Available online at http://scik.org
J. Math. Comput. Sci. 2022, 12:79
https://doi.org/10.28919/jmcs/7036
ISSN: 1927-5307

DIFFERENTIAL VALUE OF SPLITTING AND MIDDLE GRAPH OF SOME STANDARD GRAPHS

D. MURALIDHARAN[1,*], M.S. PAULRAJ[2], D. YOKESH[3]

[1]Department of Mathematics, Sri Sai Ram Institute of Technology, Chennai - 44, Tamil Nadu, India

[2]Department of Mathematics, A.M. Jain College, Chennai, Tamil Nadu, India

[3]Department of Mathematics, Kirirom Institute of Technology, Cambodia

Abstract. Let $G = (V,E)$ be a graph and X be a subset of V. Let $B(X)$ be the set of vertices in $V - X$ that has a neighbour in a set X. The differential of set X is defined as $\partial(X)$ is $|B(X)| - |X|$ and the differential of a graph is defined as

$$\partial(G) = \max\{\partial(X)/X \subset V\}.$$

In this paper, we obtain the differential value of middle and splitting graph for path, cycle, star, wheel and complete graph.

Keywords: middle graph; splitting graph.

2010 AMS Subject Classification: 05C38, 90B10, 05C69.

1. Introduction

The differential in a graph is a subject of increasing interest, both in pure and applied mathematics. In Particular, the study of the mathematical properties of the differential in graph, together with a variety of other kinds of differentials of a set, started in [5, 6]. The differential of a set D was considered in [3], where it was denoted by $\eta(D)$.

*Corresponding author

E-mail address: murali_08@yahoo.co.in

Received November 26, 2021

Throughout this paper, $G = (V,E)$ denote a simple graph. For graph, theoretical terminology was not given here, refer to Harary [4]. For a vertex $v \in V$, the *open neighbourhood* of v is the set $N(v) = \{u \in V/uv \in E\}$ and the *closed neighbourhood* of the set $N[v] = N(v) \cup \{v\}$. For a set $X \subset V$, its open neighbourhood is

$$N(X) = \bigcup_{v \in X} N(v)$$

and the closed neighbourhood is $N[X] = N(X) \cup X$. The subgraph induced by a set $S \subseteq V$ will be denoted by $\langle S \rangle$.

The *boundary* $B(X)$ of a set X is defined to be the set of vertices in $V - X$ dominated by the vertices in X, that is $B(X) = (V - X) \cap N(X)$.

The differential $\partial(X)$ of X is $|B(X)| - |X|$. The *differential* of a graph G is defined as

$$\partial(G) = \max\{\partial(X)/X \subset V\}.$$

The differential of a graph has also been investigated in [1, 2, 8, 10]. Let $T \subset V$ such $\partial(G) = \partial(T)$. Then we say that T as $\partial-$set. Sampathkumar and Waliker [9] introduced the concept of Splitting graph.

2. Splitting Graph

Definition 1. *Let $G = (V,E)$ be an arbitrary graph. For each point v of a graph G, take a new point v'. Join v' to all points of G adjacent to v. The graph $S(G)$ thus obtained is called the splitting graph.*

Theorem 2. *For any path P_n,*

$$\partial(S(P_n)) = \begin{cases} 3\left\lfloor \dfrac{n}{3} \right\rfloor & \text{for } n \equiv 0 \ (mod\ 3) \\ 3\left\lfloor \dfrac{n}{3} \right\rfloor + 1 & \text{for } n \equiv 1,2 \ (mod\ 3) \end{cases}$$

Proof. Let $\{v_1, v_2, ..., v_n\}$ be the vertices of path P_n and $\{v'_1, v'_2, ..., v'_n\}$ be the corresponding vertices to $\{v_1, v_2, ..., v_n\}$ which are added to obtain the splitting of the path $S(P_n)$.

Case (i): $n \equiv 0 (mod\ 3)$

Consider the set $V(S(C_n)) = \{v_1, v_2, ..., v_n, v'_1, v'_2, ..., v'_n\}$ and each vertex in S has the differential value 3. Let $S' = \left\{v'_2, v'_5, ..., v'_{n-1}\right\}$ and clearly $N(S') \subset N(S)$. Hence, no vertex in S' belongs to any $\partial-$set. So,

$$\partial(S(P_n)) = 3\left\lfloor \frac{n}{3} \right\rfloor.$$

Case (ii): $n \equiv 1(mod\ 3)\ n \geq 7$

It is obvious that when $n = 4$, where $\partial(S(P_4)) = 4$. Let $S = S' \cup \{v_{n-2}, v_{n-1}\}$ where $S' = \{v_2, v_5, ..., v_{n-5}\}$. Each vertex of S' gives the differential value 3. Then, $\partial(v_{n-2}) = 3$, $\partial(v_{n-2}, v_{n-1}) = 4$ and $\partial(v_{n-2}, v_n) = 3$. Hence, $S = S' \cup \{v_{n-2}, v_{n-1}\}$ is a $\partial-$set. Therefore,

$$\partial(S(P_n)) = 3\left\lfloor \frac{n}{3} \right\rfloor + 1.$$

Case (iii): $n \equiv 2(mod\ 3)$

Consider the set $S = S' \cup \{v_{n-1}, v_n\}$, where $S' = \{v_2, v_5, ..., v_{n-3}\}$. Each vertex of S' has the differential value 3 and each vertex of $\{v_{n-1}, v_n\}$ gives the differential value 1. So

$$\partial(S(P_n)) = 3\left\lfloor \frac{n}{3} \right\rfloor + 1.$$

□

Theorem 3. *For any cycle C_n,*

$$\partial(S(C_n)) = \begin{cases} 3\left\lfloor \frac{n}{3} \right\rfloor & \text{for } n \equiv 0\ (mod\ 3) \\ 3\left\lfloor \frac{n}{3} \right\rfloor + 1 & \text{for } n \equiv 1,2\ (mod\ 3) \end{cases}$$

Proof. Let $V(C_n) = \{v_1, v_2, ..., v_n\}$ be the vertices of the cycle graph of length n $(n \geq 3)$ and $\{v'_1, v'_2, ..., v'_n\}$ be the corresponding vertices to $\{v_1, v_2, ..., v_n\}$. By the definition of splitting graph, consider a vertex v'_1 which connects the vertices that are adjacent to v_1 and the vertex v'_2 which connects the vertices that are adjacent to v_2. Finally, we can consider the vertex v'_n which connects the vertices that are adjacent to v_n such that

$$V(S(C_n)) = \left\{v_1, v_2, ..., v_n, v'_1, v'_2, ..., v'_n\right\}.$$

Case(i): $n \equiv 0(mod\ 3)$

Let S be a set $\partial-$set, where $S = \{v_1, v_4, v_7, ..., v_{n-2}\}$. Each vertex in S gives the differential value of 3. Then the set $\{v'_1, v'_4, v'_7, ..., v'_{n-2}\}$ is an independent set and it is adjacent only with neighbours of S. So,

$$\partial(S(C_n)) = 3\left\lfloor \frac{n}{3} \right\rfloor.$$

Case (ii): $n \equiv 1(mod\ 3)$

Consider the differential set $S = S' \cup \{v_n\}$, where $S' = \{v_1, v_4, ..., v_{n-3}\}$. Then the set $\{v'_1, v'_4, ..., v'_{n-3}\}$ is an independent set and it is adjacent only with the vertices of $B(S)$. So,

$$\partial(S(C_n)) = 3\left\lfloor \frac{n}{3} \right\rfloor + 1.$$

Case (iii): $n \equiv 2(mod\ 3)$

Consider the differential set $S = S' \cup \{v_n\}$, where $S' = \{v_1, v_4, ..., v_{n-4}\}$. Then the set $\{v'_1, v'_4, ..., v'_{n-4}\}$ is an independent set and it is adjacent only with the vertices of $B(S)$. So,

$$\partial(S(C_n)) = 3\left\lfloor \frac{n}{3} \right\rfloor + 1.$$

□

Theorem 4. *For any star graph,* $\partial(S(K_{1,\ n-1})) = 2n-3$, $n \geq 3$.

Proof. Let $V(K_{1,\ n-1}) = \{u, v_1, v_2, ..., v_{n-1}\}$ be the star graph of n vertices. By the definition of splitting graph, consider a vertex v'_1 which connects the vertices that are adjacent to v_1 and the vertex v'_2 which connects the vertices that are adjacent to v_2 and finally, we can consider the vertex v'_{n-1} which connects the vertices that are adjacent to v_{n-1} and consider u' for a head vertex u. The head vertex u of $S(K_{1,\ n-1})$ is adjacent with all the vertices of $S(K_{1,\ n-1})$ except u' and u' is not connected with any other vertices in the graph of $S(K_{1,\ n-1})$. Hence $\partial(S(K_{1,\ n-1})) = (2n-1-1)-1 = 2n-3$. □

Theorem 5. *For any wheel graph,* $\partial(S(W_n)) = 2n-3$, $n \geq 4$.

Proof. Let $V(W_n) = \{v_1, v_2, ..., v_n\}$ be the vertices in a graph W_n and v_n be the central vertex of W_n. Let $\{v'_1, v'_2, ..., v'_n\}$ be the new vertex set corresponding to the vertex set of $\{v_1, v_2, ..., v_n\}$ such that $V(S(W_n)) = \{v_1, v_2, ..., v_n, v'_1, v'_2, ..., v'_n\}$. The vertex v_n is adjacent to all the vertices of the splitting graph $S(W_n)$ except v'_n. Hence $deg(v_n) = 2n-2 = \Delta(S(W_n))$. Thus, the differential value of the graph $S(W_n)$ is

$$\partial(S(W_n)) = \partial(\{v_n\}) = 2n-2-1 = 2n-3.$$

Hence, $\partial(S(W_n)) = 2n-3$. □

Theorem 6. *For any complete graph, $\partial(S(K_n)) = 2n-3$, $n \geq 3$.*

Proof. Let $V(K_n) = \{v_1, v_2, ..., v_n\}$ be the vertices and $\{v'_1, v'_2, ..., v'_n\}$ be the corresponding vertices to $\{v_1, v_2, ..., v_n\}$ to obtain splitting graph $S(K_n)$ such that

$$V(S(K_n)) = \{v_1, v_2, ..., v_n, v'_1, v'_2, ..., v'_n\}.$$

Clearly, $S = \{v_1\}$ be the differential set of $S(K_n)$ and hence $\partial(S(K_n)) = 2n-3$. □

3. Middle Graph

Definition 7. *The Middle graph $M(G)$ of a graph G, is the graph whose vertex set is $V(G) \cup E(G)$ where two vertices are adjacent if and only if they are either adjacent edges of G or one is a vertex and the other is an edge incident with it, refer* [7].

Theorem 8. *For any path P_n, then*

$$\partial(M(P_n)) = \begin{cases} 3\left\lfloor \dfrac{n-1}{3} \right\rfloor + 2, & n \equiv 0\ (mod\ 3) \\ 3\left\lfloor \dfrac{n-1}{3} \right\rfloor, & n \equiv 1\ (mod\ 3) \\ 3\left\lfloor \dfrac{n-1}{3} \right\rfloor + 1, & n \equiv 2\ (mod\ 3) \end{cases}$$

Proof. Let $V(P_n) = \{u_1, u_2, ..., u_n\}$ and $E(P_n) = \{v_1, v_2, ..., v_n\}$. By the definition of middle graph, $V(G) = V(P_n) \cup E(P_n)$ where $G = M(P_n)$.

Case(i): $n \equiv 0(mod\ 3)$

We observe that any vertices of $V(P_n)$ not belongs to any differential set. In this case, there are two possible differential sets.

If $N = 6k$, where $N = |V(M(P_n))|$ then $S_1 = \{u_2, u_4, ..., u_{n-1}\}$ and if $N = 6K + 3$ then

$$S_2 = \{u_2, u_4, ..., u_{n-2}\}$$

where $k \in N$ and if either S_1 or S_2 contains only the elements of $V(P_n)$ and

$$\partial(S_1) = \left\lfloor \frac{n-1}{2} \right\rfloor \qquad \text{or} \qquad \partial(S_2) = \left\lfloor \frac{n-1}{2} \right\rfloor.$$

Suppose S contains only vertices of $E(P_n) = \{v_1, v_2, ..., v_{n-1}\}$ and $S = \{v_2, v_5, ..., v_{n-1}\}$ where each vertex in S except v_{n-1} gives the differential value is 3 and the differential value of v_{n-1} is 2. Hence,

$$\partial(S) = 3 \left\lfloor \frac{n-1}{3} \right\rfloor + 2.$$

Clearly, S contains either the elements of $V(P_n)$ or $E(P_n)$ but not both. Therefore,

$$\partial(M(P_n)) = 3 \left\lfloor \frac{n-1}{3} \right\rfloor + 2.$$

Case(ii): $n \equiv 1(mod\ 3)$

We observe that any vertices of $V(P_n)$ not belongs to any differential set. In this case, there are two possible differential sets $S_1 = \{u_2, u_4, ..., u_{n-1}\}$ and $S_2 = \{u_2, u_4, ..., u_{n-2}\}$ and if either S_1 or S_2 contains only the elements of $V(P_n)$ and

$$\partial(S_1) = \left\lfloor \frac{n-1}{2} \right\rfloor \qquad \text{or} \qquad \partial(S_2) = \left\lfloor \frac{n-1}{2} \right\rfloor.$$

Suppose S contains only vertices of $E(P_n) = \{v_1, v_2, ..., v_{n-1}\}$ and $S = \{v_2, v_5, ..., v_{n-1}\}$ where each vertex in S gives the differential value is 3. Hence,

$$\partial(S) = 3 \left\lfloor \frac{n-1}{3} \right\rfloor.$$

Clearly, S contains either the elements of $V(P_n)$ or $E(P_n)$ but not both. Therefore,

$$\partial(M(P_n)) = 3 \left\lfloor \frac{n-1}{3} \right\rfloor.$$

Case(iii): $n \equiv 2(mod\ 3)$

We observe that any vertices of $V(P_n)$ not belongs to any differential set. In this case, there are two possible differential sets $S_1 = \{u_2, u_4, ..., u_{n-1}\}$ and $S_2 = \{u_2, u_4, ..., u_{n-2}\}$ and if either S_1 or S_2 contains only the elements of $V(P_n)$ and

$$\partial(S_1) = \left\lfloor \frac{n-1}{2} \right\rfloor \qquad \text{or} \qquad \partial(S_2) = \left\lfloor \frac{n-1}{2} \right\rfloor.$$

Suppose S contains only vertices of $E(P_n) = \{v_1, v_2, ..., v_{n-1}\}$ and $S = \{v_2, v_5, ..., v_{n-1}\}$ where each vertex in S gives the differential value is 3 except v_{n-1}. The differential value of v_{n-1} is 1. Therefore,

$$\partial(S) = 3 \left\lfloor \frac{n-1}{3} \right\rfloor + 1.$$

Clearly, S contains either the elements of $V(P_n)$ or $E(P_n)$ but not both. Therefore,

$$\partial(M(P_n)) = 3 \left\lfloor \frac{n-1}{3} \right\rfloor + 1.$$

□

Theorem 9. *For any cycle graph, $\partial(M(C_n)) = n$, $n \geq 3$.*

Proof. Let $V(C_n) = \{u_1, u_2, ..., u_n\}$ and $E(C_n) = \{v_1, v_2, ..., v_n\}$. By the definition of middle graph, $V(G) = V(C_n) \cup E(C_n)$ where $G = M(C_n)$.

Case(i): When n is even

Let $S = \{v_1, v_3, v_5, ..., v_{n-1}\}$. Clearly, S is a $\gamma-$set of G. Since $\partial(G) + 2\ \gamma(G) = N$ where $N = |V(G)|$ if and only if there exists a $\partial-$set which is also a $\gamma-$ set of G. Hence S is a $\partial-$set of G. So, $\partial(S) = \frac{3n}{2} - \frac{n}{2} = n$. Hence, $\partial(M(C_n)) = n$.

Case(ii): When n is odd

In this case, the possible $\partial-$sets are $S_1 = \{v_1, v_3, v_5, ..., v_{n-2}\}$ and $S_2 = \{v_1, v_2, ..., v_{n-1}\}$. In all the cases, we get, $\partial(M(C_n)) = n$. □

Theorem 10. *For any star graph, $\partial(M(K_{1,\,n-1})) = n-1$, $n \geq 3$.*

Proof. Let $V(K_{1,\,n-1}) = \{u_1, u_2, ..., u_n\}$ and $E(K_{1,\,n-1}) = \{v_1, v_2, ..., v_{n-1}\}$ where u_1 be the head vertex of the given graph $K_{1,\,n-1}$. By the definition of middle graph $G = M(K_{1,\,n-1})$ whose vertex set is $V(G) = V(K_{1,\,n-1}) \cup E(K_{1,\,n-1})$. Clearly, the degree of each vertex v_i is n and the graph induced by $\langle G - N[v_i] \rangle$ contains only isolated vertices which does not belong to any other differential set. Hence, $\partial(M(K_{1,\,n-1})) = n-1$. □

Theorem 11. *For any complete bipartite graph,* $\partial(M(K_{m,n})) = mn$.

Proof. Let (X,Y) be a bipartition of $K_{m,n}$ where $|X| = m$ and $|Y| = n$ and $m \leq n$. Let $X = \{x_1, x_2, ..., x_m\}$ and $Y = \{y_1, y_2, ..., y_n\}$ and $G = M(K_{m,n})$. Let $X_1, X_2, ..., X_m$ be the mK_{n+1} cliques and $Y_1, Y_2, ..., Y_n$ be the nK_{m+1} cliques in $M(K_{m,n})$. Further X_i and Y_j have exactly one common vertex.

Let w_{ij} be the vertex common to both X_i and Y_j. Let S be the $\partial-$set of G. Assume $w_{11} \in S$. Clearly, every vertex in the corresponding row and column is dominated by w_{11} and hence no other vertices in the corresponding row and column belong to S. Hence, all the leading diagonal vertices $\{w_{11}, w_{22}, ..., w_{mm}\}$ belongs to S.

The set S dominates all the vertices in $V(G) - S$ except $S' = \{v_{m+1}, v_{m+2}, ..., v_n\}$. Since all the vertices in S' are independent and $B(S') \subseteq B(S)$, no elements of S' are in S. Hence, $S = \{w_{11}, w_{12}, ..., w_{mm}\}$. Then

$$\begin{aligned}\partial(G) &= [|X_1| - 1 + |X_2| - 1 + ... + |X_m| - 1] - |S| \\ &= [(n-1) + (n-1) + ... + (n-1)] - m \\ &= m(n-1) - m = mn.\end{aligned}$$

□

Conflict of Interests

The author(s) declare that there is no conflict of interests.

References

[1] S. Bermudo, H. Fernau, Lower bound on the differential of a graph, Discret. Math. 312 (2012), 3236-3250.

[2] S. Bermudo, J.M. Rodriguez, J.M. Sigarreta, On the differential in graphs, Util. Math. 97 (2015), 257-270.

[3] W. Goddard, M.A. Henning, Generalized domination and independence in graphs, Congr. Numer. 123 (1997), 161-171.

[4] F. Harary, Graph Theory, Addison Wesley, Reading Mass. 1972.

[5] J.R. Lewis, Differential of graphs, Master's Thesis, East Tennessee State University, 2004.

[6] J.L. Mashburn, T.W. Haynes, S.M. Hedentnieme, S.T. Hedentnieme, P.J. Slater, Differential in graphs, Util. Math. 69 (2006), 43-54.

[7] P. Roushini Leely Pushpa, Chitra Suseendran, Secure Domination in Middle graphs, Bull. Int. Math. Virtual Inst. 9 (2019), 25-38.

[8] P. Roushini Leely Pushpa, D. Yokesh, Differential in certain classes of graphs, Tamkang J. Math. 41 (2) (2010), 129-138.

[9] E. Sampathkumar, H.B. Walker, On the Splitting graph on graph, The Karnataka University Journal Science XXV and XXVI, 1980-1981.

[10] J.M. Sigarreta, Differential in Cartesian product graphs, Ars Comb. 126 (2016), 259-267.