Available online at http://scik.org
J. Math. Comput. Sci. 2022, 12:67
https://doi.org/10.28919/jmcs/7037
ISSN: 1927-5307

EXISTENCE AND UNIQUENESS OF MILD AND STRONG SOLUTIONS OF NONLINEAR FRACTIONAL INTEGRODIFFERENTIAL EQUATION

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Abstract. In this paper, we will discuss some results on the existence and uniqueness of mild and strong solution of initial value problem of fractional order subjected to non-local conditions, by using the Banach fixed point theorem and the theory of strongly continuous cosine family under Caputo sense. Furthermore, we also prove that solution of Nonlinear Fractional Volterra Integrodifferential Equations and Nonlinear Fractional Mixed Integrodifferential Equations With Nonlocal Conditions is unique. Moreover, examples demonstrate the validity of the obtained main result and we obtain the solution for an equation, and proved that this solution is unique.

Keywords: fractional differential equation; existence and uniqueness; mild and strong solutions; Banach Fixed point theorem.

2010 AMS Subject Classification: 34A08, 34A12, 34B10.

1. Introduction

Some results on the problem of existence and uniqueness of solution of differential equations of fractional order have been discussed by some authors which can be found in [1, 2, 3, 4]. The purpose of this paper is to discuss the existence and uniqueness of solution of differential equation of fractional order, by using the Banach fixed point theorem and the theory of strongly

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Received November 26, 2021

continuous cosine family.

Now consider the fractional order non-linear differential equations with non-local conditions as follows:

$$\vartheta^{\alpha+1}(\eta)+\beta\vartheta(\eta)=\chi(\eta,\vartheta(\eta),\int_{\eta_0}^{\eta}\varphi(\eta,\nu,\vartheta(\nu))d\nu), \qquad \eta\in[\eta_0,\eta_0+\xi] \tag{1.1}$$

$$\vartheta(\eta_0)+\mu(\vartheta)=\vartheta_0, \tag{1.2}$$

For the shake of simplicity let

$$w(\eta)=\int_{\eta_0}^{\eta}\varphi(\eta,\nu,\vartheta(\nu))d\nu$$

where β is the infinitesimal generator of a $\mathscr{C}_0$ semigroup $\mathscr{T}(\eta),\eta\geq 0$, on a Banach space $\mathscr{X}$ and the nonlinear operators $\chi:[\eta_0,\eta_0+\xi]\times\mathscr{X}\times\mathscr{X}\to\mathscr{X},\mu:\Omega\to\mathscr{X},\varphi:[\eta_0,\eta_0+\xi]\times[\eta_0,\eta_0+\xi]\times\mathscr{X}\to\mathscr{X}$ are continuous and $\vartheta_o\in\mathscr{X}$.

Moreover, we consider the nonlinear fractional mixed Volterra - Fredholm integrodifferential equation:

$$\vartheta^{\alpha+1}(\eta)+\beta\vartheta(\eta)=\chi(\eta,\nu(\eta),\int_{\eta_0}^{\eta}\varphi(\eta,\nu,\vartheta(\nu))d\nu,\int_{\eta_0}^{\eta_0+\xi}\psi(\eta,\nu,\vartheta(\nu))d\nu), \tag{1.3}$$
$$\eta\in[\eta_0,\eta_0+\xi]$$

$$\vartheta(\eta_0)+\mu(\eta_1,\eta_2,........\eta_p,\vartheta(.))=\vartheta_0, \tag{1.4}$$

For shake of shortness let

$$w(\eta)=\int_{\eta_0}^{\eta}\varphi(\eta,\nu,\vartheta(\nu))d\nu \ \ and \ \ \gamma(\eta)=\int_{\eta_0}^{\eta_0+\xi}\psi(\eta,\nu,\vartheta(\nu))d\nu$$

where $0\leq\eta_0<\eta_1<\eta_2<.......\eta_p\leq\eta_0+\xi$, β is the infinitesimal generator of a $\mathscr{C}_0$ semigroup $\mathscr{T}(\eta),\eta\geq 0$, in a Banach space $\mathscr{X}$ and the nonlinear functions $\chi:[\eta_0,\eta_0+\xi]\times\mathscr{X}\times\mathscr{X}\times\mathscr{X}\to\mathscr{X},\mu:[\eta_0,\eta_0+\xi]^p\times\mathscr{X}\to\mathscr{X},\varphi,\psi:[\eta_0,\eta_0+\xi]\times[\eta_0,\eta_0+\xi]\times\mathscr{X}\to\mathscr{X}$ and $\vartheta_0\in\mathscr{X}$.

Riemann–Liouville definition [5, 6]: For $\alpha\in[n-1,n)$ the α - derivative of f is

$$D_a^{\alpha}f(x)=\frac{1}{\Gamma(n-\alpha)}\frac{d^nx}{dt^n}\int_a^{\alpha}\frac{f(x)}{(t-x)^{\alpha-n+1}}dx$$

Caputo definition[5, 6]:For $\alpha \in (n-1,n)$ the α - derivative of f is

$$ {}_{a}^{C}D_{t}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha-n)} \int_{a}^{t} \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau $$

2. Mild and Strong Solutions of Nonlinear Fractional Volterra Integrodifferential Equations

2.1. Preliminaries.

Definition 2.1.1 *A* continuous solution $\vartheta(\eta)$ of the integral equation

$$ \vartheta(\eta) = \mathscr{T}(\eta-\eta_0)[\vartheta_0 - \mu(\vartheta)] + \frac{1}{\Gamma(-\alpha-n)} \int_{\eta_0}^{\eta} \frac{\mathscr{T}(\eta-\nu)\chi^{(n)}(\nu,\vartheta(\nu),w(\nu))d\nu}{(\eta-\nu)^{-\alpha+1-n}}, \tag{2.1.1} $$

$$ \eta \in [\eta_0, \eta_0+\xi] $$

is said to be a mild solution of problem (1.1) -(1.2) on $[\eta_0, \eta_0+\xi]$

Definition 2.1.2 A function $\vartheta(\eta)$ is said to be a strong solution of problem (1.1) -(1.2) on $[\eta_0, \eta_0+\xi]$, if $\vartheta(\eta)$ is differentiate almost everywhere on $[\eta_0, \eta_0+\xi]$,$\vartheta^{\alpha} \in \mathscr{L}^1([\eta_0, \eta_0+\xi], \mathscr{X})$,$\vartheta(\eta_0)+\mu(\vartheta) = \vartheta_0$,

$$ \vartheta^{\alpha+1}(\eta) + \beta\vartheta(\eta) = \chi(\eta, \vartheta(\eta), \int_{\eta_0}^{\eta} \varphi(\eta,\nu,\vartheta(\nu))d\nu), \qquad \eta \in [\eta_0, \eta_0+\xi] \tag{2.1.2} $$

Let us denote

$$ \begin{aligned} \mathscr{M} &= \max_{\eta\in[0,\xi]} \| \mathscr{T}(\eta) \|, \\ \mathscr{L}_1 &= \max_{\eta_0\le\eta\le\eta_0+\xi} \| \chi^n(\eta,0,0) \|, \\ \Psi_1 &= \max_{\eta_0\le\nu,\eta\le\eta_0+\xi} \| \varphi(\eta,\nu,0) \|, \\ \Lambda_1 &= \max_{\vartheta\in\Omega} \| \mu(\eta) \| . \end{aligned} $$

For shake of our convenience or for further use we list the following hypothesis.

(H_1) If $\vartheta_1, \vartheta_2 \in \Upsilon$, then $\mu : \Upsilon \subset \Omega \to \mathscr{X}$ and $\exists$ a constant $\Lambda > 0$ such that

$$ \| \mu(\vartheta_1) - \mu(\vartheta_2) \| \le \| \vartheta_1(\eta) - \vartheta_2(\eta) \|_{\Upsilon} $$

(H_2) β is the infinitesimal generator of a $\mathscr{C}_0$ semigroup $\mathscr{T}(\eta), \eta \ge 0$ on X.

(H_3) The constants $\| \vartheta_0 \|, r, \xi, \mathscr{L}_{\chi}^1, \mathscr{L}_{\chi}^2, \Psi, \mathscr{M}, \mathscr{N}, \mathscr{L}_1, \Psi_1, \Lambda$ and Λ_1 satisfy the following conditions:

$\mathscr{M}\left[\| \vartheta_0 \| + \Lambda_1 + \mathscr{L}^1_\chi r\xi + \mathscr{L}^2_\chi \Psi r\xi^2 + \mathscr{L}^2_\chi \Psi_1 \xi^2 + \mathscr{L}_1 \xi\right] \leq r$, and

$[\mathscr{M}\Lambda + \mathscr{M}\mathscr{L}^1_\chi \xi + \mathscr{M}\mathscr{L}^2_\chi \Psi\xi^2] < 1.$

2.2. Main Results.

Theorem 2.2.1 If the following assumptions hold

(i) hypotheses $(H_1)-(H_3)$ hold,

(ii) $\chi : [\eta_0, \eta_0+\xi] \times \mathscr{X} \times \mathscr{X} \to \mathscr{X}$ is continuous and there exist constants $\mathscr{L}^1_\chi, \mathscr{L}^2_\chi > 0$ such that

$$\| \chi^{(n)}(\eta, \vartheta_1, \upsilon_1) - \chi^{(n)}(\eta, \vartheta_2, \upsilon_2) \| \leq \mathscr{L}^1_\chi \| \vartheta_1 - \vartheta_2 \| + \mathscr{L}^2_\chi \| \upsilon_1 - \upsilon_2 \|,$$

(iii) $\Psi : [\eta_0, \eta_0+\xi] \times [\eta_0, \eta_0+\xi] \times \mathscr{X} \to \mathscr{X}$ is continuous and there exists constant $\Psi > 0$ such that

$$\| \varphi(\eta, \nu, \vartheta_1,) - \varphi(\eta, \nu, \vartheta_2,) \| \leq \Psi(\| \vartheta_1 - \vartheta_2 \|).$$

Then problem, (1.1) -(1.2) has a unique mild solution on $[\eta_0, \eta_0+\xi]$.

Proof Let $\Phi : \Upsilon \to \Upsilon$ be an operator defined by

(2.1.3)

$$(\Phi\sigma)(\eta) = \mathscr{T}(\eta-\eta_0)[\vartheta_0 - \mu(\sigma)] + \frac{1}{\Gamma(-\alpha-n)} \int_{\eta_0}^{\eta} \frac{\mathscr{T}(\eta-\nu)\chi^{(n)}(\nu, \sigma(\nu), w(\nu))d\nu}{(\eta-\nu)^{-\alpha+1-n}},$$

$$\eta \in [\eta_0, \eta_0+\xi].$$

By our assumptions, we have

$$\| (\Phi\sigma)(\eta) \| \leq \mathscr{M}[\| \vartheta_0 \| + \Lambda_1] + \int_{\eta_0}^{\eta} \mathscr{M} \left\| \frac{\chi^{(n)}(\nu, \sigma(\nu), w(\nu))}{\Gamma(-\alpha-n)(\eta-\nu)^{-\alpha+1-n}} \right\| d\nu$$

$$\leq \mathscr{M}[\| \vartheta_0 \| + \Lambda_1]$$

$$+\mathscr{M}\int_{\eta_0}^{\eta} \left[\| \chi^{(n)}(\nu, \sigma(\nu), w(\nu)) - \chi^{(n)}(\nu, 0, 0) \| + \| \chi^{(n)}(\nu, 0, 0) \|\right] \frac{d\nu}{\| \zeta \|}$$

$$\leq \mathscr{M}[\| \vartheta_0 \| + \Lambda_1] + \mathscr{M}\int_{\eta_0}^{\eta}$$

$$\left[\mathscr{L}^1_\chi \| \sigma(\nu) \| + \mathscr{L}^2_\chi \int_{\eta_0}^{\eta} \| \varphi(\nu, \eta, \sigma(\eta)) - \varphi(\nu, \eta, 0) + \varphi(\nu, \eta, 0) \| + \mathscr{L}_1\right] \frac{d\nu}{\zeta^*}$$

$$\leq \mathscr{M}[\| \vartheta_0 \| + \Lambda_1] + \mathscr{M}\int_{\eta_0}^{\eta} \left[\mathscr{L}^1_\chi r + \mathscr{L}^2_\chi \Psi r(\nu-\eta_0) + \mathscr{L}^2_\chi \Psi_1(\nu-\eta_0) + \mathscr{L}_1\right] \frac{d\nu}{\zeta^*}$$

$$\leq \mathscr{M}\left[\| \vartheta_0 \| + \Lambda_1\right] + \mathscr{M}\left[\mathscr{L}_\chi^1 r\xi + \mathscr{L}_\chi^2 \Psi r \xi^2 + \mathscr{L}_\chi^2 \Psi_1 \xi^2 + \mathscr{L}_1 \xi\right] \frac{1}{\zeta^*} \leq \frac{r}{\zeta^*}, \tag{2.1.4}$$

where $\Gamma(-\alpha - n)(\eta - \nu)^{-\alpha+1-n} = \zeta$ and $\| \zeta \| = \zeta^*$

for $\sigma \in \Upsilon$. The equation (2.1.4) shows that $\Phi(\Upsilon) = \Upsilon$. Now for every $\sigma_1, \sigma_2 \in \Upsilon$ and $\eta \in [\eta_0, \eta_0 + \xi]$, we obtain

$$\| (\Phi\sigma_1)(\eta) - (\Phi\sigma_2)(\eta) \| \leq \| \mathscr{T}(\eta - \eta_0) \| \| \mu(\sigma_1) - \mu(\sigma_2) \|$$

$$+ \int_{\eta_0}^{\eta} \| \mathscr{T}(\eta - \nu) \| \frac{\| \left[\chi^{(n)}(\nu, \sigma_1(\nu), w(\nu)) - \chi^{(n)}(\nu, \sigma_2(\nu), w(\nu))\right] \| \, d\nu}{\| \Gamma(-\alpha - n)(\eta - \nu)^{-\alpha+1-n} \|}$$

$$\leq \mathscr{M}\Lambda \| \sigma_1 - \sigma_2 \|_\Omega + \mathscr{M} \int_{\eta_0}^{\eta} \left[\mathscr{L}_\chi^1 \| \sigma_1 - \sigma_2 \|_\Omega + \mathscr{L}_\chi^2 \Psi \| \sigma_1 - \sigma_2 \|_\Omega (\nu - \eta_0)\right] \frac{d\nu}{\zeta^*}$$

$$\leq \mathscr{M}\Lambda \| \sigma_1 - \sigma_2 \|_\Omega + \mathscr{M}[\mathscr{L}_\chi^1 (\eta - \eta_0) + \mathscr{L}_\chi^2 \Psi \xi^2] \| \sigma_1 - \sigma_2 \|_\Upsilon \frac{1}{\zeta^*}$$

$$\leq [\mathscr{M}\Lambda + \mathscr{M}\mathscr{L}_\chi^1 \xi + \mathscr{M}\mathscr{L}_\chi^2 \Psi \xi^2] \| \sigma_1 - \sigma_2 \|_\Upsilon \frac{1}{\zeta^*} \tag{2.1.5}$$

If $q = \mathscr{M}\Lambda + \mathscr{M}\mathscr{L}_\chi^1 \xi + \mathscr{M}\mathscr{L}_\chi^2 \Psi \xi^2$, then

$$\| \Phi\sigma_1 - \Phi\sigma_2 \|_\Upsilon \leq \frac{q}{\zeta^*} \| \sigma_1 - \sigma_2 \|_\Upsilon$$

where $0 < q < 1$. From this it is clear that Φ is a contraction on Υ. By the Banach fixed point theorem, the function Φ has a unique fixed point in the space Υ and this point is the mild solution of problem (1.1) -(1.2) on $[\eta_0, \eta_0 + \xi]$.

Next we prove that the problem (1.1) -(1.2) has a strong solution.

Theorem 2.2.2 As following assumptions hold

(i) hypotheses $(H_1) - (H_1)$ hold,

(ii) $\mathscr{X}$ is a reflexive Banach space with norm $\| \, . \, \|$ and $\vartheta_o \in D(\beta)$, the domain of β,

(iii) $\mu(.) \in D(\beta)$,

(iv) $\chi : [\eta_0, \eta_0 + \xi] \times \mathscr{X} \times \mathscr{X} \to \mathscr{X}$ is continuous and there exist constants $\mathscr{L}_\chi^i > 0, i = 1, 2$ and $\mathscr{L}_\eta > 0$ such that

$$\| \chi^{(n)}(\eta, \vartheta_1, \upsilon_1) - \chi^{(n)}(\eta, \vartheta_2, \upsilon_2) \| + \mathscr{L}_\eta \mid \eta_1 - \eta_2 \mid \leq \mathscr{L}_\chi^1 \| \vartheta_1 - \vartheta_2 \| + \mathscr{L}_\chi^2 \| \upsilon_1 - \upsilon_2 \|,$$

(v) $\varphi : [\eta_0, \eta_0+\xi] \times [\eta_0, \eta_0+\xi] \times \mathscr{X} \to \mathscr{X}$ is continuous and there exists constant $\Psi_\eta, \Psi > 0$ such that

$$\parallel \varphi(\eta, \nu, \vartheta_1,) - \varphi(\eta, \nu, \vartheta_2,) \parallel \leq \Psi_\eta \mid \eta_1 - \eta_2 \mid \leq \Psi \parallel \vartheta_1 - \vartheta_2 \parallel .$$

Then $\vartheta(\eta)$ is a unique strong solution of problem (1.1) -(1.2) on $[\eta_0, \eta_0+\xi]$.

Proof If all the assumptions of Theorem 2.2 are satisfied then the problem (1.1) -(1.2) has a unique mild solution belonging to Υ which is denoted by ρ.

Claim: $\rho(\eta)$ is unique strong solution of problem (1.1) -(1.2) on $[\eta_0, \eta_0+\xi]$. If

$$\mathscr{L}_2 = \max_{\eta_0 \leq \eta \leq \eta_0+\xi} \parallel \chi^n(\eta, \rho(\nu), 0) \parallel,$$

$$\Psi_2 = \max_{\eta_0 \leq \nu, \eta \leq \eta_0+\xi} \parallel \varphi(\eta, \nu, \rho(\nu)) \parallel$$

then for any $\eta \in [\eta_0, \eta_0+\xi]$ and $\rho \in R$ with $\eta+\rho \in [\eta_0, \eta_0+\xi]$, we obtain

$$\begin{aligned}
&\rho(\eta+\rho) - \rho(\eta) = \mathscr{T}(\eta-\eta_0)\,[\mathscr{T}(\rho)-I]\,\vartheta_0 \\
&-\mathscr{T}(\eta-\eta_0)\,[\mathscr{T}(\rho)-I]\,\mu(\rho) - \mathscr{T}(\eta+\rho-\eta_0)\,[\mu(\rho(\eta+\rho)) - \mu(\rho(\eta))] \\
&+\int_{\eta_0}^{\eta_0+\rho} \mathscr{T}(\eta+\rho-\nu) \times \Big[\chi^{(n)}(\nu, \rho(\nu), w(\nu)) - \chi^{(n)}(\nu, \rho(\nu), 0) \\
&+\chi^{(n)}(\nu, \rho(\nu), 0)\Big] \frac{d\nu}{\zeta^*} \\
&+\int_{\eta_0}^{\eta} \mathscr{T}(\eta-\nu)\chi^{(n)}(\nu+\rho, \rho(\nu+\rho), w(\nu+\rho)) \frac{d\nu}{\zeta^*} \\
&-\int_{\eta_0}^{\eta} \mathscr{T}(\eta-\nu)\chi^{(n)}(\nu, \rho(\nu), w(\nu)) \frac{d\nu}{\zeta^*} \\
&= \mathscr{T}(\eta-\eta_0)\,[\mathscr{T}(\rho)-I]\,\vartheta_0 \\
&-\mathscr{T}(\eta-\eta_0)\,[\mathscr{T}(\rho)-I]\,\mu(\rho) - \mathscr{T}(\eta+\rho-\eta_0)\,[\mu(\rho(\eta+\rho)) - \mu(\rho(\eta))] \\
&+\int_{\eta_0}^{\eta_0+\rho} \mathscr{T}(\eta+\rho-\nu) \times \Big[\chi^{(n)}(\nu, \rho(\nu), w(\nu)) - \chi^{(n)}(\nu, \rho(\nu), 0) \\
&+\chi^{(n)}(\nu, \rho(\nu), 0)\Big] \frac{d\nu}{\zeta^*}
\end{aligned}$$

$$+\int_{\eta_0}^{\eta} \mathscr{T}(\eta-\nu)\left[\chi^{(n)}(\nu+\rho, \rho(\nu+\rho), w(\nu+\rho)) - \chi^{(n)}(\nu, \rho(\nu), w(\nu))\right] \frac{d\nu}{\zeta^*} \tag{2.1.6}$$

where I is the identity operator.

Using our assumptions and equation (2.1.6), found that

$$\| \rho(\eta+\rho)-\rho(\eta) \| \leq \mathscr{M}\rho \| \beta\vartheta_0 \| + \mathscr{M}\rho \| \beta\mu(\rho) \| + \mathscr{M}\Lambda \| \rho(\eta+\rho)-\rho(\eta) \|$$

$$+\int_{\eta_0}^{\eta_0+\rho} \mathscr{M}\left[\mathscr{L}_\eta \mid \nu-\nu \mid + \mathscr{L}_\chi^1 \| \rho(\nu)-\rho(\nu) \| + \mathscr{L}_\chi^2 \| w(\nu) \| + \mathscr{L}_2\right]\frac{d\nu}{\zeta^*}$$

$$+\int_{\eta_0}^{\eta} \mathscr{M}\left[\mathscr{L}_\eta \mid \nu+\rho-\nu \mid + \mathscr{L}_\chi^1 \| \rho(\nu+\rho)-\rho(\nu) \| + \mathscr{L}_\chi^2 \| w(\nu+\rho)-w(\nu) \|\right]\frac{d\nu}{\zeta^*}$$

$$\leq \frac{\mathscr{M}\rho \| \beta\vartheta_0 \| + \mathscr{M}\rho \| \beta\mu(\rho) \|}{(1-\mathscr{M}\Lambda)\zeta^*}$$

$$+\frac{1}{(1-\mathscr{M}\Lambda)\zeta^*}\int_{\eta_0}^{\eta_0+\rho} \mathscr{M}\left[\mathscr{L}_\chi^2 w(\nu) + \mathscr{L}_2 w(\nu+\rho)\right]\frac{d\nu}{\zeta^*}$$

$$+\frac{1}{(1-\mathscr{M}\Lambda)\zeta^*}\int_{\eta_0}^{\eta}\left[\mathscr{L}_\eta\rho + \mathscr{L}_\chi^1 \| \rho(\nu+\rho)-\rho(\nu) \| + \mathscr{L}_\chi^2 w(\rho,\nu+\rho)\right]\frac{d\nu}{\zeta^*}$$

$$\leq \frac{\mathscr{M}\rho \| \beta\vartheta_0 \| + \mathscr{M}\rho \| \beta\mu(\rho) \|}{(1-\mathscr{M}\Lambda)\zeta^*} + \frac{1}{(1-\mathscr{M}\Lambda)\zeta^*}\mathscr{M}[\mathscr{L}_\chi^2\Psi_2\xi\theta + \mathscr{L}_2\rho]$$

$$+\frac{1}{(1-\mathscr{M}\Lambda)\zeta^*}\int_{\eta_0}^{\eta}[\mathscr{L}_\eta\rho + \mathscr{L}_\chi^1 \| \rho(\nu+\rho)-\rho(\nu) \| + \mathscr{L}_\chi^2\Psi_\eta\rho\xi + \mathscr{L}_\chi^2\Psi_2\rho]\frac{d\nu}{\zeta^*}$$

$$\leq \frac{\mathscr{M}\rho \| \beta\vartheta_0 \| + \mathscr{M}\rho \| \beta\mu(\rho) \|}{(1-\mathscr{M}\Lambda)\zeta^*}$$

$$+\frac{\mathscr{M}\mathscr{L}_\chi^2\Psi_2\rho\xi + \mathscr{M}\mathscr{L}_2\rho + \mathscr{M}\mathscr{L}_\eta\rho\xi + \mathscr{M}\mathscr{L}_\chi^2\Psi_\eta\rho\xi^2 + \mathscr{M}\mathscr{L}_\chi^2\Psi_2\rho\xi}{(1-\mathscr{M}\Lambda)\zeta^*}$$

$$+\frac{\mathscr{M}\mathscr{L}_\chi^1}{(1-\mathscr{M}\Lambda)\zeta^*}\int_{\eta_0}^{\eta} \| \rho(\nu+\rho)-\rho(\nu) \| \, d\nu$$

$$\leq P\rho + \Delta\int_{\eta_0}^{\eta} \| \rho(\nu+\rho)-\rho(\nu) \| \, d\nu\frac{1}{\zeta^*}, \tag{2.1.7}$$

where

$$P = \frac{\mathscr{M} \| \beta\vartheta_0 \| + \mathscr{M} \| \beta\mu(\rho) \| + \mathscr{M}[\mathscr{L}_\chi^2\Psi_2\xi + \mathscr{L}_2 + \mathscr{L}_\eta\xi + \mathscr{L}_\chi^2\Psi_\eta a\xi^2 + \mathscr{L}_\chi^2\Psi_2\xi]}{(1-\mathscr{M}\Lambda)\zeta^*}$$

and $\Delta = \frac{\mathscr{M}\mathscr{L}_\chi^1}{(1-\mathscr{M}\Lambda)\zeta^*}$. Using Gronwall's inequality (with $c = P\rho$), we get

$$\| \rho(\eta+\rho)-\rho(\eta) \| \leq P\rho e^{\frac{\Delta\xi}{\zeta^*}}, \quad for\ \eta \in [\eta_0, \eta_0+\xi].$$

Therefore, ρ is Lipschitz continuous on $[\eta_0, \eta_0+\xi]$. The Lipschitz continuity of ρ on $[\eta_0, \eta_0+\xi]$ combined with (iv) and (v) of Theorem 2.2, gives that

$$\eta \to \chi(\eta, \vartheta(\eta), \int_{\eta_0}^{\eta} \Psi(\eta, \nu, \vartheta(\nu))d\nu)$$

is Lipschitz continuous on $[\eta_0, \eta_0+\xi]$. Consequently, $\rho(\eta)$ is the strong solution of problem (1.1) -(1.2) on $[\eta_0, \eta_0+\xi]$

3. Existence and Uniqueness of Solutions of Nonlinear Fractional Mixed Integrodifferential Equations With Nonlocal Conditions

3.1. Preliminaries. Definition

A continuous solution $\vartheta(\eta)$ of the integral equation

$$\vartheta(\eta) = \mathscr{T}(\eta-\eta_0)\vartheta_0 - \mathscr{T}(\eta-\eta_0)\mu(\eta_1, \eta_2, ...\eta_p, \vartheta(.))$$

$$+\frac{1}{\Gamma(-\alpha-n)}\int_{\eta_0}^{\eta} \frac{\mathscr{T}(\eta-\nu)\chi^{(n)}(\nu, \vartheta(\nu), w(\nu), \gamma(\nu))d\nu}{(\eta-\nu)^{-\alpha+1-n}}, \tag{3.1.1}$$

with $\eta \in [\eta_0, \eta_0+\xi]$, is said to be a mild solution of problem (1.1) -(1.2) on $[\eta_0, \eta_0+\xi]$

For shake of our convenience or for further use we list the following hypothesis.

(H_1) If $\forall\ \vartheta_1, \vartheta_2 \in \Upsilon$, then $\exists$ a constant $\Lambda > 0$ such that

$\| \mu(\eta_1, \eta_2, ...\eta_p, \vartheta(1)) - \mu(\eta_1, \eta_2, ...\eta_p, \vartheta(2)) \| \leq \Lambda \| \vartheta_1 - \vartheta_2 \|_{\Upsilon}$

(H_2) If for some $\mathscr{M} \geq 1$, then A is the infinitesimal generator of a $\mathscr{C}_0$ semigroup $\mathscr{T}(\eta), \eta \geq 0$ on $\mathscr{X}$ such that

$\| \mathscr{T}(\eta) \| \leq \mathscr{M}$

(H_3) There are constants $\mathscr{L}_1, \Psi_1, \mathscr{H}_1$ and Λ_1 such that

$$\mathscr{L}_1 = \max_{\eta_0 \leq \eta \leq \eta_0+\xi} \| \chi^n(\eta, 0, 0, 0) \|,$$

$$\Psi_1 = \max_{\eta_0 \leq \nu, \eta \leq \eta_0+\xi} \| \varphi(\eta, \nu, 0, 0) \|,$$

$$\mathscr{H}_1 = \max_{\eta_0 \leq \nu, \eta \leq \eta_0+\xi} \| \varphi(\eta, \nu, 0, 0) \|,$$

$$\Lambda_1 = \max_{\vartheta \in E} \| \mu(\eta_1, \eta_2, ...\eta_p, \vartheta(.)) \| .$$

(H_4) The constants $\mathscr{M}\Lambda_1, \mathscr{L}, \Psi, \Psi_1, \mathscr{H}, \mathscr{H}_1, \xi$ and r satisfy the following two inequalities :

$$\mathscr{M}\left[\| \vartheta_0 \| + \Lambda_1 + \mathscr{L}r\xi + \mathscr{L}\Psi r\xi^2 + \mathscr{L}\Psi_1\xi^2 + \mathscr{L}\mathscr{H}r\xi^2 + \mathscr{L}\mathscr{H}_1\xi^2 + \mathscr{L}_1\xi\right] \leq r,$$

$$[\mathscr{M}\Lambda + \mathscr{M}\mathscr{L}\xi + \mathscr{M}\mathscr{L}\Psi\xi^2 + \mathscr{M}\mathscr{L}\mathscr{H}\xi^2] < 1.$$

3.2. Main Results.

Theorem 3.2.1 Assume that

(i) hypotheses $(H_1) - (H_4)$ hold,

(ii) $\chi : [\eta_0, \eta_0 + \xi] \times \mathscr{X} \times \mathscr{X} \times \mathscr{X} \to \mathscr{X}$ is continuous in η on $[\eta_0, \eta_0 + \xi]$ and there exists a constant $\mathscr{L} > 0$ such that

$$\| \chi^{(n)}(\eta, \vartheta_1, \upsilon_1, \sigma_1) - \chi^{(n)}(\eta, \vartheta_2, \upsilon_2, \sigma_2) \|$$
$$\leq \mathscr{L}\left(\| \vartheta_1 - \vartheta_2 \| + \| \upsilon_1 - \upsilon_2 \| + \| \sigma_1 - \sigma_2 \|\right),$$

for $\vartheta_i, \upsilon_i, \sigma_i \in \Omega_r, i = 1,2$.

(iii) $\varphi, \psi : [\eta_0, \eta_0 + \xi] \times [\eta_0, \eta_0 + \xi] \times \mathscr{X} \to \mathscr{X}$ are continuous in ν, η on $[\eta_0, \eta_0 + \xi]$ and there exist positive constants $\Psi, \mathscr{H}$ such that

$$\| \varphi(\eta, \nu, \vartheta_1,) - \varphi(\eta, \nu, \vartheta_2,) \| \leq \Psi\left(\| \vartheta_1 - \vartheta_2 \|\right),$$
$$\| \psi(\eta, \nu, \vartheta_1,) - \psi(\eta, \nu, \vartheta_2,) \| \leq \mathscr{H}\left(\| \vartheta_1 - \vartheta_2 \|\right),$$

for $\vartheta_i, \upsilon_i \in \Omega_r, i = 1,2$.

Then problem (1.1) -(1.2) has a unique mild solution on $[\eta_0, \eta_0 + \xi]$.

Proof Let $\Phi : \Upsilon \to \Upsilon$ be an operator defined by

$$(\Phi\sigma)(\eta) = \mathscr{T}(\eta - \eta_0)\vartheta_0 - \mathscr{T}(\eta - \eta_0)\mu(\eta_1, \eta_2, ...\eta_p, \sigma(.))$$
$$+ \frac{1}{\Gamma(-\alpha - n)} \int_{\eta_0}^{\eta} \frac{\mathscr{T}(\eta - \nu)\chi^{(n)}(\nu, \sigma(\nu), w(\nu), \gamma(\nu))d\nu}{(\eta - \nu)^{-\alpha + 1 - n}},$$

for $\eta \in [\eta_0, \eta_0 + \xi]$.

Claim: Φ maps Υ into itself.

By using hypotheses $(H_2) - (H_4)$ and assumptions $(ii), (iii)$, we have

$$\| (\Phi\sigma)(\eta) \| = \| \mathscr{T}(\eta - \eta_0)\vartheta_0 \| + \| \mathscr{T}(\eta - \eta_0)\mu(\eta_1, \eta_2, ...\eta_p, \sigma(.)) \|$$

$$+ \parallel \int_{\eta_0}^{\eta} \frac{\mathscr{T}(\eta-\nu)\chi^{(n)}(\nu,\sigma(\nu),w(\nu),\gamma(\nu))d\nu}{\Gamma(-\alpha-n)(\eta-\nu)^{-\alpha+1-n}} \parallel$$

$$\leq \mathscr{M} \parallel \vartheta_0 \parallel + \mathscr{M}\Lambda_1\mathscr{M}$$

$$\int_{\eta_0}^{\eta} \left[\parallel \chi^{(n)}(\nu,\sigma(\nu),w(\nu),\gamma(\nu)) - \chi^{(n)}(\nu,0,0,0) \parallel + \parallel \chi^{(n)}(\nu,0,0,0) \parallel\right] \frac{d\nu}{\parallel \zeta \parallel}$$

$$\leq \mathscr{M} \parallel \vartheta_0 \parallel + \mathscr{M}\Lambda_1\mathscr{M} \int_{\eta_0}^{\eta} [\mathscr{L}r + \mathscr{L}\xi(\Psi r + \Psi_1) + \mathscr{L}\xi(\mathscr{H}r + \mathscr{H}_1) + \mathscr{L}_1] \frac{ds}{\zeta^*}$$

$$\leq \mathscr{M}\left[\parallel \vartheta_0 \parallel + \Lambda_1 + \mathscr{L}r\xi + \mathscr{L}\Psi r\xi^2 + \mathscr{L}\Psi_1\xi^2 + \mathscr{L}\mathscr{H}r\xi^2 + \mathscr{L}\mathscr{H}_1\xi^2 + \mathscr{L}_1\xi\right] \frac{1}{\zeta^*}$$

$$\leq r\frac{1}{\zeta^*}$$

where $\Gamma(-\alpha-n)(\eta-\nu)^{-\alpha+1-n} = \zeta$ and $\parallel \zeta \parallel = \zeta^*$

Thus, Φ maps Υ into itself.

Now, for every $\sigma_1, \sigma_2 \in \Upsilon, \eta \in [\eta_0, \eta_0 + \xi]$ and using hypotheses $(H_1), (H_2), (H_4)$ and assumptions $(ii), (iii)$, we obtain

$$\parallel (\Phi\sigma_1)(\eta) - (\Phi\sigma_2)(\eta) \parallel \leq \parallel \mathscr{T}(\eta-\eta_0) \parallel$$

$$+ \parallel \mu(\eta_1,\eta_2,\ldots\ldots\ldots\eta_p,\sigma_1(.)) - \mu(\eta_1,\eta_2,\ldots\ldots\ldots\eta_p,\sigma_2(.)) \parallel$$

$$+ \int_{\eta_0}^{\eta} \parallel \mathscr{T}(\eta-\nu) \parallel\parallel \left[\chi^{(n)}(\nu,\sigma_1(\nu),w(\nu),\gamma(\nu)) - \chi^{(n)}(\nu,\sigma_2(\nu),w(\nu),\gamma(\nu))\right] \frac{d\nu}{\parallel \zeta \parallel}$$

$$\leq \mathscr{M}\Lambda \parallel \sigma_1 - \sigma_2 \parallel_\Upsilon + \mathscr{M}\mathscr{L} \parallel \sigma_1 - \sigma_2 \parallel_\Upsilon + \int_{\eta_0}^{\eta} \left[1 + \Psi\int_{\eta_0}^{\nu} d\mathscr{T} + \mathscr{H}\int_{\eta_0}^{\eta_0+\xi} d\mathscr{T}\right] \frac{d\nu}{\zeta^*}$$

$$\leq \mathscr{M}\Lambda \parallel \sigma_1 - \sigma_2 \parallel_\Upsilon + \mathscr{M}\mathscr{L} \parallel \sigma_1 - \sigma_2 \parallel_\Upsilon \xi[1 + \Psi\xi + \mathscr{H}\xi]\frac{d\nu}{\zeta^*}$$

$$\leq q \parallel \sigma_1 - \sigma_2 \parallel_\Upsilon \frac{1}{\zeta^*}$$

where $q = \mathscr{M}\Lambda + \mathscr{M}\mathscr{L}\xi + \mathscr{M}\mathscr{L}\Psi\xi^2 + \mathscr{M}\mathscr{L}\mathscr{H}\xi^2$ and hence, we obtain

$$\parallel (\Phi\sigma_1) - (\Phi\sigma_2) \parallel_\Upsilon \leq q \parallel \sigma_1 - \sigma_2 \parallel_\Upsilon \frac{1}{\zeta^*},$$

where $0 < q < 1$. Hence the operator Φ is a contraction on Υ. By using the Banach fixed point theorem,we observe that the function Φ has a unique fixed point in the space Υ and this point is the mild solution of problem (1.1) -(1.2) on $[\eta_0, \eta_0 + \xi]$.

4. Applications

We consider the fractional Volterra -fredholm partial integrodifferential equation as follows,

$$\frac{\partial^\alpha \phi(\rho,\eta)}{\partial \eta^\alpha} + \frac{\partial^2 \phi(\rho,\eta)}{\partial \rho^2}$$

$$= \theta(\eta, \phi(\rho,\eta), \int_{\eta_0}^{\eta} \varphi_1(\eta, v, \vartheta(v))dv, \int_{\eta_0}^{\eta_0+\xi} \psi_1(\eta, v, \vartheta(v))dv) \tag{4.1}$$

$$0 < \rho < 1, \quad \eta \in [\eta_0, \eta_0 + \xi]$$

with initial and boundary conditions

$$\phi(0,\eta) = \phi(1,\eta) = 0, \quad \eta \in [\eta_0, \eta_0 + \xi] \tag{4.2}$$

$$\phi(\rho,0) + \sum_{i=1}^{\theta} \phi(\rho,\eta_i) = \phi_0(\rho), 0 \le \eta_0 < \eta_1 < \eta_2 < ....\eta_p \le \eta_0 + \xi. \tag{4.3}$$

where $d : (0,1) \times [\eta_0, \eta_0 + \xi] \times R \to R, \theta : [\eta_0, \eta_0 + \xi] \times R \times R \times R \to R, \varphi_1, \psi_1 : [\eta_0, \eta_0 + \xi] \times [\eta_0, \eta_0 + \xi] \times R \to R$ are continuous functions.

Let $\mathscr{X} = \mathscr{L}^2[0,1]$ be the space of square integrable functions. Let $\beta : \mathscr{X} \to \mathscr{X}$ be an operator defined as $\beta(\rho) = \sigma''$ with dense domain
$D(\beta) = \{\sigma \in \mathscr{X} : \sigma, \sigma' \textit{ are absolutely continuous}, \ \sigma'' \in \mathscr{X} \textit{ and } \sigma(0) = \sigma(1) = 0\}$, generates an evolution system and $R_\vartheta(\eta, v)$ can be extracted from evolution system, such that $\| R_\vartheta(\eta, v) \| \le M_0, M_0 > 0$ for $v < \eta$ and $\vartheta \in \lambda \subset \mathscr{X}$. Define the functions $\chi : [\eta_0, \eta_0 + \xi] \times \mathscr{X} \times \mathscr{X} \times \mathscr{X} \to \mathscr{X}, \varphi : [\eta_0, \eta_0 + \xi] \times [\eta_0, \eta_0 + \xi] \times \mathscr{X} \to \mathscr{X}, \psi : [\eta_0, \eta_0 + \xi] \times [\eta_0, \eta_0 + \xi] \times \mathscr{X} \to \mathscr{X}$ and $\mu : [\eta_0, \eta_0 + \xi]^p \times \mathscr{X} \to \mathscr{X}$ as follows

$$\chi(\eta, \vartheta, \upsilon, \sigma)(\rho) = \theta(\eta, \vartheta(\rho), \upsilon(\rho), \sigma(\rho)),$$

$$\varphi(\eta, v, \vartheta)(\rho) = \varphi_1(\eta, v, \vartheta(\rho)),$$

$$\psi(\eta, v, \vartheta)(\rho) = \psi_1(\eta, v, \vartheta(\rho)),$$

$$\mu(\eta_1, \eta_2, ...\eta_p, \vartheta(.)) = \sum_{i=1}^{\theta} \phi(v, \eta_i)$$

for $\eta \in [\eta_0, \eta_0+\xi], \vartheta, \upsilon, \sigma \in \mathscr{X}$ and $0<\rho<1$. We assume that the functions θ, φ_1 and ψ_1 in (4.1) satisfy all the hypotheses of the Theorem 3.2. Also we suppose that

$$| \sum_{i=1}^{\theta} \phi(\rho, \eta_i) - \sum_{i=1}^{\theta} \phi(v, \eta_i) | \leq \Lambda^* \sup_{\eta \in [\eta_0, \eta_0+\xi]} | \rho(\eta) - v(\eta) |$$

for $\rho, v \in \Upsilon_1 = \mathscr{C}([\eta_0, \eta_0+\xi]; R)$ and some constant $\Lambda^* > 0$. Then the above problem (4.1) -(4.3) can be formulated abstractly as quasilinear mixed integrodifferential equation in Banach space $\mathscr{X}$:

$$\vartheta^{\alpha+1}(\eta) + \beta\vartheta(\eta) = \chi(\eta, \vartheta(\eta), \int_{\eta_0}^{\eta} \varphi(\eta, \nu, \vartheta(\nu))d\nu, \int_{\eta_0}^{\eta_0+\xi} \psi(\eta, \nu, \vartheta(\nu))d\nu),$$

$$\eta \in [\eta_0, \eta_0+\xi]$$

$$\vartheta(\eta_0) + \mu(\eta_1, \eta_2, ...\eta_p, \vartheta(.)) = \vartheta_0,$$

Since all the hypotheses of the Theorem 3.2 are satisfied, the Theorem 3.2 can be applied to guarantee the mild solution of the fractional mixed Volterra–Fredholm partial integrodifferential equations (4.1) -(4.3).

5. CONCLUSIONS

The purpose of this paper is to discuss the existence and uniqueness of solution of differential equation of fractional order, by using the Banach fixed point theorem and the theory of strongly continuous cosine family. Moreover we also discuss the existence and uniqueness of mild and strong solution of initial value problem of fractional order subjected to non-local conditions, by using the Banach fixed point theorem and the theory of strongly continuous cosine family.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

REFERENCES

[1] G. M. N guerekata, A Cauchy problem for some fractional abstract differential equation with non-local condition, Nonlinear Anal.: Theory Meth. Appl. 70(2009), 1873-1876.

[2] K. Balachandran, J. Y. Park, Non-local Cauchy problem for abstract fractional semi linear evolution equations, Nonlinear Anal.: Theory Meth. Appl. 71(2009), 4471-4475.

[3] K. M. Furati, N. Tatar, An existence result for a non-local fractional differential problem , J. Fract. Calc. 26(2004), 43-51.

[4] O. K. Jaradat, A. Al - Omari, S. Momani, Existence of the mild solution for fractional semi - linear initial value problems, Nonlinear Anal.: Theory Meth. Appl. 69(2008), 3153-3159.

[5] I. Podlubny, Fractional differential equations, Academic Press USA, (1999).

[6] S. Das, Functional fractional calculus, Springer, (2011).

[7] A. Wongcharoen, S. K. Ntouyas, J. Tariboon, On coupled systems for hilfer fractional differential equations with nonlocal integral boundary conditions, J. Math. 2020 (2020), 2875152.

[8] M. Wang, B. Jia, C. Chen, X. Zhu, F. Du, Discrete fractional Bihari inequality and uniqueness theorem of solutions of nabla fractional difference equations with non-Lipschitz nonlinearities, Appl. Math. Comput. 376 (2020), 125118.

[9] M. M. El-Borai, Semigroups and some nonlinear fractional differential equations, Appl. Math. Comput. 149 (2004) 823–831.

[10] V. Lakshmikantham, J. V. Devi, Theory of fractional differential equations in a Banach space. Eur. J. Pure Appl. Math. 1(2008), 38-45.

[11] J. Alzabut, T. Abdeljawad, F. Jarad, W. Sudsutad, A Gronwall inequality via the generalized proportional fractional derivative with applications, J. Inequal. Appl. 2019(2019), 101.