)+\frac{27}{364}\sigma_5\left(\frac{n}{6}\right)+\frac{1}{72}\sigma_3\left(\frac{n}{2}\right)\\
&+\frac{1}{360}\sigma\left(\frac{n}{2}\right)-\frac{1}{240}\sigma\left(\frac{n}{3}\right)+\frac{2}{3}W_{1,1}^{1,3}(\frac{n}{2})-\frac{1}{8736}u_4(n). \qquad (66)
\end{aligned}$$

Identity (12) follows from (3) and (66). □

3. Proof of Theorem 2

For $l\in\mathbb{N}_{\mathbf{0}}$ we set,

$$r_1(l)=\operatorname{card}\left\{(x_1,\ldots,x_4)\in\mathbb{Z}^4 : l=x_1^2+x_1x_2+x_2^2+x_3^2+x_3x_4+x_4^2\right\}, \qquad (67)$$

and let $r_2(l), r_3(l)$ and $r_4(l)$ be respectively $N(1,1,1,1)$, $N(1,1,1,2)$ and $N(1,2,2,2)$ where

(68)

$$N(a_1,a_2,a_3,a_4;n)=\operatorname{card}\left\{\begin{array}{c}(x_1,\ldots,x_8)\in\mathbb{Z}^8 : l=a_1(x_1^2+x_1x_2+x_2^2)+a_2(x_3^2+x_3x_4+x_4^2)\\ +a_3(x_5^2+x_5x_6+x_6^2)+a_4(x_7^2+x_7x_8+x_8^2).\end{array}\right\}$$

Obviously $r_i(0)=1$ for any $i\in\{1,2,3,4\}$. It is known (see for example [2]) that

$$r_1(l)=12\sigma(l)-36\sigma(\frac{l}{3}),\; l\in\mathbb{N}. \qquad (69)$$

Lomadze [21] proved that

$$r_2(l) = 24\sigma_3(l) + 216\sigma_3(\frac{l}{3}),\ l \in \mathbb{N}. \tag{70}$$

The following two formulae was proved by Köklüce [22].

$$r_3(l) = 18\sigma_3(l) - 48\sigma_3(\frac{l}{2}) - 162\sigma_3(\frac{l}{3}) + 432\sigma_3(\frac{l}{6}),\ l \in \mathbb{N} \tag{71}$$

and

$$r_4(l) = 6\sigma_3(l) - 36\sigma_3(\frac{l}{2}) - 54\sigma_3(\frac{l}{3}) + 324\sigma_3(\frac{l}{6}),\ l \in \mathbb{N}. \tag{72}$$

Proof. We just prove (ii) in detail. The remaining can be proved in a similar way.

It is clear that

$$N(1,1,1,1,2,2;n) = \sum_{\substack{l,m\in\mathbb{N}_0\\ 2l+m=n}} r_1(l)r_2(m) = r_1(0)r_2(n) + r_1(n)r_2(0) + \sum_{\substack{l,m\in\mathbb{N}\\ 2l+m=n}} r_1(l)r_2(m). \tag{73}$$

Thus using (69) and (70) we have

$$\begin{aligned}
& N(1,1,1,1,2,2;n) - (24\sigma_3(n) + 216\sigma_3(\frac{n}{2}) + 12\sigma(\frac{n}{2}) - 36\sigma(\frac{n}{6})) \\
= & \sum_{\substack{l,m\in\mathbb{N}\\ 2l+m=n}} (12\sigma(l) - 36\sigma(\frac{l}{3}))(24\sigma_3(m) + 216\sigma_3(\frac{m}{3})) \\
= & 288 \sum_{\substack{l,m\in\mathbb{N}\\ 2l+m=n}} \sigma(l)\sigma_3(m) + 2592 \sum_{\substack{l,m\in\mathbb{N}\\ 2l+m=n}} \sigma(l)\sigma_3(\frac{m}{3}) - 864 \sum_{\substack{l,m\in\mathbb{N}\\ 2l+m=n}} \sigma(\frac{l}{3})\sigma_3(m) \\
& -7776 \sum_{\substack{l,m\in\mathbb{N}\\ 2l+m=n}} \sigma(\frac{l}{3})\sigma_3(\frac{m}{3}) \\
= & 288W_{2,1}^{1,3}(n) + 2592W_{2,3}^{1,3}(n) - 864W_{6,1}^{1,3}(n) - 7776W_{2,1}^{1,3}(\frac{n}{3})
\end{aligned}$$

Appealing to (5), (9) and (10) and adding $24\sigma_3(n) + 216\sigma_3(\frac{n}{2}) + 12\sigma(\frac{n}{2}) - 36\sigma(\frac{n}{6})$ to both sides of the equation we obtain the desired result.

For (i) use $N(1,1,1,1,1,2;n) = \sum_{\substack{l,m\in\mathbb{N}_0\\ l+m=n}} r_1(l)r_3(m)$

and

for (iii) use $N(1,1,1,2,2,2;n) = \sum_{\substack{l,m\in\mathbb{N}_0 \\ l+m=n}} r_1(l)r_4(m).$

$\square$

Denoting the right hand side of (18) by $S(1,1,1,1,2,2;n)$, we give the first ten values of $N(1,1,1,1,2,2;n)$ and $S(1,1,1,1,2,2;n)$ in Table 1 to illustrate the equations.

Table 1 The first ten values of $S(1,1,1,1,2,2;n)$ and $N(1,1,1,1,2,2;n)$

n	$S(1,1,1,1,2,2;n)$	$\sigma_5(n)$	$u_1(n)$	$u_2(n)$	$N(1,1,1,1,2,2;n)$
1	24	1	101	-2	24
2	228	33	362	-36	228
3	1176	244	-27	606	1176
4	4380	1057	1660	-2216	4380
5	14544	3126	-3138	2484	14544
6	36804	8052	-486	2172	36804
7	76800	16808	7192	-7408	76800
8	175692	33825	13352	4464	175692
9	244824	59293	8181	-162	244824
10	522648	103158	-12804	9768	522648

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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