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FIXED POINTS FOR INTIMATE MAPPINGS

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Abstract. In this paper, we introduce (ψ,ϕ)-weak contraction condition that involves cubic terms of distance function. We prove some fixed point theorems for pairs of intimate mappings satisfying newly introduced contraction condition and generalize the result of Murthy and Prasad [14] and Jain *et al.* [8]. At the end, an application for integral type contraction condition is given.

Keywords: weak contraction; (ψ,ϕ)-weak contraction; intimate mappings; integral type contraction condition.
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1. Introduction and Preliminaries

Banach contraction principle [2] which is known as the basic tool of fixed point theory ensures the existence of a unique fixed point for every contraction mapping T(say) defined on a complete metric space E. The mapping T in Banach contraction principle is always uniformly continuous. For the last ten decades, authors are continuously trying to extend and generalize the Banach contraction principle in various directions.

One of the directions of generalization of Banach contraction principle concerns the coincidence points and common fixed points of pair of mappings satisfying contractive conditions.

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The use of notion of commutative mappings in fixed point theory literature became a turning moment. The first attempt to relax commutative condition of mapping to weak commutative condition was initiated by Sessa [17]. In 1986, Jungck [10] further weakened the notion of commutativity/weak commutative to compatible mappings. In 1993, Jungck, Murthy and Cho [12] further generalized the notion of compatible mappings to compatible mappings of type (A). The process of generalizing the concept of compatible mappings still going on.
Now, we recall some basic concepts which are useful for our work.

Definition 1.1. Let (E,d) be a metric space. Two mappings $S,T : E \to E$ are said to be compatible [10] if and only if

$$\lim_{n\to\infty} d(STu_n, TSu_n) = 0,$$

whenever $\{u_n\}$ is a sequence in E such that $\lim_{n\to\infty} Su_n = \lim_{n\to\infty} Tu_n = z$, for some $z \in E$.

Definition 1.2. Let (E,d) be a metric space. Two mappings $S,T : E \to E$ are said to be compatible of type (A) [12] if

$$\lim_{n\to\infty} d(SSu_n, TSu_n) = 0 \text{ and } \lim_{n\to\infty} d(TTu_n, STu_n) = 0,$$

whenever $\{u_n\}$ is a sequence in E such that $\lim_{n\to\infty} Su_n = \lim_{n\to\infty} Tu_n = z$, for some $z \in E$.

In 2001, Shahu and Dhagat [16] generalized the concept of compatible mappings of type (A) and introduce the concept of intimate mappings as follows.

Definition 1.3. Let S and T be two self mappings on a metric space E. Then S and T are said to be

(a) $T-$intimate if

$$\alpha d(TSu_n, Tu_n) \leq \alpha d(SSu_n, Su_n),$$

where $\alpha = \liminf$ or $\limsup$ and $\{u_n\}$ is a sequence in E such that $\lim_n Su_n = \lim_n Tu_n = z$, for some $z \in E$.

(a) $S-$intimate if

$$\alpha d(STu_n, Su_n) \leq \alpha d(TTu_n, Tu_n),$$

where $\alpha = \liminf$ or $\limsup$ and $\{u_n\}$ is a sequence in E such that $\lim_n Su_n = \lim_n Tu_n = z$, for some $z \in E$.

Preposition 1.1. *If S and T are compatible mappings of type (A), then S and T are S-intimate mappings or T-intimate mappings.*

Proof. *Since $d(STu_n, Su_n) \le d(STu_n, TTu_n) + d(TTu_n, Tu_n)$, for $n \ge 1$.*
Therefore, $\alpha d(STu_n, Su_n) \le \alpha 0 + \alpha d(TTu_n, Tu_n)$,
implies that $\alpha d(STu_n, Su_n) \le \alpha d(TTu_n, Tu_n)$, whenever $\{u_n\}$ is a sequence in E such that $\lim_{n\to\infty} Su_n = \lim_{n\to\infty} Tu_n = z$, for some $z \in E$.
Thus, the pair (S,T) is $S-$intimate. Similarly, one can prove that the pair (S,T) is $T-$intimate.

Converse of the above Preposition 1.1 may not true.

Example 1.1. *Let $E = [0,5]$ be a metric space with usual metric d and $S,T : E \to E$ be defined as follows*

$$Su = \frac{3}{u+3}, \quad \text{for all } u \in E, \qquad Tu = \frac{2}{u+2}, \quad \text{for all } u \in E.$$

Let $\{u_n\}$ be a sequence such that $u_n = \frac{1}{n}$, then $\lim_{n\to\infty} Su_n = 1$ and $\lim_{n\to\infty} Tu_n = 1$.
Also, $d(STu_n, Su_n) \to \frac{1}{4}$ and $d(TTu_n, Tu_n) \to \frac{1}{3}$ as $n \to \infty$. Therefore,

$$\lim_{n\to\infty} d(STu_n, Su_n) < \lim_{n\to\infty} d(TTu_n, Tu_n).$$

Hence, the pair (S,T) is $S-$intimate, but $d(STu_n, TTu_n) \to \frac{1}{12}$, as $n \to \infty$, so S and T are not compatible mappings of type (A).

Preposition 1.2. *Let S and T be two self mappings on a metric space (E,d). Suppose that pair (S,T) is a pair of T-intimate mappings and $Su = Tu = z$, $z \in E$.Then $d(Tz,z) \le d(Sz,z)$.*

Proof. *Assume that $u_n = u$, for all $n \ge 1$. So, $Su_n \to Su = z$ and $Tu_n \to Tu = z$.*
Since the pair (S,T) is $T-$intimate, then

$$\begin{aligned} d(TSu, Tu) &= \lim_{n\to\infty} d(TSu_n, Tu_n) \\ &\le \lim_{n\to\infty} d(SSu_n, Su_n) \\ &= d(SSu, Su), \end{aligned}$$

i.e., $d(Tz,z) \leq d(Sz,z)$.

2. Fixed Points

In 1969, Boyd and Wong [5] introduced ϕ contraction condition of the form $d(Tu,Tv) \leq \phi(d(u,v))$ for all $u,v \in E$, where T is a self map on a complete metric space E and $\phi : [0,\infty) \to [0,\infty)$ is an upper semi continuous function from right such that $0 \leq \phi(t) < t$ for all $t > 0$. In 1997, Alber and Guerre- Delabriere [1] generalized ϕ contraction to $\phi-$weak contraction in Hilbert spaces, which was further extended and proved by Rhoades [15] in complete metric space.

A self map T on a complete metric space is said to be a $\phi-$ weak contraction if for each $u,v \in E$, there exists a continuous non-decreasing function $\phi : [0,\infty) \to [0,\infty)$ satisfying $\phi(t) > 0$, for all $t > 0$ and $\phi(t) = 0$ if and only if $t = 0$ such that

$$d(Tu,Tv) \leq d(u,v) - \phi(d(u,v)). \tag{2.1}$$

The function ϕ in the above inequality (2.1) is known as control function or altering distance function. The notion of control function was given by Khan *et al.* [13] as follows.

Definition 2.1. [13] An altering distance is a function $\phi : [0,\infty) \to [0,\infty)$ satisfying the following

(i) ϕ is an increasing and continuous function,
(ii) $\phi(t) = 0$ if and only if $t = 0$.

In 2009, Zhang and Song [18] gave the notion of generalized $\phi-$ weak contraction by generalizing the concept of $\phi-$weak contraction.

Definition 2.2. [18] Two self mappings S and T on a metric space (E,d) are said to be generalized $\phi-$weak contractions if there exists a mapping $\phi : [0,\infty) \to [0,\infty)$ with $\phi(t) > 0$ for all $t > 0$ and $\phi(0) = 0$ such that

$$d(Su,Tv) \leq M(u,v) - \phi(M(u,v)) \text{ for all } u,v \in E,$$

where $M(u,v) = \max\{d(u,v), d(u,Su), d(v,Tv), \frac{d(u,Tv)+d(v,Su)}{2}\}$.

In 2013, Murthy and Prasad [14] introduced a weak contraction that involves cubic terms of distance function.

Theorem 2.1. [14] *Let T be a self-map on a complete metric space E satisfying:*

$$\begin{aligned}[1+pd(u,v)]d^2(Tu,Tv) \le p\max\Big\{&\frac{1}{2}[d^2(u,Tu)d(v,Tv)+d(u,Tu)d^2(v,Tv)],\\ &d(u,Tu)d(u,Tv)d(v,Tu), d(u,Tv)d(v,Tu)d(v,Tv)\Big\}\\ &+m(u,v)-\phi(m(u,v))\end{aligned}$$

where

$$\begin{aligned}m(u,v)=\max\Big\{&d^2(u,v), d(u,Tu)d(v,Tv), d(u,Tv)d(v,Tu),\\ &\frac{1}{2}[d(u,Tu)d(u,Tv)+d(v,Tu)d(v,Tv)]\Big\},\end{aligned}$$

where $p \ge 0$ is a real number and $\phi : [0,\infty) \to [0,\infty)$ is a continuous function with $\phi(t) = 0$ iff $t = 0$ and $\phi(t) > 0$ for each $t > 0$. Then T has a unique fixed point in E.

In 2017, Jain *et al.* [7] generalized Theorem 2.1 for pairs of commuting mappings and in 2018, Jain and Kumar [8] generalized Theorem 2.1 for the pairs of intimate mappings.

In this paper, we will prove fixed point theorems for pairs of intimate mappings by using the control function $\psi \in \Psi$ and generalize the result of Jain and Kumar [8] and Murthy and Prasad [14], where Ψ is a collection of all functions $\psi : [0,\infty)^4 \to [0,\infty)$ satisfying the following conditions:

- (ψ_1) ψ is non decreasing and upper semi continuous in each coordinate variables,
- (ψ_2) $\Delta(t) = \max\{\psi(t,t,0,0), \psi(0,0,0,t), \psi(0,0,t,0), \psi(t,t,t,t)\} \le t$, for each $t > 0$.

Let Φ be a collection of all the functions $\phi : [0,\infty) \to [0,\infty)$ satisfying the following conditions:

- (ϕ_1) ϕ is a continuous function,
- (ϕ_2) $\phi(t) > t$ for each $t > 0$ and $\phi(0) = 0$.

Now, we prove our main results.

Theorem 2.2. *Let (E,d) be a metric space and f, g, S and T be four self mappings on E satisfying the following conditions:*

(C_1) *$S(E) \subset g(E)$ and $T(E) \subset f(E)$,*

(C_2) *pair (f,S) is $f-$intimate and (g,T) is $g-$intimate,*

(C_3) *$f(E)$ is a complete subspace,*

(C_4) *for $\psi \in \Psi$, $\phi \in \Phi$, real number $p > 0$ and for all $u, v \in E$,*

$$
\begin{aligned}
&[1+pd(fu,gv)]d^2(Su,Tv) \leq \\
&\qquad p\psi\Big(d^2(fu,Su)d(gv,Tv), d(fu,Su)d^2(gv,Tv),\\
&\qquad\qquad d(fu,Su)d(fu,Tv)d(gv,Su),\\
&\qquad\qquad d(fu,Tv)d(gv,Su)d(gv,Tv)\Big)\\
&\qquad +m(fu,gv) - \phi(m(fu,gv)),
\end{aligned}
$$

where

$$
\begin{aligned}
m(fu,gv) = \max\Big\{&d^2(fu,gv), d(fu,Su)d(gv,Tv), d(fu,Tv)d(gv,Su),\\
&\frac{1}{2}[d(fu,Su)d(fu,Tv)+d(gv,Su)d(gv,Tv)]\Big\}.
\end{aligned}
$$

Then f, g, S and T have a unique common fixed point in E.

Proof. *Let $u_0 \in E$ be an arbitrary point. Since $S(E) \subset g(E)$ and $T(E) \subset f(E)$, therefore one can find u_1 and u_2 such that $Su_0 = gu_1 = v_0$ and $Tu_1 = fu_2 = v_1$. Continuing in this fashion, one can construct sequences such that*

$$v_{2n} = Su_{2n} = gu_{2n+1} \quad \text{and} \quad v_{2n+1} = Tu_{2n+1} = fu_{2n+2}, \tag{2.2}$$

for each $n = 0,1,2,3....$ First, we shall prove that $\lim_{n\to\infty} d(v_n, v_{n+1}) = 0$. For simplicity, let us denote

$$\gamma_n = d(v_n, v_{n+1)}, n = 0,1,2,3,.... \tag{2.3}$$

Now, we prove that $\{\gamma_n\}$ is non-increasing sequence, i.e., $\gamma_{n+1} \leq \gamma_n$ for $n = 1,2,3,....$

Case I. If n is even. By taking $u = u_{2n}$ and $v = u_{2n+1}$ in (C_4), we get

$$
\begin{aligned}
[1+pd(fu_{2n},gu_{2n+1})]d^2(Su_{2n},Tu_{2n+1}) \leq p\,\psi\Big(& d^2(fu_{2n},Su_{2n})d(gu_{2n+1},Tu_{2n+1}),\\
& d(fu_{2n},Su_{2n})d^2(gu_{2n+1},Tu_{2n+1}),\\
& d(fu_{2n},Su_{2n})d(fu_{2n},Tu_{2n+1})d(gu_{2n+1},Su_{2n}),\\
& d(fu_{2n},Tu_{2n+1})d(gu_{2n+1},Su_{2n})d(gu_{2n+1},Tu_{2n+1})\Big)\\
& + m(fu_{2n},gu_{2n+1}) - \phi(m(fu_{2n},gu_{2n+1})),
\end{aligned}
$$

where

$$
\begin{aligned}
m(fu_{2n},gu_{2n+1}) = \max\Big\{ & d^2(fu_{2n},gu_{2n+1}), d(fu_{2n},Su_{2n})d(gu_{2n+1},Tu_{2n+1}),\\
& d(fu_{2n},Tu_{2n+1})d(gu_{2n+1},Su_{2n},\\
& \frac{1}{2}[d(fu_{2n},Su_{2n})d(fu_{2n},Tu_{2n+1})+\\
& d(gu_{2n+1},Su_{2n})d(gu_{2n+1},Tu_{2n+1})]\Big\}.
\end{aligned}
$$

Using equations (2.2) and (2.3) in the above inequality, we have

$[1+p\gamma_{2n-1}]\gamma_{2n}{}^2 \leq p\psi\big(\gamma_{2n-1}{}^2\gamma_{2n}, \gamma_{2n-1}\gamma_{2n}{}^2, 0, 0\big)+$

$$
m(v_{2n-1},v_{2n}) - \phi\left(m(v_{2n-1},v_{2n})\right), \tag{2.4}
$$

where

$m(v_{2n-1},v_{2n}) = \max\left\{\gamma_{2n-1}{}^2, \gamma_{2n-1}\gamma_{2n}, 0, \frac{1}{2}\left[\gamma_{2n-1}d(v_{2n-1},v_{2n+1})+0\right]\right\}.$

By triangular inequality, we have

$$
\begin{aligned}
d(v_{2n-1},v_{2n+1}) \leq & d(v_{2n-1},v_{2n}) + d(v_{2n},v_{2n+1})\\
= & \gamma_{2n-1}+\gamma_{2n}.
\end{aligned}
$$

Hence,

$$
m(v_{2n-1},v_{2n}) \leq \max\left\{\gamma_{2n-1}{}^2, \gamma_{2n-1}\gamma_{2n}, 0, \frac{1}{2}\left[\gamma_{2n-1}(\gamma_{2n-1}+\gamma_{2n})\right]\right\}. \tag{2.5}
$$

Now we claim that $\{\gamma_{2n}\}$ is non-increasing. Suppose it is not possible, i.e., $\gamma_{2n-1} < \gamma_{2n}$, then by using the inequality (2.5) with the property of ϕ and ψ, equation (2.4) reduces to

$$[1+p\gamma_{2n-1}]{\gamma_{2n}}^2 \leq p\gamma_{2n-1}{\gamma_{2n}}^2 + \gamma_{2n-1}\gamma_{2n} - \phi(\gamma_{2n-1}\gamma_{2n}),$$

i.e., ${\gamma_{2n}}^2 < \gamma_{2n}^2$, a contradiction. Therefore, $\gamma_{2n} \leq \gamma_{2n-1}$. In a similar way, if n is odd, then we can obtain $\gamma_{2n+1} \leq \gamma_{2n}$. It follows that the sequence $\{\gamma_n\}$ is non-increasing. Now we prove that $\lim_{n\to\infty}\gamma_n = 0$.

Suppose $\lim_{n\to\infty}\gamma_n \neq 0$, i.e.,

$$\lim_{n\to\infty}\gamma_n = t, \text{ for some } t > 0. \tag{2.6}$$

Taking $n \to \infty$,using the inequality (2.5), the equation (2.6) with the property of ϕ, ψ the inequality (2.4) reduces to

$$[1+pt]t^2 \leq pt^3 + t^2 - \phi(t^2).$$

This implies that $\phi(t^2) \leq 0$, a contradiction to the definition of ϕ.

Therefore,

$$\lim_{n\to\infty}\gamma_n = \lim_{n\to\infty} d(v_n, v_{n-1}) = 0. \tag{2.7}$$

Now, we prove that sequence $\{v_n\}$ is a Cauchy sequence in E. Let us assume that $\{v_n\}$ is not a Cauchy sequence, so there exists an $\varepsilon > 0$, for which, one can find two sequences of positive integers $\{m(k)\}$ and $\{n(k)\}$ such that

$$d(v_{m(k)}, v_{n(k)}) \geq \varepsilon \quad and \quad d(v_{m(k)}, v_{n(k)-1}) < \varepsilon, \tag{2.8}$$

for all positive integers $k, n(k) > m(k) > k$.

Now, $\varepsilon \leq d(v_{m(k)}, v_{n(k)}) \leq d(v_{m(k)}, v_{n(k)-1}) + d(v_{n(k)-1}, v_{n(k)})$.

Letting $k \to \infty$, we get

$$\lim_{k\to\infty} d(v_{m(k)}, v_{n(k)}) = \varepsilon. \tag{2.9}$$

From the triangular inequality, we have,

$$|d(v_{n(k)}, v_{m(k)+1}) - d(v_{m(k)}, v_{n(k)})| \leq d(v_{m(k)}, v_{m(k)+1}).$$

Taking limit as $k \to \infty$ and using equations (2.7) and (2.9), we have

$$\lim_{k\to\infty} d(v_{n(k)}, v_{m(k)+1}) = \varepsilon. \tag{2.10}$$

Again from the triangular inequality, we have

$$|d(v_{m(k)}, v_{n(k)+1}) - d(v_{m(k)}, v_{n(k)})| \leq d(v_{n(k)}, v_{n(k)+1})$$

Taking limit as $k \to \infty$ and using equations (2.7) and (2.9), we have

$$\lim_{k\to\infty} d(v_{m(k)}, v_{n(k)+1}) = \varepsilon. \tag{2.11}$$

Similarly, on using triangular inequality, we have

$$|d(v_{m(k)+1}, v_{n(k)+1}) - d(v_{m(k)}, v_{n(k)})| \leq d(v_{m(k)}, v_{m(k)+1}) + d(v_{n(k)}, v_{n(k)+1})$$

Taking limit as $k \to \infty$ in the above inequality and using equations (2.7) and (2.9), we have

$$\lim_{k\to\infty} d(v_{n(k)+1}, v_{m(k)+1}) = \varepsilon. \tag{2.12}$$

On taking $u = u_{m(k)}$ and $v = u_{n(k)}$ and using equation (2.2) in (C_4), we get

$$\begin{aligned}
&[1 + pd(v_{m(k)-1}, v_{n(k)-1})]d^2(v_{m(k)}, v_{n(k)}) \leq \\
&\quad p\psi\Big(d^2(v_{m(k)-1}, v_{m(k)})d(v_{n(k)-1}, v_{n(k)}), \\
&\qquad d(v_{m(k)-1}, v_{m(k)})d^2(v_{n(k)-1}, v_{n(k)}), \\
&\qquad d(v_{m(k)-1}, v_{m(k)})d(v_{m(k)-1}, v_{n(k)})d(v_{n(k)-1}, v_{m(k)}), \\
&\qquad d(v_{m(k)-1}, v_{n(k)})d(v_{n(k)-1}, v_{m(k)})d(v_{n(k)-1}, v_{n(k)})\Big) \\
&\quad + m(v_{m(k)-1}, v_{n(k)-1}) - \phi\left(m(v_{m(k)-1}, v_{n(k)-1})\right),
\end{aligned}$$

where

$$m(v_{m(k)-1},v_{n(k)-1}) = \max\Big\{d^2(v_{m(k)-1},v_{n(k)-1}),$$
$$d(v_{m(k)-1},v_{m(k)})d(v_{n(k)-1},v_{n(k)}),$$
$$d(v_{m(k)-1},v_{n(k)})d(v_{n(k)-1},v_{m(k)}),$$
$$\frac{1}{2}\Big[d(v_{m(k)-1},v_{m(k)})d(v_{m(k)-1},v_{n(k)})+$$
$$d(v_{n(k)-1},v_{m(k)})d(v_{n(k)-1},v_{n(k)})\Big]\Big\}.$$

Letting $k\to\infty$ and using equations (2.7)-(2.12) with the property of ϕ and ψ, we obtain

$$[1+p\varepsilon]\varepsilon^2 \le p\psi(0,0,0,0)+\varepsilon^2-\phi(\varepsilon^2) < \varepsilon^2,$$

which is a contradiction. Thus, the sequence $\{v_n\}$ is a Cauchy sequence in E. Since $f(E)$ is a complete subspace, therefore, there exists $z\in f(E)$ such that $v_{2n+1} = Tu_{2n+1} = fu_{2n+2}$ converges to z as $n\to\infty$. Consequently, one can find a point $t\in E$ such that $ft = z$. Now, $\{v_n\}$ is a Cauchy sequence containing a convergent subsequence $\{v_{2n+1}\}$, so, the sequence $\{v_n\}$ is also convergent. $\{v_{2n}\}$ being a subsequence of a convergent sequence $\{v_n\}$, is also convergent. So, the sub sequences $\{Su_{2n}\}$, $\{fu_{2n+2}\}$, $\{Tu_{2n+1}\}$, and $\{gu_{2n+1}\}$ also converges to the same point z.

We claim that $St = z$. For this substituting $u = t$ and $v = u_{2n+1}$ in (C_4) and letting $n\to\infty$, we have

$$[1+pd(ft,z)]d^2(St,z) \le p\psi(0,0,0,0)+m(ft,z)-\phi(m(ft,z)),$$

where

$$m(ft,z) = \max\Big\{d^2(ft,z), d(ft,St)d(z,z), d(ft,z)d(z,St),$$
$$\frac{1}{2}[d(ft,St)d(ft,z)+d(z,St)d(z,z)]\Big\} = 0.$$

On solving, we get $d^2(St,z) = 0$ which implies that $St = z$. Therefore, we have $z = ft = St$. Since $S(E)\subset g(E)$, therefore for this z there exists a point $x\in E$ such that $gx = St = z$.

Now, we prove that $Tx = z$. For this, taking $u = t$, $v = x$ in (C_4), we get

$$[1+pd(ft,gx)]d^2(St,Tx) \leq p\psi(0,0,0,0)+m(ft,gx)-\phi(m(ft,gx)),$$

where

$$m(ft,gx) = \max\Big\{d^2(ft,gx), d(ft,St)d(gx,Tx), d(ft,Tx)d(gx,St), \frac{1}{2}[d(ft,St)d(ft,Tx)+d(gx,St)d(gx,Tx)]\Big\} = 0.$$

On simplification, we get $d^2(z,Tx) = 0$, which implies that $z = Tx$, i.e., $gx = z = Tx$.

Thus, $z = ft = St = gx = Tx$. The pair (f,S) is $f-$intimate and $St = ft = z$, so by Proposition 1.2, we have $d(fz,z) \leq d(Sz,z)$.

Let us suppose that $Sz \neq z$, then from the inequality (C_4), we have

$$[1+pd(fz,gx)]d^2(Sz,Tx) \leq p\psi(0,0,0,0)+m(fz,gx)-\phi(m(fz,gx)),$$

where

$$m(fz,gx) = \max\Big\{d^2(fz,gx), d(fz,Sz)d(gx,Tx), d(fz,Tx)d(gx,Sz), \frac{1}{2}[d(fz,Sz)d(fz,Tx)+d(gx,Sz)d(gx,Tx)]\Big\} = 0.$$

After simplification, we get $d^2(Sz,z) < 0$, a contradiction.Therefore, $Sz = z$.Similarly, we get $gz = Tz = z$. For the uniqueness, let us suppose thatw, z, $w \neq z$, be two common fixed points of f,g,S and T. Taking $u = w$, $v = z$ n (C_4), we get,

$$[1+pd(w,z)]d^2(w,z) \leq p\psi(0,0,0,0)+m(w,z)-\phi(m(w,z)),$$

where

$$m(w,z) = \max\Big\{d^2(w,z), d(w,w)d(z,z), d(w,z)d(z,w), \frac{1}{2}[d(w,w)d(w,z)+d(z,w)d(z,z)]\Big\} = d^2(w,z).$$

After simplification the above inequality reduce to $d^2(w,z) < 0$, a contradiction. So, $w = z$. This proves the uniqueness of the common fixed point of f,g,S and T. This completes the proof.

On substituting $f = g = S$ and $S = T$ in Theorem 2.2, one can deduce the following corollary which generalizes the result of Murthy and Prasad [14] and Jain *et al.* [7].

Corollary 2.1. *Let (E,d) be a metric space. Suppose that $S,T:E\to E$ are two mappings satisfying the following conditions*

(C_{1^*}) $T(E)\subset S(E)$,

(C_{2^*}) *the pair (S,T) is $S-$intimate,*

(C_{3^*}) *$T(E)$ is a complete subspace,*

(C_{4^*}) *for all $u,v\in E$, real number $p>0$, $\psi\in\Psi$, $\phi\in\Phi$,*

$$\begin{aligned}[1+pd(Su,Sv)]d^2(Tu,Tv) \leq & p\,\psi\bigg(d^2(Su,Tu)d(Sv,Tv), d(Su,Tu)d^2(Sv,Tv), \\ & d(Su,Tu)d(Su,Tv)d(Sv,Tu), d(Su,Tv)d(Sv,Tu)d(Sv,Tv)\bigg) \\ & + m(Su,Sv)-\phi(m(Su,Sv)),\end{aligned}$$

where

$$\begin{aligned}m(Su,Sv)=\max\Big\{ & d^2(Su,Sv), d(Su,Tu)d(Sv,Tv), d(Su,Tv)d(Sv,Tu), \\ & \frac{1}{2}[d(Su,Tu)d(Su,Tv)+d(Sv,Tu)d(Sv,Tv)]\Big\}.\end{aligned}$$

Then S and T have a unique common fixed point in E.

Next, we generalize the above Theorem 2.1 and Theorem 2.2 for six intimate mappings.

Theorem 2.3. *Let A, B,P,Q, S and T be six self-mappings on metric space (E,d) satisfying the conditions*

(P_1) *$A(E)\subset PQ(E)$ and $B(E)\subset ST(E)$,*

(P_2) *$PQ=QP$, $ST=TS$, $AT=TA$ and $BQ=QB$,*

(P_3) *the pair (A,ST) is $ST-$intimate and the pair (B,PQ) is $PQ-$intimate,*

(P_4) *$ST(E)$ is a complete subspace.*

(P_5) *for $\psi\in\Psi$, $\phi\in\Phi$, real number $p>0$ and for all $u,v\in E$,*

$$[1+pd(STu,PQv)]d^2(Au,Bv) \leq$$

$$p\psi\Big(d^2(STu,Au)d(PQv,Bv), d(STu,Au)d^2(PQv,Bv),$$

$$d(STu,Au)d(STu,Bv)d(PQv,Au), d(STu,Bv)d(PQv,Au)d(PQv,Bv)\Big)$$

$$+m(STu,Bv)-\phi(m(STu,PQv)),$$

where

$$m(STu,PQv) = \max\Big\{d^2(STu,PQv), d(STu,Au)d(PQv,Bv), d(STu,Bv)d(PQv,Au),$$

$$\frac{1}{2}[d(STu,Au)d(STu,Bv)+d(PQv,Au)d(PQv,Bv)]\Big\}.$$

Then A,B,S,T, P, and Q have a unique common fixed point in E.

Proof. *Let $u_0 \in E$ be arbitrary point. Using (P_1), one can find $u_1, u_2 \in E$ such that $Au_0 = PQu_1 = v_0$ and $Bu_1 = STu_2 = v_1$. Following it, one can construct sequences such that*

$$v_{2n} = Au_{2n} = PQu_{2n+1} \quad \text{and} \quad v_{2n+1} = Bu_{2n+1} = STu_{2n+2}, \tag{2.13}$$

for each $n = 0,1,2,3....$ For simplicity, let us denote

$$\beta_n = d(v_n, v_{n+1}), n = 0,1,2,3,.... \tag{2.14}$$

First, we prove that $\{\beta_n\}$ is non-increasing sequence, i.e., $\beta_{n+1} \leq \beta_n$ for $n = 1,2,3,....$

Case I. If n is even. By taking $u = u_{2n}$ and $v = u_{2n+1}$ in (P_5), we get

$$[1+pd(STu_{2n},PQu_{2n+1})]d^2(Au_{2n},Bu_{2n+1})$$

$$\leq p\,\psi\Big(d^2(STu_{2n},Au_{2n})d(PQu_{2n+1},Bu_{2n+1}),$$

$$d(STu_{2n},Au_{2n})d^2(PQu_{2n+1},Bu_{2n+1}),$$

$$d(STu_{2n},Au_{2n})d(STu_{2n},Bu_{2n+1})d(PQu_{2n+1},Au_{2n}),$$

$$d(STu_{2n},Bu_{2n+1})d(PQu_{2n+1},Au_{2n})d(PQu_{2n+1},Bu_{2n+1})\Big)$$

$$+m(STu_{2n},PQu_{2n+1})-\phi(m(STu_{2n},PQu_{2n+1})),$$

where

$$m(STu_{2n},PQu_{2n+1}) = \max\Big\{d^2(STu_{2n},PQu_{2n+1}), d(STu_{2n},Au_{2n})d(PQu_{2n+1},Bu_{2n+1}),$$
$$d(STu_{2n},Bu_{2n+1})d(PQu_{2n+1},Au_{2n},$$
$$\frac{1}{2}[d(STu_{2n},Au_{2n})d(STu_{2n},Bu_{2n+1})+$$
$$d(PQu_{2n+1},Au_{2n})d(PQu_{2n+1},Bu_{2n+1})]\Big\}.$$

Using equations (2.13) and (2.14) in the above inequality, we have

$$[1+p\beta_{2n-1}]{\beta_{2n}}^2 \leq p\psi\big({\beta_{2n-1}}^2\beta_{2n}, \beta_{2n-1}{\beta_{2n}}^2, 0, 0\big)+$$

$$m(v_{2n-1},v_{2n}) - \phi\big(m(v_{2n-1},v_{2n})\big), \tag{2.15}$$

where $m(v_{2n-1},v_{2n}) = \max\big\{{\beta_{2n-1}}^2, \beta_{2n-1}\beta_{2n}, 0, \frac{1}{2}\big[\beta_{2n-1}d(v_{2n-1},v_{2n+1})+0\big]\big\}$.
Using triangular inequality, we get

$$d(v_{2n-1},v_{2n+1}) \leq d(v_{2n-1},v_{2n}) + d(v_{2n},v_{2n+1}) = \beta_{2n-1}+\beta_{2n}.$$

Hence,

$$m(v_{2n-1},v_{2n}) \leq \max\big\{{\beta_{2n-1}}^2, \beta_{2n-1}\beta_{2n}, 0, \frac{1}{2}\big[\beta_{2n-1}(\beta_{2n-1}+\beta_{2n})\big]\big\}. \tag{2.16}$$

Now we claim that $\{\beta_{2n}\}$ *is non-increasing. Suppose it is not possible, i.e.,* $\beta_{2n-1} < \beta_{2n}$*, then by using the inequality (2.16) with the property of* ϕ *and* ψ*, equation (2.15) reduces to*

$$[1+p\beta_{2n-1}]{\beta_{2n}}^2 \leq p\beta_{2n-1}{\beta_{2n}}^2 + \beta_{2n-1}\beta_{2n} - \phi(\beta_{2n-1}\beta_{2n}),$$

i.e., ${\beta_{2n}}^2 < \beta_{2n}^2$ *, a contradiction. Therefore,* $\beta_{2n} \leq \beta_{2n-1}$*. In a similar way, if n is odd, then we can obtain* $\beta_{2n+1} \leq \beta_{2n}$*. It follows that the sequence* $\{\beta_n\}$ *is non-increasing. Now we prove that* $\lim_{n\to\infty}\beta_n = 0$*. Suppose* $\lim_{n\to\infty}\beta_n \neq 0$*, i.e.,*

$$\lim_{n\to\infty}\beta_n = t, \text{ for some } t > 0. \tag{2.17}$$

Taking $n \to \infty$ *in inequality (2.15) and using the inequality (2.16), the equation (2.17) with the property of* ϕ*,* ψ*, we have*

$$[1+pt]t^2 \leq pt^3 + t^2 - \phi(t^2).$$

This implies that $\phi(t^2) \leq 0$, a contradiction to the definition of ϕ.

Therefore,

$$\lim_{n\to\infty} \beta_n = \lim_{n\to\infty} d(v_n, v_{n-1}) = 0. \tag{2.18}$$

Let us assume that $\{v_n\}$ is not a Cauchy sequence, so there exists an $\varepsilon > 0$, for which, one can find two sequences of positive integers $\{m(k)\}$ and $\{n(k)\}$ such that

$$d(v_{m(k)}, v_{n(k)}) \geq \varepsilon \quad \text{and} \quad d(v_{m(k)}, v_{n(k)-1}) < \varepsilon, \tag{2.19}$$

for all positive integers $k, n(k) > m(k) > k$.

Now

$$\varepsilon \leq d(v_{m(k)}, v_{n(k)}) \leq d(v_{m(k)}, v_{n(k)-1}) + d(v_{n(k)-1}, v_{n(k)}).$$

Letting $k \to \infty$, we get

$$\lim_{k\to\infty} d(v_{m(k)}, v_{n(k)}) = \varepsilon. \tag{2.20}$$

Now from the triangular inequality, we have,

$$|d(v_{n(k)}, v_{m(k)+1}) - d(v_{m(k)}, v_{n(k)})| \leq d(v_{m(k)}, v_{m(k)+1}).$$

Taking limit as $k \to \infty$ and using (2.18) and (2.20), we have

$$\lim_{k\to\infty} d(v_{n(k)}, v_{m(k)+1}) = \varepsilon. \tag{2.21}$$

Again from the triangular inequality, we have

$$|d(v_{m(k)}, v_{n(k)+1}) - d(v_{m(k)}, v_{n(k)})| \leq d(v_{n(k)}, v_{n(k)+1})$$

Taking limit as $k \to \infty$ and using (2.18) and (2.20), we have

$$\lim_{k\to\infty} d(v_{m(k)}, v_{n(k)+1}) = \varepsilon. \tag{2.22}$$

Similarly, on using triangular inequality, we have

$$|d(v_{m(k)+1}, v_{n(k)+1}) - d(v_{m(k)}, v_{n(k)})| \leq d(v_{m(k)}, v_{m(k)+1}) + d(v_{n(k)}, v_{n(k)+1})$$

Taking limit as $k \to \infty$ in the above inequality and using (2.18) and (2.20), we have

$$\lim_{k\to\infty} d(v_{n(k)+1}, v_{m(k)+1}) = \varepsilon. \tag{2.23}$$

On taking $u = u_{m(k)}$ and $v = u_{n(k)}$ and using equation (2.13) in (P_5), we get

$$\begin{aligned}
[1 + pd(v_{m(k)-1}, v_{n(k)-1})]d^2(v_{m(k)}, v_{n(k)}) \leq \\
p\psi\Big(& d^2(v_{m(k)-1}, v_{m(k)})d(v_{n(k)-1}, v_{n(k)}), \\
& d(v_{m(k)-1}, v_{m(k)})d^2(v_{n(k)-1}, v_{n(k)}), \\
& d(v_{m(k)-1}, v_{m(k)})d(v_{m(k)-1}, v_{n(k)})d(v_{n(k)-1}, v_{m(k)}), \\
& d(v_{m(k)-1}, v_{n(k)})d(v_{n(k)-1}, v_{m(k)})d(v_{n(k)-1}, v_{n(k)})\Big) \\
& +m(v_{m(k)-1}, v_{n(k)-1}) - \phi\big(m(v_{m(k)-1}, v_{n(k)-1})\big),
\end{aligned}$$

where

$$\begin{aligned}
m(v_{m(k)-1}, v_{n(k)-1}) = \max\Big\{ & d^2(v_{m(k)-1}, v_{n(k)-1}), \\
& d(v_{m(k)-1}, v_{m(k)})d(v_{n(k)-1}, v_{n(k)}), \\
& d(v_{m(k)-1}, v_{n(k)})d(v_{n(k)-1}, v_{m(k)}), \\
& \frac{1}{2}\Big[d(v_{m(k)-1}, v_{m(k)})d(v_{m(k)-1}, v_{n(k)}) + \\
& d(v_{n(k)-1}, v_{m(k)})d(v_{n(k)-1}, v_{n(k)})\Big]\Big\}.
\end{aligned}$$

Letting $k \to \infty$ and using equations (2.18)-(2.23) with the property of ϕ and ψ, we obtain

$$[1 + p\varepsilon]\varepsilon^2 \leq p\psi(0,0,0,0) + \varepsilon^2 - \phi(\varepsilon^2) < \varepsilon^2,$$

which is a contradiction. So, the sequence $\{v_n\}$ is a Cauchy sequence in E. Since $ST(E)$ is a complete subspace, so there exists a point $z \in St(E)$ such that $v_{2n+1} = Bu_{2n+1} = STu_{2n+2}$ converges to z as $n \to \infty$. Consequently, one can find a point $t \in E$ such that $STt = z$. Now, $\{v_n\}$ is a Cauchy sequence containing a convergent subsequence $\{v_{2n+1}\}$, so, the sequence $\{v_n\}$ is also convergent. $\{v_{2n}\}$ being a subsequence of a convergent sequence $\{v_n\}$, is also convergent.

So, the sub sequences $\{Au_{2n}\}$, $\{STu_{2n+2}\}$, $\{Bu_{2n+1}\}$, *and* $\{PQu_{2n+1}\}$ *also converges to the same point* z.*We claim that* $At = z$. *For this substituting* $u = t$ *and* $v = u_{2n+1}$ *in* (P_5) *and letting* $n \to \infty$, *we have*

$$[1+pd(STt,z)]d^2(At,z) \le p\psi(0,0,0,0) + m(STt,z) - \phi(m(STt,z)),$$

where

$$\begin{aligned} m(STt,z) = \max\Big\{ & d^2(STt,z), d(STt,At)d(z,z), \\ & d(STt,z)d(z,At), \\ & \frac{1}{2}[d(STt,At)d(STt,z) + d(z,At)d(z,z)]\Big\} = 0. \end{aligned}$$

On solving it, we get $d^2(At,z) = 0$ *which implies that* $At = z$.*Therefore, we have* $z = STt = At$.*Since* $A(E) \subset PQ(E)$, *therefore for this* z *there exists a point* $x \in E$ *such that* $PQx = At = z$. *Now, we claim that* $Bx = z$. *For this taking* $u = t$, $v = x$ *in* (P_5), *we get*

$$[1+pd(STt,PQx)]d^2(At,Bx) \le p\psi(0,0,0,0) + m(STt,PQx) - \phi(m(STt,PQx)),$$

where

$$\begin{aligned} m(STt,PQx) = \max\Big\{ & d^2(STt,PQx), d(STt,At)d(PQx,Bx), d(STt,Bx)d(PQx,At), \\ & \frac{1}{2}[d(STt,At)d(STt,Bx) + d(PQx,At)d(PQx,Bx)]\Big\} = 0. \end{aligned}$$

After simplification, we get $d^2(z,Bx) = 0$, *which implies that* $z = Bx$, *i.e.,* $PQx = z = Bx$. *Thus,* $z = STt = At = PQx = Bx$. *The pair* (A,ST) *is* $ST-$*intimate and* $STt = At = z$, *so by Proposition 1.2, we have* $d(STz,z) \le d(Az,z)$. *Let us suppose that* $Az \ne z$, *then from the inequality* (P_5), *we have*

$$[1+pd(STz,PQx)]d^2(Az,Bx) \le p\psi(0,0,0,0) + m(STz,PQx) - \phi(m(STz,PQx)),$$

where

$$\begin{aligned} m(STz,PQx) = \max\Big\{ & d^2(STz,PQx), d(STz,Az)d(PQx,Bx), d(STz,Bx)d(PQx,Az), \\ & \frac{1}{2}[d(STz,Az)d(STz,Bx) + d(PQx,Az)d(PQx,Bx)]\Big\} = 0. \end{aligned}$$

After simplification, we get $d^2(Az,z) < 0$, a contradiction. Therefore, $Az = z$i.e., $STz = Az = z$. Similarly, we get $PQz = Bz = z$. Suppose that $Tz \neq z$. Substituting $u = Tz$, $v = u_{2n+1}$ in (P_5) and letting $n \to \infty$, we have

$$[1+pd(STTz,z)]d^2(ATz,z) \leq p\psi(0,0,0,0)+m(STTz,z)-\phi(m(STTz,z)),$$

where

$$m(STTz,z) = \max\Big\{d^2(STTz,z), d(STTz,ATz)d(z,z), d(STTz,z)d(z,ATz), \frac{1}{2}[d(STTz,ATz)d(STTz,z)+d(z,ATz)d(z,z)]\Big\} = d^2(Tz,z),$$

since $STTz = TSTz = Tz$ and $ATz = TAz = Tz$. On simplifying, we get

$$[1+pd(Tz,z)]d^2(Tz,z) \leq d^2(Tz,z)-\phi(d^2(Tz,z)) < 0,$$

which is a contradiction. Therefore, $Tz = z$ and $z = STz = Sz$.
Now we claim that $Qz = z$. For this taking $u = u_{2n}$, $v = Qz$ in (P_5) and letting $n \to \infty$, we get,

$$[1+pd(z,PQQz)]d^2(z,BQz) \leq p\psi(0,0,0,0)+m(z,PQQz)-\phi(m(z,PQQz)),$$

where

$$m(z,PQQz) = \max\Big\{d^2(z,PQQz), d(z,z)d(PQQz,BQz), d(z,BQz)d(PQQz,z), \frac{1}{2}[d(z,z)d(z,BQz)+d(PQQz,z)d(PQQz,BQz)]\Big\} = d^2(z,Qz),$$

since $PQQz = QPQz = Qz$ and $BQz = QBz$. After simplifying the above inequality, we obtain $d^2(z,Qz) = 0$, which implies that $Qz = z$. $z = PQz = Pz$.The uniqueness can be proved easily. Hence, z is a unique common fixed point of A,B,S, T, P and Q.

Remark 2.1. *If we consider the function $\psi : [0,\infty)^4 \to [0,\infty)$ defined by*

$$\psi(t_1,t_2,t_3,t_4) = \max\{\tfrac{1}{2}[t_1+t_2],t_3,t_4\},$$

in Theorem 2.2 and Theorem 2.3, then we conclude that our results generalize the Theorem 2.1 and the results of Jain et al. [8].

3. Application

In 2001, Branciari [6] obtained Banach contraction principle for mapping satisfying an integral type contraction condition. On the similar lines, we analyze our results for mappings satisfying a generalized $(\phi - \psi)-$weak contraction condition of integral type.

Theorem 3.1. *Let f,g,S and T be four self mappings on a metric space (E,d) satisfying the conditions (C_1), (C_2), (C_3) and*

(C_5) *for $u,v \in E$,*

$$\int_o^{M(u,v)} \gamma(t)\,dt \leq \int_o^{N(u,v)} \gamma(t)\,dt,$$

where

$$M(u,v) = [1 + pd(fu,gv)]d^2(Su,Tv),$$

$$N(u,v) = p\,\psi\Bigg(d^2(fu,Su)d(gv,Tv), d(fu,Su)d^2(gv,Tv), d(fu,Su)d(fu,Tv)d(gv,Su),$$
$$d(fu,Tv)d(gv,Su)d(gv,Tv)\Bigg) + m(fu,gv) - \phi(m(fu,gv)),$$

where

$$m(fu,gv) = \max\Big\{d^2(fu,gv), d(fu,Su)d(gv,Tv), d(fu,Tv)d(gv,Su),$$
$$\frac{1}{2}[d(fu,Su)d(fu,Tv) + d(gv,Su)d(gv,Tv)]\Big\},$$

$\psi \in \Psi$, $\phi \in \Phi$, $p > 0$ is a real number and $\gamma : [0,\infty) \to [0,\infty)$ is a Lebesgue integrable function which is summable on each compact subset of $[0,\infty)$ such that for each $\varepsilon > 0$, $\int_o^{\varepsilon} \gamma(t)\,dt > 0$. Then f,g,S and T have a unique common fixed point.

Proof. On putting $\gamma(t) = c$ (some non zero constant), it reduces to Theorem 2.2 .

Theorem 3.2. *Let A,B,S,T,P and Q be six self mappings on a metric space (E,d) satisfying the conditions (P_1), (P_2), (P_3), (P_4) and*

(P_6) *for $u,v \in E$,*

$$\int_o^{M(u,v)} \gamma(t)\,dt \leq \int_o^{N(u,v)} \gamma(t)\,dt,$$

where

$$M(u,v) = [1 + pd(STu, PQv)]d^2(Au, Bv),$$

$$N(u,v) = p\,\psi\Big(d^2(STu,Au)d(PQv,Bv), d(STu,Au)d^2(PQv,Bv), d(STu,Au)d(STu,Bv)d(PQv,Au), d(STu,Bv)d(PQv,Au)d(PQv,Bv)\Big) + m(STu,Bv) - \phi(m(STu,PQv)),$$

where

$$m(STu,PQv) = \max\Big\{d^2(STu,PQv), d(STu,Au)d(PQv,Bv), d(STu,Bv)d(PQv,Au), \frac{1}{2}[d(STu,Au)d(STu,Bv) + d(PQv,Au)d(PQv,Bv)]\Big\}.$$

$\psi \in \Psi$, $\phi \in \Phi$, $p > 0$ *is a real number and* $\gamma : [0,\infty) \to [0,\infty)$ *is a Lebesgue integrable function which is summable on each compact subset of* $[0,\infty)$ *such that for each* $\varepsilon > 0$, $\int_o^{\varepsilon} \gamma(t)\,dt > 0$. *Then* A,B,S,T,P *and* Q *have a unique common fixed point.*

Proof. On putting $\gamma(t) = c$ *(some non zero constant), it reduces to Theorem 2.3 .*

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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