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KANNAN'S AND CHATTERJEE'S TYPE FIXED POINT THEOREMS USING ψ-POSITIVE FUNCTIONS IN C^*- ALGEBRA VALUED B-METRIC SPACES

R. A. RASHWAN[1], SALEH OMRAN[2], ASMAA FANGARY[2,*]

[1]Department of Mathematics, Faculty of Science, Assuit University, Assuit 71516, Egypt

[2]Department of Mathematics, Faculty of Science, South Valley University, Qena, 83523, Egypt

Abstract. The aim of this present paper is to obtain some fixed point theorems such as Kannan and Chatterjee type and their extension for a self mappings in a complete C^*- algebra valued b-metric space by using positive functions on C^*-algebras.

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1. Introduction

The Banach contraction principle [7] is a fundamental result in fixed point theory and has a great many application, and they are scattered throughout almost all branches of mathematics. It has been extensively used in proving existence and uniqueness of solutions to various functional equations, particularly differential and integral equations.

Many generalizations of Banach fixed point theorem were obtained by many authors have extended, generalized and improved Banach fixed point theorem in different ways. One of the most important of these generalizations is Kennan's fixed point theorem. In 1968, Kannan [4]

*Corresponding author

E-mail address: asmaa.fangary44@yahoo.com

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showed that a contractive mapping with a fixed point need not be necessarily continuous in proving the following result:

theorem [4] Let (X,d) be a complete metric space and $T : X \longrightarrow X$ be a mapping such that:

$$d(Tx,Ty) \preceq k[d(Tx,x)+d(Ty,y)]$$

for all $x,y \in X$ and $k \in [0,\frac{1}{2})$. Then T has a unique fixed point. The importance of the above result lies in the fact that Kannan's theorem characterizes the completeness of the metric space. Theorem 1.1 is one of the several generalizations of the Banach contraction principle which were derived either by changing the contraction condition or by changing the space to a more generalized space.

In [9], the concept of C^*-algebra-valued b-metric spaces was introduced. The main idea consists in using the set of all positive elements of a unital C^*-algebra instead of the set of real numbers. They presented some fixed point results for mapping under contractive or expansive conditions in these spaces.

In this paper, we give some fixed point theorems for self-map with a new contractive condition depend on ψ-positive contractive mapping.

To begin with, we collect some definitions and basic facts on the theory of C^*-algebras, which will be needed in the sequel. Suppose that $\mathbb{A}$ is an unital algebra with the unit I. An involution on $\mathbb{A}$ is a conjugate-linear map $a \longrightarrow a^*$ on $\mathbb{A}$ such that $(a^*)^* = a$ and $(ab)^* = b^*a^*$ for all $a,b \in \mathbb{A}$. The pair$(\mathbb{A},*)$ is called a $*$-algebra [1]. A Banach $*$-algebra is a $*$-algebra $\mathbb{A}$ together with a complete submultiplicative norm such that $\|a^*\| = \|a\|\ \forall a \in \mathbb{A}$. A C^*-algebra is a Banach $*$-algebra such that $\|a^*a\| = \|a\|^2\ \forall a \in \mathbb{A}$.

An element $x \in \mathbb{A}$ is a positive element, denote it by $x \succeq 0_{\mathbb{A}}$, if $x \in \mathbb{A}_h$ and $\sigma(x) \subset [0,\infty]$, where $\sigma(x)$ is the spectrum of x and $\mathbb{A}_h = \{x \in \mathbb{A} : x^* = x\}$. Using positive elements, one can define a partial ordering $\preceq$ on $\mathbb{A}_h$ as follows: $x \preceq y$ if and only if $y - x \succeq 0_{\mathbb{A}}$. From now, by $\mathbb{A}_+$ we denote the set $\{x \in \mathbb{A} : x \succeq 0_{\mathbb{A}}\}$ and $|x| = (x^*x)^{\frac{1}{2}}$, $\acute{\mathbb{A}_+}$= $\{a \in \mathbb{A}_+ : ab = ba\ \forall\ b \in \mathbb{A}_+\}$.

2. PRELIMINARIES

Defination [9] Let X be a nonempty set, and $b \in \acute{\mathbb{A}}$ such that $b \succeq I$. Suppose the mapping $d : X \times X \longrightarrow \mathbb{A}$ satisfies:

(1) $0_{\mathbb{A}} \preceq d_{\mathbb{A}}(x,y)$ for all $x,y \in X$ and $d_{\mathbb{A}}(x,y) = 0_{\mathbb{A}} \Leftrightarrow x = y$.

(2) $d_{\mathbb{A}}(x,y) = d_{\mathbb{A}}(y,x)$ for all $x,y \in X$.

(3) $d_{\mathbb{A}}(x,y) \preceq b[d_{\mathbb{A}}(x,z) + d_{\mathbb{A}}(z,y)]$ for all $x,y,z \in X$.

Then d is called a C^*-algebra-valued b-metric on X and $(X,\mathbb{A},d)$ is a C^*-algebra-valued b-metric space.

Defination [9] Let $(X,\mathbb{A},d)$ be a C^*-algebra-valued b-metric space. Suppose that $\{x_n\} \subset X$ and $x \in X$. If for any $c \succ 0_{\mathbb{A}}$ there is a natural number N such that for all $n > N$, $d_{\mathbb{A}}(x_n,x) \prec c$, then $\{x_n\}$ is said to be converge with respect to $\mathbb{A}$, and $\{x_n\}$ converges to x and x is the limit of $\{x_n\}$. We denote it by $\lim\limits_{n \longrightarrow +\infty} \{x_n\} = x$.

If for any $c \succ 0_{\mathbb{A}}$ there is $\mathbb{N}$ such that for all $n,m > \mathbb{N}$, $d_{\mathbb{A}}(x_n,x_m) \prec c$, then $\{x_n\}$ is said to be a Cauchy with respect to $\mathbb{A}$.

We say $(X,\mathbb{A},d)$ is a complete C^*-algebra-valued b- metric space if every Cauchy sequence with respect to $\mathbb{A}$ is convergent.

Example 1.1 Let $X = \mathbb{C}$ and $\mathbb{A} = M_n(\mathbb{C})$ the set of all $n \times n$-matrices with entries in $\mathbb{C}$.

Defin

$$d(a,b) = \begin{pmatrix} |a_1 - b_1|^p & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & |a_n - b_n|^p \end{pmatrix}$$

where $a = (a_i)_{i=1}^n$, $b = (b_i)_{i=1}^n$ are two $m \times n$-matrices, $a_i, b_i \in \mathbb{C}$ for all $i = 1, \cdots, n$.

One can define a partial ordering on $(\preceq M_n(\mathbb{C}))$ as following $a_i \preceq b_i$ if and only if $Re(a_i) \leq Re(b_i)$ and $Im(a_i) \leq Im(b_i) \quad \forall i = 1, \cdots, n$. And an element $a \geq 0_{\mathbb{A}}$ is positive in $M_n(\mathbb{C})$ if and only if $Re(a_i) \geq 0$ and $Im(a_i) \geq 0$ for all $i = 1, \cdots, n$. $(X,\mathbb{C},d)$ is C^*-algebra-valued b-metric space.

One can prove that

$$d(a,c) \preceq 2^p(d(a,b) + d(b,c)),$$

for all $a,b,c \in M_n(\mathbb{R})$.

We need only to use the following inequality

$$|x - z|^p \leq 2^p(|x - y|^p + |y - z|^p).$$

Where $b = 2^p I_{M_n(\mathbb{R})} \succ I_{M_n(\mathbb{R})} \quad \forall p > 1$, where $I_{M_n(\mathbb{R})}$ is the unite element in $M_n(\mathbb{R})$.

Lemma 1.1 [6, 1] Suppose that $\mathbb{A}$ is a unital C^*-algebra with a unit I.

(1) For any $a \in \mathbb{A}_+$ we have $a \preceq I \Leftrightarrow \|a\| \leq 1$.

(2) For all $a, b \in \mathbb{A}$, $0_{\mathbb{A}} \preceq a \preceq b$ implies that $\|a\| \preceq \|b\|$.

(3) If $a \in \mathbb{A}_+$ with $\|a\| < \frac{1}{2}$, then $I - a$ is invertable and $\|a(I-a)^{-1}\| < 1$.

(4) Suppose that $a, b \in \mathbb{A}$ with $a, b \succeq 0_{\mathbb{A}}$ and $ab = ba$, then $ab \succeq 0_{\mathbb{A}}$, where $0_{\mathbb{A}}$ is the zero element in $\mathbb{A}$.

(5) by $\acute{\mathbb{A}}$ we denote the set $\{a \in \mathbb{A} : ab = ba \quad \forall b \in \mathbb{A}\}$ Let $a \in \acute{\mathbb{A}}$, if $b, c \in \mathbb{A}$ with $b \succeq c \succeq 0_{\mathbb{A}}$ $(I-a)^{-1}b \succeq (I-a)^{-1}c$.

Defination [1] If $\psi_{\mathbb{A}} : \mathbb{A} \longrightarrow \mathbb{B}$ is a linear mapping in C^*-algebras. It is said to be positive function if $\psi_{\mathbb{A}}(\mathbb{A}_+) \subseteq \mathbb{B}_+$, where $\mathbb{A}_+$ the positive cone in $\mathbb{A}$, and $\mathbb{B}_+$ the positive cone in $\mathbb{B}$.

Proposition 1.1 [1] Let $\mathbb{A}$ be a C^*-algebra with I, then the positive function is bounded, continuous, contractive and $\psi_{\mathbb{A}}(1) = \|\psi_{\mathbb{A}}\|$.

Defination [2] Suppose that $\mathbb{A}$ and $\mathbb{B}$ are C^*-algebra. Amapping $\psi_{\mathbb{A}} : \mathbb{A} \longrightarrow \mathbb{B}$ is said to be C^*-homomorphism if :

(1) $\psi_{\mathbb{A}}(ax + by) = a\psi_{\mathbb{A}}(x) + b\psi_{\mathbb{A}}(y)$.

(2) $\psi_{\mathbb{A}}(xy) = \psi_{\mathbb{A}}(x)\psi_{\mathbb{A}}(y)$.

(3) $\psi_{\mathbb{A}}(x^*) = \psi_{\mathbb{A}}(x)^*$.

(4) $\psi_{\mathbb{A}}$ the unit in $\mathbb{A}$ to the unit in $\mathbb{B}$

Corollary [2] Every C^*-homomorphism is contractive and hence bounded.

Corollary [2] Suppose that $\psi_{\mathbb{A}}$ is C^*-isomorphism from $\mathbb{A}$ to $\mathbb{B}$, then $\sigma((\psi_{\mathbb{A}}(x))) = \sigma(x)$ and $\|\psi_{\mathbb{A}}(x)\| = \|x\|$.

Defination [5] Let $\psi_{\mathbb{A}} : \mathbb{A}_+ \longrightarrow \mathbb{A}_+$ be a positive function and having the following constraints:

(1) $\psi_{\mathbb{A}}$ is continuous and nondecreasing.

(2) $\psi_{\mathbb{A}}(c) = 0_{\mathbb{A}}$ if and only if $c = 0_{\mathbb{A}}$.

(3) $\psi_{\mathbb{A}}(c) \prec c \quad \forall c \succ 0_{\mathbb{A}} \quad (c \in \mathbb{A}_+)$.

(4) $\sum_{k=0}^{n} b^k \psi_{\mathbb{A}}^k \longrightarrow 0_{\mathbb{A}}$ at n $\longrightarrow +\infty$ where $b \in \acute{\mathbb{A}}_+$ with $b \succeq I$.

Theorem [1]

(1) The set $\mathbb{A}_+$ is equal to $\{a^*a : a \in \mathbb{A}\}$.

(2) If $a, b \in \mathbb{A}_{sa}$ and $c \in \mathbb{A}$, then $a \preceq b \Rightarrow c^*ac \preceq c^*bc$.

(3) If $\mathbb{A}$ is unital and a, b are positive invertible elements, then $a \preceq b \Rightarrow 0_{\mathbb{A}} \preceq a^{-1} \preceq b^{-1}$.

3. Main Results

In this section, we give some basic fixed point theorems for self-map with contractive conditions in complete C^*-algebra-valued b-metric spaces.

Theorem 3.1 Let $(X, \mathbb{A}, d_{\mathbb{A}})$ be a complete C^*-algebra valued b-metric space. Let $T : X \longrightarrow X$ be a self mapping satisfy the following contraction condition

$$d_{\mathbb{A}}(Tx, Ty) \preceq \psi_{\mathbb{A}}(d_{\mathbb{A}}(Tx, x) + d_{\mathbb{A}}(Ty, y)),$$

where $\psi_{\mathbb{A}} : \mathbb{A}_+ \longrightarrow \mathbb{A}_+$ satisfy the condition $\|\psi_{\mathbb{A}}\| < \frac{1}{2}$. Then T has a unique fixed point.

Proof. Let $x_0 \in X$ be arbitrary point and construct a sequence $\{x_n\}_{n=0}^{\infty} \subseteq X$ by the way: $x_1 = Tx_0, x_2 = Tx_1, \ldots.., x_{n+1} = Tx_n$

$$\begin{aligned}
d_{\mathbb{A}}(x_{n+1}, x_n) &= d_{\mathbb{A}}(Tx_n, Tx_{n-1}) \\
&\preceq \psi_{\mathbb{A}}(d_{\mathbb{A}}(Tx_n, x_n) + d_{\mathbb{A}}(Tx_{n-1}, x_{n-1})) \\
&= \psi_{\mathbb{A}}(d_{\mathbb{A}}(Tx_n, x_n) + \psi_{\mathbb{A}}(d_{\mathbb{A}}(Tx_{n-1}, x_{n-1})).
\end{aligned}$$

Implies $(I - \psi_{\mathbb{A}})d_{\mathbb{A}}(x_{n+1}, x_n) \preceq \psi_{\mathbb{A}}(d_{\mathbb{A}}(Tx_{n-1}, x_{n-1}))$.
Implies $d_{\mathbb{A}}(x_{n+1}, x_n) \preceq ((I - \psi_{\mathbb{A}})^{-1}\psi_{\mathbb{A}})(d_{\mathbb{A}}(Tx_{n-1}, x_{n-1}))$.

$$\begin{aligned}
d_{\mathbb{A}}(x_{n+1}, x_n) &\preceq ((I - \psi_{\mathbb{A}})^{-1}\psi_{\mathbb{A}})(d_{\mathbb{A}}(x_n, x_{n-1})) \\
&= ((I - \psi_{\mathbb{A}})^{-1}\psi_{\mathbb{A}})(d_{\mathbb{A}}(Tx_{n-1}, Tx_{n-2})) \\
&\preceq ((I - \psi_{\mathbb{A}})^{-1}\psi_{\mathbb{A}})^2(d_{\mathbb{A}}(x_{n-1}, x_{n-2})) \\
&\quad \vdots \\
&\preceq ((I - \psi_{\mathbb{A}})^{-1}\psi_{\mathbb{A}})^n(d_{\mathbb{A}}(x_1, x_0)).
\end{aligned}$$

Let $\phi_{\mathbb{A}} = (I - \psi_{\mathbb{A}})^{-1}\psi_{\mathbb{A}}$, since $\|\psi_{\mathbb{A}}\| < \frac{1}{2}$ implies $\|(I - \psi_{\mathbb{A}})^{-1}\psi_{\mathbb{A}}\| < 1$.
Implies $d_{\mathbb{A}}(x_{n+1}, x_n) \preceq \phi_{\mathbb{A}}^n(d_{\mathbb{A}}(x_1, x_0))$.
For any $m \geq 1, p \geq 1$, it follows that

$$\begin{aligned}
d_{\mathbb{A}}(x_m, x_{m+p}) &\preceq bd_{\mathbb{A}}(x_m, x_{m+1}) + b^2 d_{\mathbb{A}}(x_{m+1}, x_{m+2}) + \cdots \\
&\quad + b^{p-1}d_{\mathbb{A}}(x_{m+p-2}, x_{m+p-1}) + b^{p-1}d_{\mathbb{A}}(x_{m+p-1}, x_{m+p}) \\
&= b\phi_{\mathbb{A}}^m(d_{\mathbb{A}}(x_1, x_0)) + b^2\phi_{\mathbb{A}}^{m+1}(d_{\mathbb{A}}(x_1, x_0)) + \cdots \\
&\quad + b^{p-1}\phi_{\mathbb{A}}^{m+p-2}(d_{\mathbb{A}}(x_1, x_0)) + b^{p-1}\phi_{\mathbb{A}}^{m+p-1}(d_{\mathbb{A}}(x_1, x_0))
\end{aligned}$$

$$= \sum_{k=1}^{p-1} b^k \phi_{\mathbb{A}}^{m+k-1}(d_{\mathbb{A}}(x_1,x_0)) + b^{p-1}\phi_{\mathbb{A}}^{m+p-1}(d_{\mathbb{A}}(x_1,x_0))$$
$$\longrightarrow 0_{\mathbb{A}}(at \quad m,p \longrightarrow +\infty).$$

Therefore $\{x_n\}$ is a Cauchy sequence with respect to $\mathbb{A}$. By the completeness of $(X,\mathbb{A},d_{\mathbb{A}})$, there exists an $x \in X$ such that $\lim_{n\longrightarrow+\infty} x_n = x$.

$$\begin{aligned}
0_{\mathbb{A}} \preceq d_{\mathbb{A}}(Tx,x) &\preceq b[d_{\mathbb{A}}(Tx,Tx_n) + d_{\mathbb{A}}(Tx_n,x)] \\
&= bd_{\mathbb{A}}(Tx,Tx_n) + bd_{\mathbb{A}}(x_{n+1},x) \\
&\preceq b\psi_{\mathbb{A}}(d_{\mathbb{A}}(Tx,x) + d_{\mathbb{A}}(Tx_n,x_n)) + bd_{\mathbb{A}}(Tx_n,x) \\
&= b\psi_{\mathbb{A}}(d_{\mathbb{A}}(Tx,x)) + b\psi_{\mathbb{A}}(d_{\mathbb{A}}(Tx_n,x_n)) + bd_{\mathbb{A}}(Tx_n,x)
\end{aligned}$$

Implies $d_{\mathbb{A}}(Tx,x) \preceq ((I-b\psi_{\mathbb{A}})^{-1}b\psi_{\mathbb{A}})(d_{\mathbb{A}}(Tx_n,x_n)) + ((I-b\psi_{\mathbb{A}})^{-1})(d_{\mathbb{A}}(Tx_n,x)) \longrightarrow 0_{\mathbb{A}}(n \longrightarrow +\infty)$,
Implies $d_{\mathbb{A}}(Tx,x) = 0_{\mathbb{A}}$ implies $Tx = x$.
Hence, $Tx = x$, i.e., x is a fixed point of T.
To prove the uniquness suppose that $y(\neq x)$ is another fixed point of T. Since

$$\begin{aligned}
0_{\mathbb{A}} \preceq d_{\mathbb{A}}(x,y) = d_{\mathbb{A}}(Tx,Ty) &\preceq \psi_{\mathbb{A}}(d_{\mathbb{A}}(Tx,x) + d_{\mathbb{A}}(Ty,y)) \\
&= \psi_{\mathbb{A}}(d_{\mathbb{A}}(Tx,x)) + \psi_{\mathbb{A}}(d_{\mathbb{A}}(Ty,y)) \\
&\preceq 0_{\mathbb{A}}.
\end{aligned}$$

This is contradiction $\Longrightarrow d_{\mathbb{A}}(x,y) = 0_{\mathbb{A}} \Longrightarrow x = y$. □

Theorem 3.2 Let $(X,\mathbb{A},d_{\mathbb{A}})$ be a complete C^*-algebra valued b-metric space. Let $T : X \longrightarrow X$ be a self mapping satisfy the following contraction condition

$$d_{\mathbb{A}}(Tx,Ty) \preceq \psi_{\mathbb{A}}\left(\frac{d_{\mathbb{A}}(x,y)}{2} + \frac{d_{\mathbb{A}}(Tx,x)+d_{\mathbb{A}}(Ty,y)}{2}\right),$$

where $\psi_{\mathbb{A}} : \mathbb{A}_+ \longrightarrow \mathbb{A}_+$ satisfy the condition $\|\psi_{\mathbb{A}}\| < \frac{1}{4}$. Then T has a unique fixed point.

Proof. Let $x_0 \in X$ be arbitrary point and construct a sequence $\{x_n\}_{n=0}^{\infty} \subseteq X$ by the way: $x_1 = Tx_0, x_2 = Tx_1, ...,x_{n+1} = Tx_n$.

$$\begin{aligned}
d_{\mathbb{A}}(x_{n+1},x_n) &= d_{\mathbb{A}}(Tx_n,Tx_{n-1}) \\
&\preceq \psi_{\mathbb{A}}\left(\frac{d_{\mathbb{A}}(x_n,x_{n-1})}{2} + \frac{d_{\mathbb{A}}(Tx_n,x_n)+d_{\mathbb{A}}(Tx_{n-1},x_{n-1})}{2}\right) \\
&= \psi_{\mathbb{A}}\left(\frac{d_{\mathbb{A}}(x_n,x_{n-1})}{2} + \frac{d_{\mathbb{A}}(x_{n+1},x_n)}{2} + \frac{d_{\mathbb{A}}(x_n,x_{n-1})}{2}\right) \\
&= \psi_{\mathbb{A}}\left(d_{\mathbb{A}}(x_n,x_{n-1}) + \frac{d_{\mathbb{A}}(x_{n+1},x_n)}{2}\right).
\end{aligned}$$

Implies $(I-\frac{I}{2}\psi_{\mathbb{A}})(d_{\mathbb{A}}(x_{n+1},x_n)) \preceq \psi_{\mathbb{A}}(d_{\mathbb{A}}(x_n,x_{n-1}))$.

Implies

$$\begin{aligned} d_{\mathbb{A}}(x_{n+1},x_n) &\preceq (I-\tfrac{I}{2}\psi_{\mathbb{A}})^{-1}\psi_{\mathbb{A}})(d_{\mathbb{A}}(x_n,x_{n-1})) \\ &\preceq (I-\tfrac{I}{2}\psi_{\mathbb{A}})^{-1}\psi_{\mathbb{A}})^2(d_{\mathbb{A}}(x_{n-1},x_{n-2})) \\ &\vdots \\ &\preceq (I-\tfrac{I}{2}\psi_{\mathbb{A}})^{-1}\psi_{\mathbb{A}})^n(d_{\mathbb{A}}(x_1,x_0)). \end{aligned}$$

Let $\phi_{\mathbb{A}} = ((I-\frac{I}{2}\psi_{\mathbb{A}})^{-1}\psi_{\mathbb{A}})$, since $\|\psi_{\mathbb{A}}\| < \frac{1}{2}$ implies $\|((I-\frac{I}{2}\psi_{\mathbb{A}})^{-1}\psi_{\mathbb{A}})\| < 1$.

Implies $d_{\mathbb{A}}(x_{n+1},x_n) \preceq \phi_{\mathbb{A}}^n(d_{\mathbb{A}}(x_1,x_0))$.

For any $m \geq 1, p \geq 1$, it follows that

$$\begin{aligned} d_{\mathbb{A}}(x_m,x_{m+p}) &\preceq bd_{\mathbb{A}}(x_m,x_{m+1}) + b^2 d_{\mathbb{A}}(x_{m+1},x_{m+2}) + \cdots \\ &\quad + b^{p-1}d_{\mathbb{A}}(x_{m+p-2},x_{m+p-1}) + b^{p-1}d_{\mathbb{A}}(x_{m+p-1},x_{m+p}) \\ &= b\phi_{\mathbb{A}}^m(d_{\mathbb{A}}(x_1,x_0)) + b^2\phi_{\mathbb{A}}^{m+1}(d_{\mathbb{A}}(x_1,x_0)) + \cdots \\ &\quad + b^{p-1}\phi_{\mathbb{A}}^{m+p-2}(d_{\mathbb{A}}(x_1,x_0)) + b^{p-1}\phi_{\mathbb{A}}^{m+p-1}(d_{\mathbb{A}}(x_1,x_0)) \\ &= \sum_{k=1}^{p-1} b^k \phi_{\mathbb{A}}^{m+k-1}(d_{\mathbb{A}}(x_1,x_0)) + b^{p-1}\phi_{\mathbb{A}}^{m+p-1}(d_{\mathbb{A}}(x_1,x_0)) \\ &\longrightarrow 0_{\mathbb{A}}(at \quad m,p \longrightarrow +\infty). \end{aligned}$$

Therefore $\{x_n\}$ is a Cauchy sequence with respect to $\mathbb{A}$. By the completeness of $(X,\mathbb{A},d_{\mathbb{A}})$, there exists an $x \in X$ such that $\lim_{n \longrightarrow +\infty} x_n = x$.

$$\begin{aligned} 0_{\mathbb{A}} \preceq d_{\mathbb{A}}(Tx,x) &\preceq b[d_{\mathbb{A}}(Tx,Tx_n) + d_{\mathbb{A}}(Tx_n,x)] \\ &= bd_{\mathbb{A}}(Tx,Tx_n) + bd_{\mathbb{A}}(Tx_n,x) \\ &\preceq b\psi_{\mathbb{A}}(\tfrac{d_{\mathbb{A}}(x,x_n)}{2} + \tfrac{d_{\mathbb{A}}(Tx,x)+d_{\mathbb{A}}(Tx_n,x_n)}{2}) + bd_{\mathbb{A}}(Tx_n,x) \\ &= b\psi_{\mathbb{A}}(\tfrac{d_{\mathbb{A}}(x,x_n)}{2}) + b\psi_{\mathbb{A}}(\tfrac{d_{\mathbb{A}}(Tx,x)}{2}) + b\psi_{\mathbb{A}}(\tfrac{d_{\mathbb{A}}(x_{n+1},x_n)}{2}) + bd_{\mathbb{A}}(x_{n+1},x). \end{aligned}$$

Implies $(I-b\frac{I}{2}\psi_{\mathbb{A}})(d_{\mathbb{A}}(Tx,x)) \preceq b\psi_{\mathbb{A}}(\frac{d_{\mathbb{A}}(x,x_n)}{2}) + b\psi_{\mathbb{A}}(\frac{d_{\mathbb{A}}(x_{n+1},x_n)}{2}) + bd_{\mathbb{A}}(x_{n+1},x)$.

Implies $d_{\mathbb{A}}(Tx,x) \preceq ((I-b\frac{I}{2}\psi_{\mathbb{A}})^{-1})(b\psi_{\mathbb{A}}(\frac{d_{\mathbb{A}}(x,x_n)}{2}) + b\psi_{\mathbb{A}}(\frac{d_{\mathbb{A}}(x_{n+1},x_n)}{2}) + bd_{\mathbb{A}}(x_{n+1},x)) \longrightarrow 0_{\mathbb{A}}(at \quad n \longrightarrow +\infty)$.

Implies $d_{\mathbb{A}}(Tx,x) = 0_{\mathbb{A}} \Longrightarrow Tx = x$.

hence, $Tx = x$, i.e., x is a fixed point of T.

To prove the uniquness suppose that $y(\neq x)$ is another fixed point of T. Since

$$0_{\mathbb{A}} \preceq d_{\mathbb{A}}(x,y) = d_{\mathbb{A}}(Tx,Ty) \preceq \psi_{\mathbb{A}}(\tfrac{d_{\mathbb{A}}(x,y)}{2} + \tfrac{d_{\mathbb{A}}(Tx,x)+d_{\mathbb{A}}(Ty,y)}{2})$$
$$= \psi_{\mathbb{A}}(\tfrac{d_{\mathbb{A}}(x,y)}{2}).$$

Implies $\|d_{\mathbb{A}}(x,y)\| \leq \|\psi_{\mathbb{A}}(\frac{d_{\mathbb{A}}(x,y)}{2})\| = \|\frac{d_{\mathbb{A}}(x,y)}{2}\|$.
Implies $d_{\mathbb{A}}(x,y) = 0_{\mathbb{A}} \Longrightarrow x = y$. □

Theorem 3.3 Let $(X,\mathbb{A},d_{\mathbb{A}})$ be a complete C^*-algebra valued b-metric space. Let $T: X \longrightarrow X$ be a self mapping satisfy the following contraction condition

$$d_{\mathbb{A}}(Tx,Ty) \preceq \psi_{\mathbb{A}}(d_{\mathbb{A}}(Tx,y) + d_{\mathbb{A}}(Ty,x)),$$

where $\psi_{\mathbb{A}} : \mathbb{A}_+ \longrightarrow \mathbb{A}_+$ satisfy the conditions $\psi_{\mathbb{A}}(b) \in \acute{\mathbb{A}}_+$ for all $b \in \acute{\mathbb{A}}_+$ and $\|\psi_{\mathbb{A}}(b)\psi_{\mathbb{A}}\| \prec \frac{1}{2}$. Then T has a unique fixed point.

Proof. Let $x_0 \in X$ be arbitrary point and construct a sequence $\{x_n\}_{n=0}^{\infty} \subseteq X$ by the way: $x_1 = Tx_0, x_2 = Tx_1, \ldots.., x_{n+1} = Tx_n$

$$\begin{aligned} d_{\mathbb{A}}(x_{n+1},x_n) &= d_{\mathbb{A}}(Tx_n,Tx_{n-1}) \\ &\preceq \psi_{\mathbb{A}}(d_{\mathbb{A}}(Tx_n,x_{n-1}) + d_{\mathbb{A}}(Tx_{n-1},x_n)) \\ &= \psi_{\mathbb{A}}(d_{\mathbb{A}}(x_{n+1},x_{n-1})) \\ &\preceq \psi_{\mathbb{A}}(bd_{\mathbb{A}}(x_{n+1},x_n) + bd_{\mathbb{A}}(x_n,x_{n-1})) \\ &= \psi_{\mathbb{A}}(b)\psi_{\mathbb{A}}(d_{\mathbb{A}}(x_{n+1},x_n)) + \psi_{\mathbb{A}}(b)\psi_{\mathbb{A}}(d_{\mathbb{A}}(x_n,x_{n-1})) . \end{aligned}$$

Implies $[(I-\psi_{\mathbb{A}}(b))\psi_{\mathbb{A}}](d_{\mathbb{A}}(x_{n+1},x_n)) \preceq \psi_{\mathbb{A}}(b)\psi_{\mathbb{A}}(d_{\mathbb{A}}(x_n,x_{n-1}))$.

$$d_{\mathbb{A}}(x_{n+1},x_n) \preceq [(I-\psi_{\mathbb{A}}(b))\psi_{\mathbb{A}}]^{-1}(\psi_{\mathbb{A}}(b)\psi_{\mathbb{A}}(d_{\mathbb{A}}(x_n,x_{n-1})).$$

Let $\phi_{\mathbb{A}} = [(I-\psi_{\mathbb{A}}(b))\psi_{\mathbb{A}}]^{-1}(\psi_{\mathbb{A}}(b)\psi_{\mathbb{A}})$. since $\|\psi_{\mathbb{A}}(b)\psi_{\mathbb{A}}\| < \frac{1}{2}$.
Implies $\|((I-\psi_{\mathbb{A}}(b)\psi_{\mathbb{A}})^{-1}\psi_{\mathbb{A}}(b)\psi_{\mathbb{A}})\| < 1$.
Then

$$\begin{aligned} d_{\mathbb{A}}(x_{n+1},x_n) &\preceq \phi_{\mathbb{A}}(d_{\mathbb{A}}(x_n,x_{n-1})) \\ &\preceq \phi_{\mathbb{A}}^2(d_{\mathbb{A}}(x_{n-1},x_{n-2})) \\ &\vdots \\ &\preceq \phi_{\mathbb{A}}^n(d_{\mathbb{A}}(x_1,x_0)). \end{aligned}$$

For any $m \geq 1, p \geq 1$, it follows that

$$\begin{aligned} d(x_m,x_{m+p}) \preceq{} & bd_{\mathbb{A}}(x_m,x_{m+1}) + b^2 d_{\mathbb{A}}(x_{m+1},x_{m+2}) + \cdots \\ & + b^{p-1} d_{\mathbb{A}}(x_{m+p-2},x_{m+p-1}) + b^{p-1} d_{\mathbb{A}}(x_{m+p-1},x_{m+p}) \end{aligned}$$

$$
\begin{aligned}
&= b\phi_{\mathbb{A}}^{m}(d_{\mathbb{A}}(x_1,x_0)) + b^2\phi_{\mathbb{A}}^{m+1}(d_{\mathbb{A}}(x_1,x_0)) + \cdots \\
&\quad + b^{p-1}\phi_{\mathbb{A}}^{m+p-2}(d_{\mathbb{A}}(x_1,x_0)) + b^{p-1}\phi_{\mathbb{A}}^{m+p-1}(d_{\mathbb{A}}(x_1,x_0)) \\
&= \sum_{k=1}^{p-1} b^k\phi_{\mathbb{A}}^{m+k-1}(d_{\mathbb{A}}(x_1,x_0)) + b^{p-1}\phi_{\mathbb{A}}^{m+p-1}(d_{\mathbb{A}}(x_1,x_0)) \\
&\quad \longrightarrow 0_{\mathbb{A}}(at \quad m,p \longrightarrow +\infty).
\end{aligned}
$$

Therefore $\{x_n\}$ is a Cauchy sequence with respect to $\mathbb{A}$. By the completeness of $(X,\mathbb{A},d_{\mathbb{A}})$, there exists an $x \in X$ such that $\lim_{n\longrightarrow+\infty} x_n = x$.

$$
\begin{aligned}
0_{\mathbb{A}} \preceq d_{\mathbb{A}}(Tx,x) &\preceq b[d_{\mathbb{A}}(Tx,Tx_n) + d_{\mathbb{A}}(Tx_n,x)] \\
&= bd_{\mathbb{A}}(Tx,Tx_n) + bd_{\mathbb{A}}(x_{n+1},x) \\
&\preceq b\psi_{\mathbb{A}}(d_{\mathbb{A}}(Tx,x_n) + d_{\mathbb{A}}(Tx_n,x)) + bd_{\mathbb{A}}(Tx_n,x) \\
&= b\psi_{\mathbb{A}}(d_{\mathbb{A}}(Tx,x_n)) + b\psi_{\mathbb{A}}(d_{\mathbb{A}}(Tx_n,x)) + bd_{\mathbb{A}}(Tx_n,x) \\
&\preceq b\psi_{\mathbb{A}}(bd_{\mathbb{A}}(Tx,x) + bd_{\mathbb{A}}(x,x_n)) + b\psi_{\mathbb{A}}(d_{\mathbb{A}}(Tx_n,x)) + bd_{\mathbb{A}}(Tx_n,x).
\end{aligned}
$$

Implies

$d_{\mathbb{A}}(Tx,x) \preceq ((I - b\psi_{\mathbb{A}}(b)\psi_{\mathbb{A}})^{-1})(b\psi_{\mathbb{A}}(b)\psi_{\mathbb{A}}((d_{\mathbb{A}}(Tx_n,x_n))) + ((I - b\psi_{\mathbb{A}}(b)\psi_{\mathbb{A}})^{-1})(b\psi_{\mathbb{A}}(d_{\mathbb{A}}(Tx_n,x))) + ((I - b\psi_{\mathbb{A}}(b)\psi_{\mathbb{A}})^{-1})(b(d_{\mathbb{A}}(Tx_n,x))) \longrightarrow 0_{\mathbb{A}}(n \longrightarrow +\infty)$.

Implies $d_{\mathbb{A}}(Tx,x) = 0_{\mathbb{A}}$ implies $Tx = x$.

Hence, $Tx = x$, i.e., x is a fixed point of T.

To prove the uniquness suppose that $y(\neq x)$ is another fixed point of T. Since

$$
\begin{aligned}
0_{\mathbb{A}} \preceq d_{\mathbb{A}}(x,y) = d_{\mathbb{A}}(Tx,Ty) &\preceq \psi_{\mathbb{A}}(d_{\mathbb{A}}(Tx,y) + d_{\mathbb{A}}(Ty,x)) \\
&= \psi_{\mathbb{A}}(d_{\mathbb{A}}(x,y)) + \psi_{\mathbb{A}}(d_{\mathbb{A}}(x,y)).
\end{aligned}
$$

$\Longrightarrow \|d_{\mathbb{A}}(x,y)\| \leq \|\psi_{\mathbb{A}}(d_{\mathbb{A}}(x,y))\| + \|\psi_{\mathbb{A}}(d_{\mathbb{A}}(x,y))\| \leq \|d_{\mathbb{A}}(x,y)\| + \|d_{\mathbb{A}}(x,y)\| = 2\|d_{\mathbb{A}}(x,y)\|$.

This is contradiction $\Longrightarrow d_{\mathbb{A}}(x,y) = 0_{\mathbb{A}} \Longrightarrow x = y$. □

Theorem 4.4 Let $(X,\mathbb{A},d_{\mathbb{A}})$ be a complete C^*-algebra valued b-metric space. Let $T : X \longrightarrow X$ be a self mapping satisfy the following contraction condition

$$d_{\mathbb{A}}(Tx,Ty) \preceq \psi_{\mathbb{A}}(d_{\mathbb{A}}(x,y) + d_{\mathbb{A}}(Tx,y) + d_{\mathbb{A}}(Ty,x)),$$

where $\psi_{\mathbb{A}} : \mathbb{A}_+ \longrightarrow \mathbb{A}_+$ satisfy the conditions $\psi_{\mathbb{A}}(b) \in \acute{\mathbb{A}}_+$ for all $b \in \acute{\mathbb{A}}_+$ and $\|b\psi_{\mathbb{A}}(b)\psi_{\mathbb{A}}\| < \frac{1}{2}$. Then T has a unique fixed point.

Proof. Let $x_0 \in X$ be arbitrary point and construct a sequence $\{x_n\}_{n=0}^{\infty} \subseteq X$ by the way: $x_1 = Tx_0, x_2 = Tx_1, \ldots\ldots, x_{n+1} = Tx_n$

$$\begin{aligned} d_{\mathbb{A}}(x_{n+1},x_n) &= d_{\mathbb{A}}(Tx_n,Tx_{n-1}) \\ &\preceq \psi_{\mathbb{A}}(d_{\mathbb{A}}(x_n,x_{n-1})+d_{\mathbb{A}}(Tx_n,x_{n-1})+d_{\mathbb{A}}(Tx_{n-1},x_n)) \\ &= \psi_{\mathbb{A}}(d_{\mathbb{A}}(x_n,x_{n-1})+d_{\mathbb{A}}(x_{n+1},x_{n-1})) \\ &\preceq \psi_{\mathbb{A}}(d_{\mathbb{A}}(x_n,x_{n-1}))+\psi_{\mathbb{A}}(bd_{\mathbb{A}}(x_{n+1},x_n)+bd_{\mathbb{A}}(x_n,x_{n-1})) \\ &= \psi_{\mathbb{A}}(d_{\mathbb{A}}(x_n,x_{n-1}))+\psi_{\mathbb{A}}(b)\psi_{\mathbb{A}}(d_{\mathbb{A}}(x_{n+1},x_n)) \\ &\quad +\psi_{\mathbb{A}}(b)\psi_{\mathbb{A}}(d_{\mathbb{A}}(x_n,x_{n-1}))\,. \end{aligned}$$

Implies $[(I-\psi_{\mathbb{A}}(b))\psi_{\mathbb{A}}](d_{\mathbb{A}}(x_{n+1},x_n)) \preceq [\psi_{\mathbb{A}}+\psi_{\mathbb{A}}(b)\psi_{\mathbb{A}}](d_{\mathbb{A}}(x_n,x_{n-1}))$.

Then

$$d_{\mathbb{A}}(x_{n+1},x_n) \preceq [(I-\psi_{\mathbb{A}}(b))\psi_{\mathbb{A}}]^{-1}((\psi_{\mathbb{A}}+\psi_{\mathbb{A}}(b)\psi_{\mathbb{A}})(d_{\mathbb{A}}(x_n,x_{n-1})).$$

Let $\phi_{\mathbb{A}} = [(I-\psi_{\mathbb{A}}(b))\psi_{\mathbb{A}}]^{-1}(\psi_{\mathbb{A}}+\psi_{\mathbb{A}}(b)\psi_{\mathbb{A}})$.

Then

$$\begin{aligned} d_{\mathbb{A}}(x_{n+1},x_n) &\preceq \phi_{\mathbb{A}}(d_{\mathbb{A}}(x_n,x_{n-1})) \\ &\preceq \phi_{\mathbb{A}}^2(d_{\mathbb{A}}(x_{n-1},x_{n-2})) \\ &\quad\vdots \\ &\preceq \phi_{\mathbb{A}}^n(d_{\mathbb{A}}(x_1,x_0)). \end{aligned}$$

For any $m \geq 1, p \geq 1$, it follows that

$$\begin{aligned} d_{\mathbb{A}}(x_m,x_{m+p}) &\preceq bd_{\mathbb{A}}(x_m,x_{m+1})+b^2d_{\mathbb{A}}(x_{m+1},x_{m+2})+\cdots \\ &\quad +b^{p-1}d_{\mathbb{A}}(x_{m+p-2},x_{m+p-1})+b^{p-1}d_{\mathbb{A}}(x_{m+p-1},x_{m+p}) \\ &= b\phi_{\mathbb{A}}^m(d_{\mathbb{A}}(x_1,x_0))+b^2\phi_{\mathbb{A}}^{m+1}(d_{\mathbb{A}}(x_1,x_0))+\cdots \\ &\quad +b^{p-1}\phi_{\mathbb{A}}^{m+p-2}(d_{\mathbb{A}}(x_1,x_0))+b^{p-1}\phi_{\mathbb{A}}^{m+p-1}(d_{\mathbb{A}}(x_1,x_0)) \\ &= \sum_{k=1}^{p-1} b^k\phi_{\mathbb{A}}^{m+k-1}(d_{\mathbb{A}}(x_1,x_0))+b^{p-1}\phi_{\mathbb{A}}^{m+p-1}(d_{\mathbb{A}}(x_1,x_0)) \\ &\longrightarrow 0_{\mathbb{A}}(at \quad m,p \longrightarrow +\infty). \end{aligned}$$

Therefore $\{x_n\}$ is a Cauchy sequence with respect to $\mathbb{A}$. By the completeness of $(X,\mathbb{A},d_{\mathbb{A}})$, there exists an $x \in X$ such that $\lim_{n\longrightarrow+\infty} x_n = x$.

$$\begin{aligned} 0_{\mathbb{A}} \preceq d_{\mathbb{A}}(Tx,x) &\preceq b[d_{\mathbb{A}}(Tx,Tx_n)+d_{\mathbb{A}}(Tx_n,x)] \\ &= bd_{\mathbb{A}}(Tx,Tx_n)+bd_{\mathbb{A}}(x_{n+1},x) \\ &\preceq b\psi_{\mathbb{A}}(d_{\mathbb{A}}(x,x_n)+d_{\mathbb{A}}(Tx,x_n)+d_{\mathbb{A}}(Tx_n,x))+bd_{\mathbb{A}}(Tx_n,x) \\ &= b\psi_{\mathbb{A}}(d_{\mathbb{A}}(x,x_n))+b\psi_{\mathbb{A}}(d_{\mathbb{A}}(Tx,x_n))+b\psi_{\mathbb{A}}(d_{\mathbb{A}}(Tx_n,x))+bd_{\mathbb{A}}(Tx_n,x) \\ &\preceq b\psi_{\mathbb{A}}(d_{\mathbb{A}}(x,x_n))+b\psi_{\mathbb{A}}(bd_{\mathbb{A}}(Tx,x)+bd_{\mathbb{A}}(x,x_n)) \end{aligned}$$

$$+b\psi_{\mathbb{A}}(d_{\mathbb{A}}(Tx_n,x))+bd_{\mathbb{A}}(Tx_n,x).$$

Implies

$d_{\mathbb{A}}(Tx,x) \preceq ((I - b\psi_{\mathbb{A}}(b)\psi_{\mathbb{A}})^{-1})(b\psi_{\mathbb{A}}(d_{\mathbb{A}}(x,x_n))) + ((I - b\psi_{\mathbb{A}}(b)\psi_{\mathbb{A}})^{-1})(b\psi_{\mathbb{A}}(b)\psi_{\mathbb{A}}((d_{\mathbb{A}}(x,x_n))) + ((I-b\psi_{\mathbb{A}}(b)\psi_{\mathbb{A}})^{-1})(b\psi_{\mathbb{A}}(d_{\mathbb{A}}(Tx_n,x))) + ((I-b\psi_{\mathbb{A}}(b)\psi_{\mathbb{A}})^{-1})(b(d_{\mathbb{A}}(Tx_n,x))) \longrightarrow 0_{\mathbb{A}}(n \longrightarrow +\infty)$.

Implies $d_{\mathbb{A}}(Tx,x) = 0_{\mathbb{A}}$ implies $Tx = x$.

Hence, $Tx = x$, i.e., x is a fixed point of T.

To prove the uniquness suppose that $y(\neq x)$ is another fixed point of T. Since

$$\begin{aligned}0_{\mathbb{A}} \preceq d_{\mathbb{A}}(x,y) = d_{\mathbb{A}}(Tx,Ty) &\preceq \psi_{\mathbb{A}}(d_{\mathbb{A}}(x,y)+d_{\mathbb{A}}(Tx,y)+d_{\mathbb{A}}(Ty,x))\\ &= \psi_{\mathbb{A}}(d_{\mathbb{A}}(x,y))+\psi_{\mathbb{A}}(d_{\mathbb{A}}(x,y))+\psi_{\mathbb{A}}(d_{\mathbb{A}}(x,y)).\end{aligned}$$

Implies

$$\begin{aligned}\|d_{\mathbb{A}}(x,y)\| &\leq \|\psi_{\mathbb{A}}(d_{\mathbb{A}}(x,y))\|+\|\psi_{\mathbb{A}}(d_{\mathbb{A}}(x,y))\|+\|\psi_{\mathbb{A}}(d_{\mathbb{A}}(x,y))\|\\ &\leq \|d_{\mathbb{A}}(x,y)\|+\|d_{\mathbb{A}}(x,y)\|+\|d_{\mathbb{A}}(x,y)\| = 3\|d_{\mathbb{A}}(x,y)\|.\end{aligned}$$

This is contradiction $\Longrightarrow d_{\mathbb{A}}(x,y) = 0_{\mathbb{A}} \Longrightarrow x = y$. □

4. Conclusion

C^*-algebras is an interesting subject in the functional analysis and operator theory which plays an important role in fixed point theory.

In this paper, we introduced a new insight of C^*-algebra-valued b-metric space by using ψ positive function. Also, We define some contraction mapping and prove the existence and the uniqueness of some fixed point theorems such as Kannan and Chatterjee and their extension.

Conflict of Interests

The author(s) declare that there is no conflict of interests.

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