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## OPTIMIZATION OF THE CRAMER LUNDBERG MODEL BASED VALUE FUNCTION OF REINSURANCE WITH RANDOM CLAIMS AND NEW PREMIUM ARRIVAL

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**Abstract:** In general, an insurance company who experiences two opposing cash flows incoming cash premiums and outgoing claims that is also known as classical risk process that satisfies Cramér–Lundberg model. However the arrival of the new premium holders and their cash flow over a period of time was not considered in most works. In this model, we considered the arrival of new premiums with expectation of surplus process until ruin time with dynamic reinsurance strategy. For attaining this condition, we formulated a Value function which is bounded and satisfied by the Hamilton Jacobi Bellman (HJB) partial differential equation. We apply the policy iteration method to find the maximum the surplus level and corresponding dynamic reinsurance strategy under excess of loss, quota share and stop loss reinsurance problems.

**Keywords:** Cramér–Lundberg model; ruin probability; insurance; reinsurance; stochastic control; quota share; excess of loss; stop loss reinsurance.

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## 1. INTRODUCTION

Reinsurance is a method of sharing a part of loss and premiums of a insurer company by another company. Insurance company purchase reinsurance. Reinsurance allows insurance companies to remain solvent after major claims events, such as major disasters like hurricanes and wildfires. Reinsurance has roles in risk management, tax mitigation and other reasons. The company that purchases the reinsurance policy is called a ceding company or cedent or cedant. The company issuing the reinsurance policy is known as the reinsurer. Cedent company pays a premium to the reinsurer company, who in exchange pays a part of the claims incurred by Cedent.

Proportional reinsurance is a type of reinsurance where one or more reinsurers take a stated percentage share of each policy that an insurer issues then the reinsurer will receive that stated percentage of the premiums and will pay the stated percentage of claims in exchange the reinsurer will allow a ceding commission to the insurer to cover the costs incurred by the insurer. Example: Quota share. A non-proportional reinsurance is another type of reinsurance where the reinsurer only pays to the insurer if the total claims occurred in given period is more than a fixed amount, which is called the retention. Example: Excess of loss, stop loss.

The policy iteration method is an algorithm that manipulates the policy directly instead finding it indirectly via the optimal value functions. It determines the value function of a policy. It executes a policy and finds the expected infinite discounted reward that will be gained. It can be obtained by solving a system of linear equations. Once we know the value of each state under the current policy, we look whether value could be improved by changing the first action taken. If yes, we change the policy to take the new action whenever it is in that situation. This step must improve the performance of the policy. When no improvements are possible, then the policy must be optimal.

Many papers model optimal reinsurance or optimal investment solving various issues in risk theory. In these models, the insurer takes reinsurance and invests its capital in the insurance market. Some models use stochastic control theory and related methods to minimize the probability of ruin or the maximum expected utility of returning surplus. Taksar and Markussen [1] considered the optimal reinsurance policy which minimizes the ruin probability of the cedent.

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Bai and Guo [2] model the problem of maximizing the expected exponential utility of terminal surplus in proportional reinsurance. Asmussen et al. [3] models present the dynamic method of excess-of-loss reinsurance retention level and the dividend distribution policy for the purpose of maximizing the expected present value of the dividends. Irgens and Paulsen [4] present a model for optimal reinsurance and investment strategy with a jump diffusion process in risk market. Arian Cani and Stefan Thonhauser[5] developed a model of dynamic reinsurance and optimal strategy that maximizes the surplus.

This study is related to problems of optimization in reinsurance. Reinsurance is an insurance for insurers that is the transfer of risk from a direct insurer the cedent to a second insurance carrier the reinsurer. Insurer passes some of its premium income to a reinsurer who covers certain proportion of the claims that occur. In the literature it is proved that reinsurance would be a good method for sharing our losses or profits. It reduces risk for cedent and also reduces the probability of a direct insurer's ruin.

In this literature, we optimize the reinsurance that is an interesting research topic in the areas of insurance mathematics. Here the objective is to determine value function corresponding to which strategy it is optimal. The method is used here is iteration procedure that solve the Hamilton Jacobi Equation. And later we apply the method in Excess of loss and problems. An optimal insurance arrangement for the insurer company with some constraint from the reinsurer company. Many new concepts and methodologies for optimal reinsurance have been studied for everyone's perspective. Researchers introduce the optimum reinsurance strategies. Some researchers introduced the method of obtaining premium and objectives and the risk processes. Borch [6] proved that stop loss reinsurance can be used to minimize the retained loss. Arrow [7] said that stop loss may maximize expected utility of terminal wealth of an insurer. Kaluszka [8] take the combination of stop loss and quota share can be used to find optimal strategies to minimize a cedent's retained risk. Centeno [9] defined the optimum excess of loss retentions for two dependent risks using two objective functions maximizing insurer's expected utility of wealth net of reinsurance with respect to an exponential utility function and maximizing the adjustment coefficient of retained business respectively. The concepts of dynamic reinsurance is a classical problem of maximizing the dividends of an insurer before to ruin in a compound Poisson

processes such a model was introduced by Azcue and Muler [10]. for general reinsurance schemes. Mnif and Sulem [11] has studied the excess of loss reinsurance optimization problems.

## 2. PROBLEM FORMULATION

The surplus process keeps on increasing because of premiums are deposited over time at a constant rate  $c > 0$ . It decreases since the claims are arriving according to Poisson process  $N = (N_t)_{t \geq 0}$  with intensity  $\lambda > 0$ . The sequence  $\{Y_n, n \in \mathbb{N}\}$  of claims is a positive independently and identically distributed random variables with a density function  $f_y(\cdot)$ . Similarly, during the same period the occurrence of the new premium arrival are also according to the  $M = (M_t)_{t \geq 0}$ . The sequence  $\{Z_n, n \in \mathbb{M}\}$  of claims is a positive independently and identically distributed random variables with a density function  $f_z(\cdot)$ . With finite mean  $\mu$ . all the random variables  $\{Y_n, Z_n, n \in \mathbb{N}\}$  and  $N$  are independent.

In this model, for a diffusion risk model based on the dynamic reinsurance, we will apply policy iteration procedure to optimize the cost function associated to the Hamilton Jacobi Bellman Equation. Let  $(\Omega, \mathcal{F}, P)$  be a probability space. In 1903, Cramér–Lundberg introduced an equation in ruin theory. It is also called classical compound-Poisson risk model.

$$X_t^u = x + ct - R(t) + P(t) \quad t \geq 0 \quad (1)$$

$$\text{where } R(t) = \sum_{n=1}^{N(t)} Y_n \text{ and } P(t) = \sum_{n=1}^{M(t)} Z_n \quad (2)$$

Where  $x$  denotes initial investment,  $R(t)$  and  $P(t)$  are the rate of claim and premium arrival respectively.

The time of ruin is one of interesting problems in classical ruin theory. The time of ruin is the first time that the surplus becomes negative. Let  $X^u = (X_t^u)_{t \geq 0}$  be the surplus process and  $\tau_x^u$  be the time of ruin to any strategy  $u$  then,

$$\tau_x^u = \inf\{t \geq 0: X_t^u < 0 \mid X_0^u = x\} \quad (3)$$

The probability of ruin  $\psi(x)$  is defined as a function of initial capital  $x \geq 0$ .

$$\psi(x) = P\{\tau_x < \infty\} \quad (4)$$

Let  $\delta > 0$  a discount rate then the expected value of the surplus corresponding to a strategy  $u$  is called return function. It is defined as follow

$$V^u(x) = E_x \left[ \int_0^{\tau_x^u} e^{-\delta t} X_t^u dt \right] \quad (5)$$

We would like to get return function with maximum surplus i.e our problem is to find

$$V(x) = \sup_{u \in U} V^u(x) \quad (6)$$

$V(x)$  is called Value function.

### 3. RESULTS

**Proposition 1: The bounding values for the value function  $V(x)$  when  $x \geq 0$**

**Upper Bound:**  $V(x) \leq \frac{x}{\delta} + \frac{c+z}{\delta^2}$  for  $x \geq 0$

**Lower Bound:**  $V(x) \geq \frac{x}{\delta} - \frac{\lambda\pi(y-z)-c}{\delta^2} [1 - e^{\frac{-\delta x}{\lambda\pi(y-z)-c}}]$  for  $x \geq 0$

Proof:

For any given strategy  $u = \{u_t\}_{t \geq 0}$ ,  $c(u_s) \leq c$  for all  $s \geq 0$ . So the equation from the equation (1) implies

$$X_t^{u_t} \leq x + c t + P(t) \text{ for all } t \geq 0.$$

Thus the value of the function under this condition can be given as

$$\begin{aligned} V^u(x) &\leq \int_0^{\infty} e^{-\delta t} (x + (c+z)t) dt = \frac{x}{\delta} + \frac{c}{\delta^2} + \frac{z}{\delta^2} \\ \Rightarrow V^u(x) &\leq \frac{x}{\delta} + \frac{c+z}{\delta^2} \end{aligned} \quad (7)$$

Through the concept of supermom the obtained value function  $V(x)$  satisfied the upper bound value.

Similarly the condition for lower bound occurs when the claim was considered as the full continuous reinsurance model that was subjective to the ruin time concept that yield the following result for  $X^{u_0}$  as,

$$X_t^{u_0} = x + (c - \lambda\pi(y-z))t$$

With time of ruin  $\tau_x^{u_0} = \frac{x}{\lambda\pi(y-z)-c}$

$$\begin{aligned} \text{Then, } V^{u_0}(x) &= \frac{x}{\delta} - \frac{\lambda\pi(y-z)-c}{\delta^2} [1 - e^{\frac{-\delta x}{\lambda\pi(y-z)-c}}] \\ \Rightarrow V(x) &\geq \frac{x}{\delta} - \frac{\lambda\pi(y-z)-c}{\delta^2} [1 - e^{\frac{-\delta x}{\lambda\pi(y-z)-c}}] \end{aligned} \quad (8)$$

**Proposition 2:** when  $x > y > z \geq 0$ , the value function  $V$ , satisfies

$$V(x) - V(y) + V(z) \leq \frac{x-y+z}{\delta} + C(x, y, z)V(x-y+z), \text{ where } C(x, y, z) \rightarrow 0$$

$$\text{as } |x - y + z| \rightarrow 0$$

$$V(x) - V(y) + V(z) \geq \frac{x-y+z}{\delta+\lambda} + \epsilon$$

Proof:

For a given and  $\epsilon_1 > 0$  for any given  $x > 0$ . Consider a strategy  $u$  with  $\epsilon$ -optimal was given by

$$V^u(x) \leq E_x \left[ \int_0^{\tau_x^u} e^{-\delta t} X_t^u dt \right] + \epsilon_1$$

Similarly for initial capital  $y$  with  $x > y > z \geq 0$  that is up to time  $\tau_z^u$ ,

$$V^u(y) \leq E_y \left[ \int_0^{\tau_y^u} e^{-\delta t} X_t^u dt \right] + \epsilon_2$$

$$\begin{aligned} \Rightarrow V(x) - V(y) + V(z) &\leq E_x \left[ \int_0^{\tau_x^u} e^{-\delta t} X_t^u dt \right] - E_y \left[ \int_0^{\tau_y^u} e^{-\delta t} X_t^u dt \right] \\ &\quad + E_z \left[ \int_0^{\tau_z^u} e^{-\delta t} X_t^u dt \right] + \epsilon \end{aligned} \quad (9)$$

Where the terms  $E_x$ ,  $E_y$ , and  $E_z$  are the initial values for respective process. With the path-wise argument the time for ruin probability is same for all the process through the fixed path  $\omega$ . All processes move parallel and will be ruined at some time, and then the inequality statement in preposition can be written as,

$$\begin{aligned} V(x) - V(y) + V(z) &\leq E_x \left[ \int_0^{\tau_x^u} e^{-\delta t} X_t^u dt \right] - E_y \left[ \int_0^{\tau_y^u} e^{-\delta t} X_t^u dt \right] + E_z \left[ \int_0^{\tau_z^u} e^{-\delta t} X_t^u dt \right] \\ &\quad + E_x \left[ \mathbf{1}_{\mathcal{C}} \int_{\tau_y^u - \tau_z^u}^{\tau_x^u} e^{-\delta t} X_t^u dt \right] + \epsilon \\ &\leq \frac{x-y+z}{\delta} + E_x \left[ \mathbf{1}_{\mathcal{C}} \int_0^{\tau_y^u} e^{-\delta t} X_t^u dt \right] + \epsilon \\ &\leq \frac{x-y+z}{\delta} + E[\mathbf{1}_{\mathcal{C}}] V(x-y+z) + \epsilon \end{aligned} \quad (10)$$

The optimal value of strategy depends on the starting values of  $x$ ,  $y$  and  $z$  that varies on  $\mathcal{C}$ . The second inequality in the preposition 2 is observed as:

$$\begin{aligned} &E_x \left[ \int_0^{\tau_y^u} e^{-\delta t} X_t^u dt \right] - E_y \left[ \int_0^{\tau_y^u} e^{-\delta t} X_t^u dt \right] + E_z \left[ \int_0^{\tau_y^u} e^{-\delta t} X_t^u dt \right] \\ &= E \left[ \int_0^{\tau_y^u} e^{-\delta t} (x - y + z) dt \right] \leq \int_0^{\infty} e^{-\delta t} (x - y + z) dt \end{aligned}$$

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Since by using the path-wise argument  $\mathcal{g} = \{\omega \in \Omega | \tau_x^u(\omega) = \tau_y^u(\omega) = \tau_z^u(\omega)\}$

When  $y \geq 0$  and  $\epsilon > 0$ , the allowable strategy  $\bar{u}$ , and  $x > y$ , then

$$\begin{aligned} V(x) - V(y) + V(z) &\geq E_x \left[ \int_0^{\tau_x^{\bar{u}}} e^{-\delta t} X_t^{\bar{u}} dt \right] - E_y \left[ \int_0^{\tau_y^{\bar{u}}} e^{-\delta t} X_t^{\bar{u}} dt \right] \\ &\quad + E_z \left[ \int_0^{\tau_z^{\bar{u}}} e^{-\delta t} X_t^{\bar{u}} dt \right] - \epsilon \end{aligned} \quad (11)$$

Similarly when the ruin functions are same for both the investment and the claims and the first claim takes place at time T, then

$$\begin{aligned} V(x) - V(y) + V(z) &\geq E_x \left[ \int_0^{\tau_x^{\bar{u}}} e^{-\delta t} X_t^{\bar{u}} dt \right] - E_y \left[ \int_0^{\tau_y^{\bar{u}}} e^{-\delta t} X_t^{\bar{u}} dt \right] + E_z \left[ \int_0^{\tau_z^{\bar{u}}} e^{-\delta t} X_t^{\bar{u}} dt \right] \\ &\quad + E_x \left[ \mathbf{1}_{\mathcal{g}^c} \int_{\tau_y^u - \tau_z^u}^{\tau_x^u} e^{-\delta t} X_t^u dt \right] - \epsilon \geq E \left[ \int_0^T e^{-\delta t} (x - y + z) dt \right] \\ V(x) - V(y) + V(z) &\geq \frac{x - y + z}{\delta + \lambda} - \epsilon \end{aligned} \quad (12)$$

**Lemma 1:** The obtained value function was lower bound by  $\frac{x}{\delta} - \frac{c + z - \lambda\mu}{\delta(\delta + \lambda)}$  which states that

$$V(x) \geq \frac{x}{\delta} - \frac{c + z - \lambda\mu}{\delta(\delta + \lambda)}$$

Proof:

When the function  $n(x)$  is differentiated through the Dynkin's formula

$$E_x(e^{-\delta t \wedge \tau} n(X_{t \wedge \tau})) = n(x) + E_x \int_0^{t \wedge \tau} e^{-\delta s} [Ln(X_s) - \delta n(X_s)] ds$$

We assume that,

$$n(x) = \begin{cases} \frac{x}{\delta} & x \geq 0, \\ 0 & x < 0 \end{cases}$$

$Ln(X_s) - \delta n(X_s) \geq -X_s + \frac{c + z - \lambda\mu}{\delta}$ , by using the formula.

$$\begin{aligned} &E_x(e^{-\delta t \wedge \tau} n(X_{t \wedge \tau})) + E_x \int_0^{t \wedge \tau} e^{-\delta s} X_s ds \\ &\geq n(x) + E_x \int_0^{t \wedge \tau} e^{-\delta s} \left[ \frac{c + z - \lambda\mu}{\delta} \right] ds \\ &\geq n(x) + E_x \int_0^{t \wedge T_1} e^{-\delta s} \left[ \frac{c + z - \lambda\mu}{\delta} \right] ds \end{aligned} \quad (13)$$

Since  $n(X_{t,\tau})$  is monotone and bounded convergence

$$E_x\left(\int_0^\tau e^{-\delta s} X_s ds\right) \geq n(x) + \frac{c+z-\lambda\mu}{\delta(\delta+\lambda)}$$

Also, we know that,  $n(x) = \begin{cases} \frac{x}{\delta}, & x \geq 0, \\ 0, & x < 0 \end{cases}$

$$\Rightarrow V(x) \geq E_x\left(\int_0^\tau e^{-\delta s} X_s ds\right) \geq \frac{x}{\delta} + \frac{c+z-\lambda\mu}{\delta(\delta+\lambda)} \quad (14)$$

**Lemma 2:** The local Lipchitz continuous condition was observed in the value function.

Proof:

When  $x > 0$  and  $\epsilon > 0$ , then the allowable strategy  $u$ ,

$$V(x) \leq E_x\left[\int_0^{t_x^u} e^{-\delta t} X_t^u dt\right] + \epsilon$$

Similarly when the investment is considered to be  $y$  and let  $T$  be the time for the first claim under the condition,  $T > \frac{x-y+z}{c(u)}$  the value function  $V$  is of the form,

$$V(y) \geq E_y\left[\int_0^{t_y^u} e^{-\delta t} X_t^u dt\right]$$

Under the condition that investment was higher than the claims and the premium arrivals

i.e.  $x > y, z$  which implies that  $x > y - z \geq 0$

$$\begin{aligned} \Rightarrow 0 &\leq V(x) - V(y) + V(z) \\ &\leq V(x)\left(1 - e^{-(\delta+\lambda)\frac{x-y+z}{c(u)}}\right) - e^{-\lambda\frac{x-y+z}{c(u)}}\left(\frac{c+\delta y - \delta z - (c+\delta x)e^{-\delta\frac{x-y+z}{c(u)}}}{\delta^2}\right) + \epsilon \\ &= \left(V(x)\frac{\delta+\lambda}{c(u)}(x-y+z) + \mathcal{O}(x-y+z)^2\right) + \frac{x}{c(u)}(x-y+z) \\ &\quad + \mathcal{O}((x-y+z)^2) + \epsilon \end{aligned} \quad (15)$$

$\Rightarrow$  Vis lipschitz Continuous.

**Lemma 3:** The value function  $V$  satisfies Hamilton Jacobi Equation (HJB).

$$\begin{aligned} \sup_{u \in U} \left\{ x + (c(u) + z)V'(x) - (\delta + \lambda)V(x) \right. \\ \left. + \lambda \int_0^{\rho(x,u)} V(x - r(y, z, u)) dF_Y(y) \right\} = 0 \end{aligned} \quad (16)$$



**Proof:**

In the dynamic programming methods of solving optimization problems for every  $F_t$  and a stopping time  $S \geq 0$ . Value function can be obtained from the following equation

$$V(x) = \sup_{u \in U} E_x \left[ \int_0^{\tau_x^u \wedge S} e^{-\delta t} X_t^u dt + e^{-\delta(\tau_x^u \wedge S)} V(X_{\tau_x^u \wedge S}^u) \right] \quad (17)$$

Let  $x > 0, z > 0, h > 0$  and strategy  $u \in U$  be as follows  $\tilde{u} = (u_t)_{t \geq 0}$  such that  $u_t = u$  in  $t \in [0, h]$  and  $u_t = \tilde{u}_{t-h}$  for  $t > 0$  for some  $\tilde{u} \in U$ . For small  $h > 0$  such that  $x + (c(u) + z)h > 0$ .

Let  $T$  be time of the first claim occurrence and set  $S = \min\{T, h\}$ . Then the equation form

$$0 \geq E_x \left[ \int_0^S e^{-\delta t} (x + (c(u) + z)t) dt + e^{-\delta t} V(X_S^{\tilde{u}}) - V(x) \right] \quad (18)$$

Since  $u$  is a constant control it applies on the interval  $[0, S]$  we can apply Rolski [13] and get that  $V$  lies in the domain of the generator where generator  $A^u$  is a *constantly* controlled process  $X^u$  Rolski [13] implies

$$A^u(n(x)) = c(u)n'(x) - \lambda n(x) + \lambda \int_0^{\rho(x,u)} n(x - r(x, y, z)) dF_Y(y)$$

Since absolutely continuous map  $t \rightarrow V(x + (c(u) + z)t)$  and the *boundary* is empty and we proved the integrable condition. So Dynkin formula, and equation (18) implies

$$\begin{aligned} 0 \geq E_x \left[ \int_0^S e^{-\delta t} (x + (c(u) + z)t) dt \right. \\ \left. + \int_0^S e^{-\delta t} [(c(u)V'(x + (c(u) + z)t) - (\delta + \lambda)V(x + (c(u) + z)t) \right. \\ \left. + \lambda \int_0^{\rho(x+(c(u)+z)t, u)} V(x + (c(u) + z)t) \right. \\ \left. - r(y, z, u)) dF_Y(y)] dt \right] \quad (19) \end{aligned}$$

Divide both sides by  $h$  we have

$$\begin{aligned} 0 \geq \frac{1}{h} E_x \left[ \int_0^S e^{-\delta t} (x + (c(u) + z)t) dt \right. \\ \left. + \int_0^S e^{-\delta t} [(c(u)V'(x + (c(u) + z)t) - (\delta + \lambda)V(x + (c(u) + z)t) \right. \\ \left. + \lambda \int_0^{\rho(x+(c(u)+z)t, u)} V(x + (c(u) + z)t) \right. \\ \left. - r(y, z, u)) dF_Y(y)] dt \right] \end{aligned}$$

The first term in the right is Riemann integrable and  $V$  is continuous so  $h \rightarrow 0$  implies

$$0 \geq x - (\delta + \lambda)V(x) - (\delta + \lambda)V(x) + \lambda \int_0^{\rho(x,u)} V(x + (c(u) + z)t) \\ - r(y, z, u))dF_Y(y) + \lim_{h \rightarrow 0} \frac{1}{h} E_x \left[ \int_0^S e^{-\delta t} c(u) V'(x + (c(u) + z)t) dt \right]$$

Consider the limit

$$\lim_{h \rightarrow 0} \frac{1}{h} E_x \left[ \int_0^S e^{-\delta t} c(u) V'(x + (c(u) + z)t) dt \right] \\ = \lim_{h \rightarrow 0} \frac{1}{h} e^{-\lambda h} \left[ \int_0^h e^{-\delta t} c(u) V'(x + (c(u) + z)t) dt \right] \\ + \frac{1}{h} \left[ \int_0^h \lambda e^{-\lambda s} \left[ \int_0^S e^{-\delta t} c(u) V'(x + (c(u) + z)t) dt \right] ds \right] = (c(u) + z) V'(x) \quad (20)$$

Applying *Lebesgue's Differentiation concepts* and Wheeden and Zygmund[14] Since  $V'(x)$  is Lebesgue integrable because of bounded and monotonic properties .

$$\Rightarrow \lim_{h \rightarrow 0} \frac{1}{h} \left[ \int_0^h \lambda e^{-\lambda s} \left[ \int_0^S e^{-\delta t} c(u) V'(x + (c(u) + z)t) dt \right] ds \right] = 0$$

Since  $s = 0$  integrand  $ds$  is also zero  $u \in U$  is arbitrary

$$0 \geq \sup_{u \in U} \left\{ x + (c(u) + z)V'(x) - (\delta + \lambda)V(x) + \lambda \int_0^{\rho(x,u)} V(x - r(y, z, u))dF_Y(y) \right\} \quad (21)$$

In this step we will prove that left hand side of equation(16) is less than or equal to zero. Let  $S = \min\{T, h\}$ , where  $h > 0$  and let  $u^1 = (u_t^1)t \geq 0$  be  $h^2$ -optimal strategy so the right hand side of (17) can be written as

$$V(x) = \sup_{u \in U} E_x \left[ \int_0^S e^{-\delta t} \left( x + \int_0^t (c(u_s) + z) ds \right) dt + e^{-\delta S} V(X_S^u) \right] \\ < E_x \left[ \int_0^S e^{-\delta t} \left( x + \int_0^t (c(u_s^1) + z) ds \right) dt + e^{-\delta S} V(X_S^{u_s^1}) \right] + h^2 + \epsilon h \quad (22)$$

Where  $\epsilon > 0$  is arbitrary. Suppose  $T_1 \sim \text{Exp}(\lambda)$  now above equation becomes

$$0 < E_x \left[ \int_0^S e^{-\delta t} \left( x + \int_0^t (c(u_s^1) + z) ds \right) dt \right] + (e^{-(\delta + \lambda)h} - 1) E_x \left[ V \left( x + \int_0^h (c(u_s^1) + z) ds \right) \right] \\ + E_x \left[ \int_0^h \lambda e^{-\lambda t} \int_0^{\rho(x + \int_0^t (c(u_s^1) + z) ds, u_t^1)} V \left( x + \int_0^t (c(u_s^1) + z) ds \right) - r(y, z, u_t^1) dF_Y(y) dt \right] + \\ E_x \left[ V \left( x + \int_0^h (c(u_s^1) + z) ds - V(x) \right) \right] + h^2 + \epsilon h \\ = A + B + C + D + h^2 + \epsilon h \quad (23)$$

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Divide the above equation (21) by  $h$ , then using the dominated convergence Theorem the expression B can be expressed as

$$\lim_{h \rightarrow 0} \frac{1}{h} (e^{-(\delta+\lambda)h} - 1) E_x \left[ V \left( x + \int_0^h (c(u_s^1) + z) ds \right) \right] = -(\delta + \lambda)V(x) \quad (24)$$

The equation (22) shows the continuity of  $V$ . similarly for  $C$ ,

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{1}{h} E_x \left[ \int_0^h \lambda e^{-\lambda t} \int_0^{\rho(x + \int_0^t (c(u_s^1) + z) ds, u_t^1)} V \left( x + \int_0^t (c(u_s^1) + z) ds \right) \right. \\ \left. - r(y, z, u_t^1) dF_Y(y) dt \right] \end{aligned} \quad (25)$$

By using the Wheeden and Zygmund (1977)

$$= \lambda \int_0^{\rho(x, u_0^1)} V(x - r(y, z, u_0^1)) dF_Y(y) \quad (26)$$

Using absolute continuity property of  $V$ ,  $D$  becomes

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{1}{h} E_x \left[ V \left( x + \int_0^h (c(u_s^1) + z) ds \right) V(x) \right] \\ = \lim_{h \rightarrow 0} \frac{1}{h} E_x \left[ \int_0^{\int_0^h (c(u_s^1) + z) ds} V'(x + y - z) dy \right] \\ = \lim_{h \rightarrow 0} \frac{1}{h} E_x \left[ \int_0^h c(u_t^1 + z) V' \left( x + \int_0^t (c(u_s^1) + z) ds \right) dt \right] \\ = (c(u_0^1) + z) V'(x) \end{aligned} \quad (27)$$

Similarly A can be obtained as

$$\lim_{h \rightarrow 0} \frac{1}{h} E_x \left[ \int_0^S e^{-\delta t} \left( x + \int_0^t (c(u_s^1) + z) ds \right) dt \right] = x \quad (28)$$

Finally, we get

$$\begin{aligned} 0 \leq x + (c(u_0^1) + z) V'(x) - (\delta + \lambda)V(x) \\ + \lambda \int_0^{\rho(x, u_0^1)} V(x - r(y, z, u_0^1)) dF_Y(y) \} + \epsilon \end{aligned} \quad (29)$$

Since  $\epsilon$  is arbitrary the equation (30) provide the proof for lemma 3.

$$\text{Sup}_u \left\{ x + (c(u) + z) V'(x) - (\delta + \lambda)V(x) + \lambda \int_0^{\rho(x, u)} V(x - r(y, z, u)) dF_Y(y) \right\} = 0 \quad (30)$$

Let  $U = \{u \in U \mid c(u) \geq 0\}$  and  $V(x)$  is monotonic

$$V'(x) = \inf_{u \in \bar{U}} \left\{ \frac{(\delta + \lambda)V(x) - x - \lambda \int_0^{\rho(x,u)} V(x - r(y,z,u)) dF_Y(y)}{c(u) + z} \right\} \quad (31)$$

$$\begin{aligned} &\leq \frac{(\delta + \lambda)V(x) - x - \lambda \int_0^{\rho(x,u)} V(x - r(y,z,u)) dF_Y(y)}{c(u) + z} \\ &\leq \frac{(\delta + \lambda) \left( \frac{x}{\delta} + \frac{c+z}{\delta^2} \right) - x - \lambda \int_0^x \frac{(x-y+z)}{\delta} dF_Y(y)}{c + z} \\ \Rightarrow V'(x) &\leq \frac{(\delta + \lambda) \frac{c+z}{\delta^2} + \frac{\lambda\mu}{\delta} + H(x)}{c + z} \end{aligned} \quad (32)$$

Where  $H(x) = \frac{\lambda}{\delta} (x + z)(1 - F_Y(x)) \geq 0$  also  $H(x) = \frac{\lambda}{\delta} \int_x^\infty (x+z) dF_Y(y) \leq \frac{\lambda\mu}{\delta}$

equality (32) becomes now

$$\Rightarrow V'(x) \leq \frac{(\delta + \lambda) \frac{c+z}{\delta^2} + \frac{2\lambda\mu}{\delta}}{c+z} \quad (33)$$

This completes the characterization of the value function  $V(x)$ .

**Theorem 1:** Let  $g(0) > 0$  be a random initial value, then the exclusive a.e. differential solution to

$$w'(x) = \inf \left\{ \frac{(\delta + \lambda)w(x) - x - \lambda \int_0^{\rho(x,u)} w(x - r(y,z,u)) dF_Y(y)}{c(u) + z} \right\} \text{ with } g(0) = w(0)$$

Proof:

Let  $x_0 \geq 0$  and a continuous function  $f: [0, x_0] \rightarrow \mathbb{R}$  be given. Fix  $h > 0$  and set

$$\mathbb{C} = \{w: [x_0, x_0 + h] \rightarrow \mathbb{R}\}$$

For  $w$  to be continuous,  $w(x_0) = f(x_0)$ , then the operator  $Tw(x) = f(x_0) +$

$$\int_{x_0}^x \inf_{u \in \bar{U}} \left\{ \frac{(\delta + \lambda)w(s) - s - \lambda \int_0^{\rho(s-x_0,u)} w(s - r(y,z,u)) dF_Y(y)}{c(u) + z} - \frac{\lambda \int_{\rho(s-x_0,u)}^{\rho(s,u)} w(s - r(y,z,u)) dF_Y(y)}{c(u) + z} \right\} ds$$

The minimizer will exist in the form of  $u(s)$ , then we get

$$Tw_1(x) - Tw_2(x) \leq$$

$$\int_{x_0}^x \left\{ \frac{(\delta + \lambda)(w_1(s) - w_2(s)) - \lambda \int_0^{\rho(s-x_0, u^2(s))} (w_1(s - r(y,z, u^2(s))) - w_2(s - r(y,z, u^2(s)))) dF_Y(y)}{c(u^2(s)) + z} \right\} ds$$

$$\leq h \frac{(\delta + 2\lambda)}{L} \sup_{s \in [x_0, x_0 + h]} |w_1(s) - w_2(s)|$$

When the role of  $w_1$  and  $w_2$ , the obtained value of  $h$  is  $\frac{L}{2(\delta + 2\lambda)}$

$$|Tw_1(x) - Tw_2(x)| \leq \frac{1}{2} \sup_{s \in [x_0, x_0 + h]} |w_1(s) - w_2(s)| \quad (34)$$

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By this construction, we can perceive that the solution is absolutely continuous on  $\mathbb{R}^+$ , since one may alter the grid for the construction procedure and finally the analytical characterization of  $V$  is obtained.

**Theorem 2:** Let  $\mathbb{R} \rightarrow \mathbb{R}$  with  $w(x) = 0$  when  $x < 0$  that are bounded linearly by  $\frac{x}{\delta} + \frac{c+z}{\delta^2}$  which is an absolute continuous solution to (31), then  $v(x) = w(x)$ .

Proof: Let  $t > 0$  then  $u = u_t$  and the path of  $(X_t^u)$  are subjected to variation, then using the stietjes integral, the following expression was obtained as,

$$\begin{aligned} e^{\delta t \wedge \tau_x^u} w(X_{t \wedge \tau_x^u}^u) - w(x) &= \int_0^{t \wedge \tau_x^u} e^{-\delta s} [-\delta w(X_s^u) + (c(u_s) + z)w'(X_s^u)] ds \\ &+ \sum_{T_x \leq t \wedge \tau_x^u} e^{-\delta T_x} [w(X_{T_x}^u) - w(X_{T_x^-}^u)] \end{aligned}$$

For the process  $N$ , defined as  $N_t$  for  $t \geq 0$  is given by

$$N_t = \sum_{T_x \leq t} e^{-\delta T_x} [w(X_{T_x}^u) - w(X_{T_x^-}^u)] = -\lambda \int_0^t e^{-\delta s} \left[ \int_0^{\rho(x_s^u, u_s)} w(X_s^u - r(y, z, u_s)) dF_Y(y) - w(X_s^u) \right] ds$$

With expectation the above equation can be given as

$$\begin{aligned} E_x[e^{-\delta t \wedge \tau_x^u} w(X_{t \wedge \tau_x^u}^u)] &= w(x) + E_x \left[ \int_0^{t \wedge \tau_x^u} e^{-\delta s} [-(\delta + \lambda)w(X_s^u) + (c(u_s) + z)w'(X_s^u)] \right. \\ &\left. + \lambda \left[ \int_0^{\rho(x_s^u, u_s)} w(X_s^u - r(y, z, u_s)) dF_Y(y) \right] ds \right] \end{aligned}$$

We know that

$$w'(X_s^u) = \inf \left\{ \frac{(\delta + \lambda)w(X_s^u) - x - \lambda \int_0^{\rho(x, u)} w(X_s^u - r(y, z, u)) dF_Y(y)}{c(u) + z} \right\}$$

This implies that,

$$E_x[e^{-\delta t \wedge \tau_x^u} w(X_{t \wedge \tau_x^u}^u)] \leq w(x) - E_x \left[ \int_0^{t \wedge \tau_x^u} e^{-\delta s} X_s^u ds \right] \quad (35)$$

From Schmidli (2008, Lem.2.2), we know that controlled surplus tends to infinity. So use the bounded convergence to get the following,

$$E_x \left[ \int_0^{t \wedge \tau_x^u} e^{-\delta s} X_s^u ds \right] \leq w(x)$$

Hence,  $v(x) = w(x)$ .

#### 4. NUMERICAL ILLUSTRATIONS

For analysing the value function through the different types of reinsurance strategies, we consider two types of claim distributions in gamma and exponential form. Let us consider  $F_Y(y) = \gamma^2 y e^{-\gamma y}$  in gamma function and  $F_Y(y) = \beta e^{-\beta y}$  in exponential form. The optimized strategies are estimated for quota share, excess loss and stop loss. In the current work, the expected principle value read as  $C = (1+\eta)\lambda\mu$  and the reinsurance scheme was given as  $r(y,z,u) = uyz$  with the control parameter.

For deriving numerical approximations to the value function and to the optimal strategy, we implemented the program we have illustrated in the introduction to this section. In contrast to the case of excess of loss reinsurance, the proportional situation turned out to be numerically demanding, requiring lots of computational efforts for arriving at passably satisfying results.

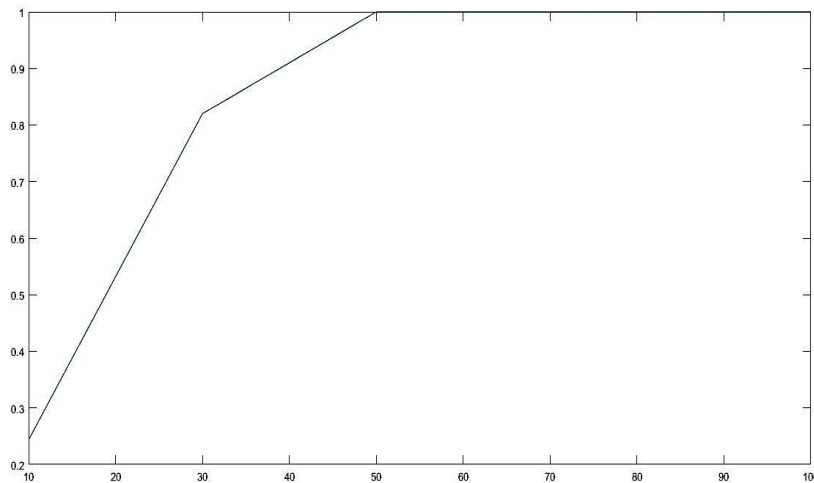
With  $V^{SR}$ , the strategy  $U^1(x)$  can be computed as

$$U^1(x) = \arg \max_{u \in \bar{U}} \left\{ \begin{aligned} &x + (c(u) + z) \frac{\partial}{\partial x} V^{SR}(x) - (\delta + \lambda) V^{SR}(x) \\ &+ \lambda \int_0^{\rho(x,u)} V^{SR}(x - r(y,z,u)) dF_Y(y) \end{aligned} \right\} \quad (36)$$

With value of C, we can write

$$V(x) = \frac{x}{\delta} + \frac{\eta\mu\lambda + Z}{\delta^2} \quad (37)$$

The parameter set for quota share and non-proportional reinsurance was given in the table 1 for gamma distribution of claims. Using equation (5) and (36), along the table 1 values, the strategy U for the quota share was given in figure 1 that represent the gamma distribution of claims. Here the convergence or optimization of strategy occurs at the value of  $x = 50$ .



**Figure 1:** Illustration of optimal strategy in quota share

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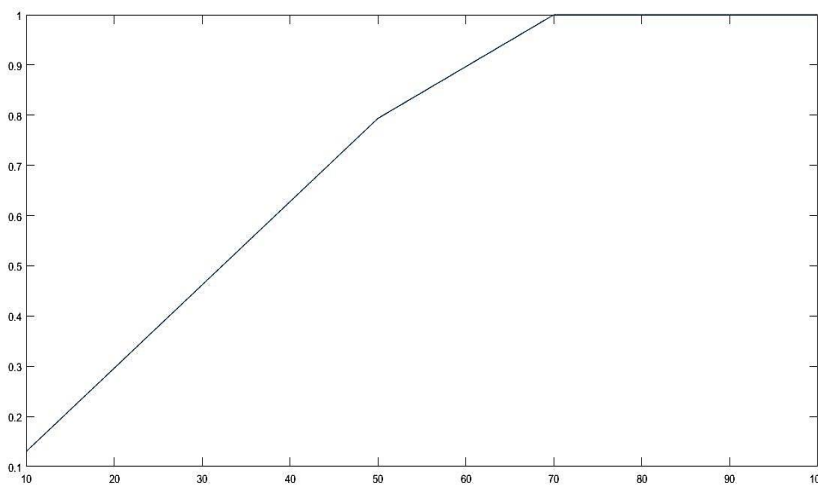
According to the graphical representation the value function is continuously increasing with respect to the investment level. Since the optimal strategy in quota share is both the function are proportional. Similarly with the equation (5) and (36), the excess loss strategy for the model was plotted and shown in figure 2 based on the values of table 2. This is again for the gamma distribution of the claims and the convergence was likely to occur at the value of  $x=70$ . For the stop loss reinsurance, we made the assumption that stop loss can occur only when there was more than 70% of excess loss. With the stop loss reinsurance the convergence was attained only at the range of 80. This value of  $x$  was greater than that of both the quota share loss and the excess loss.

**Table 1:** Set of parameters for Quota share reinsurance

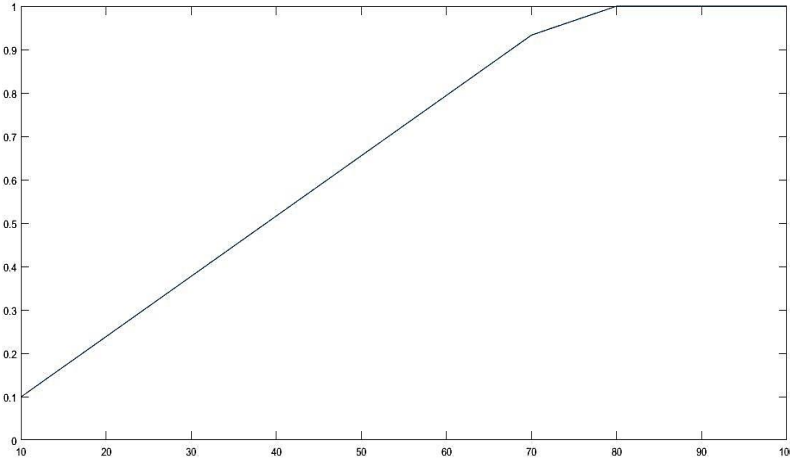
$\Upsilon$	$\eta$	$\theta$	$\lambda$	$\delta$
0.2	0.1	0.11	1	0.1

**Table 2:** Set of parameters for Non-proportional reinsurance

$\Upsilon$	$\eta$	$\theta$	$\lambda$	$\delta$
0.2	0.08	0.15	1	0.1

**Figure 2:** Illustration of optimal strategy in XL reinsurance

According to the above graphical representation the value function is continuously increasing with respect to the investment level. But at some certain level the value function will get the XL this doesn't affect the investment since this non-proportional relation.



**Figure 3:** Illustration of optimal strategy in stop loss

According to the above graphical representation the value function is continuously increasing with respect to the investment level. But at some certain level the value function will get the XL this doesn't affect the investment since this non-proportional relation. If the excess loss strategy is continues more than 70% then stop loss.

Now let us consider  $c(u) = \lambda\mu(u(1 + \theta) - (\Theta - \eta))$ ,

$$\begin{aligned}
 u^* &= \frac{(\delta + \lambda)V(x) - x + \lambda\mu(\theta - \eta)V'(x) - \lambda \int_0^{\frac{x}{u^*(x)}} V(x - u^*(x)y + z) dF_Y(y)}{\lambda\mu(1 + \theta)V'(x)} \\
 &\approx \frac{(\delta + \lambda)\left(\frac{x}{\delta} + \frac{\eta\mu\lambda + Z}{\delta^2}\right) - x + \lambda\mu(\theta - \eta)\frac{1}{\delta} - \lambda \int_0^{\frac{x}{u^*(x)}} \left[\frac{(x - u^*(x)y + z)}{\delta} + \frac{\eta\mu\lambda + Z}{\delta^2}\right] dF_Y(y)}{\lambda\mu(1 + \theta)\frac{1}{\delta}} \\
 u^* &\approx \frac{\theta + u^*}{1 + \theta} \tag{38}
 \end{aligned}$$

When considering the exponential form of claims, the optimal strategy for the quota share was estimated through the set of parameters in table 3. Then by using the equation (37) and (38), the convergence of x value was observed at the value of 6 as in figure 4. Similarly through the set of



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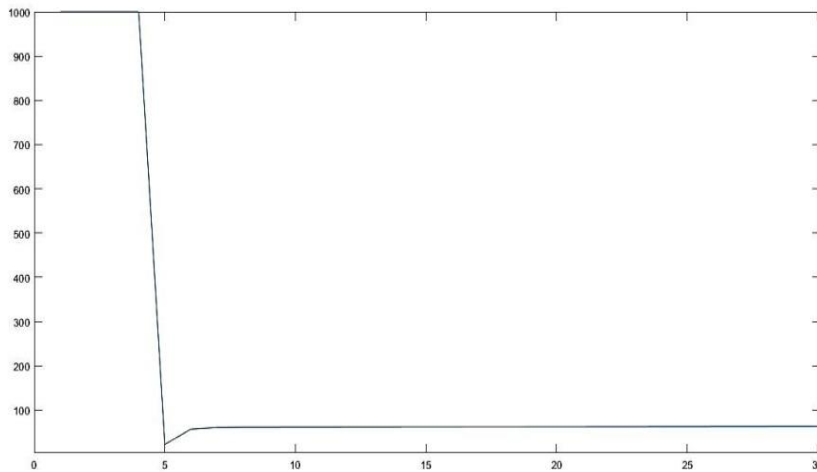
parameters for the non-proportional reinsurance in table 4, the optimum strategy for the XL reinsurance and stop loss reinsurance as shown in figure 5 and 6.

**Table 3:** Set of parameters for Quota share reinsurance

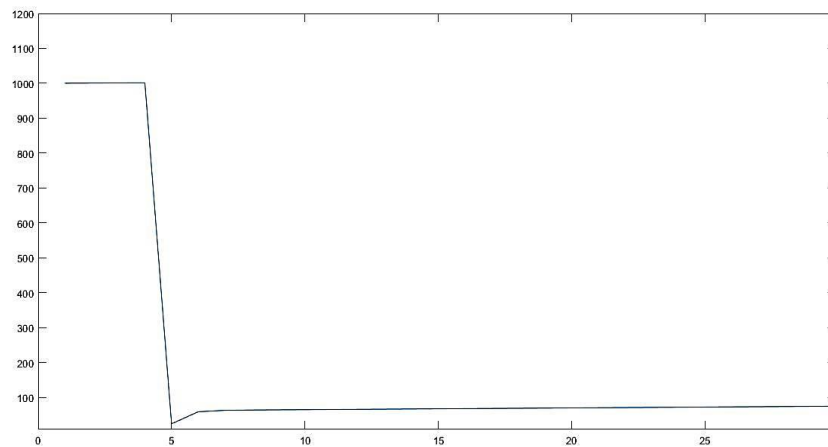
$\beta$	$\eta$	$\theta$	$\lambda$	$\delta$
1	0.6	0.61	1	0.01

**Table 4:** Set of parameters for Quota share reinsurance

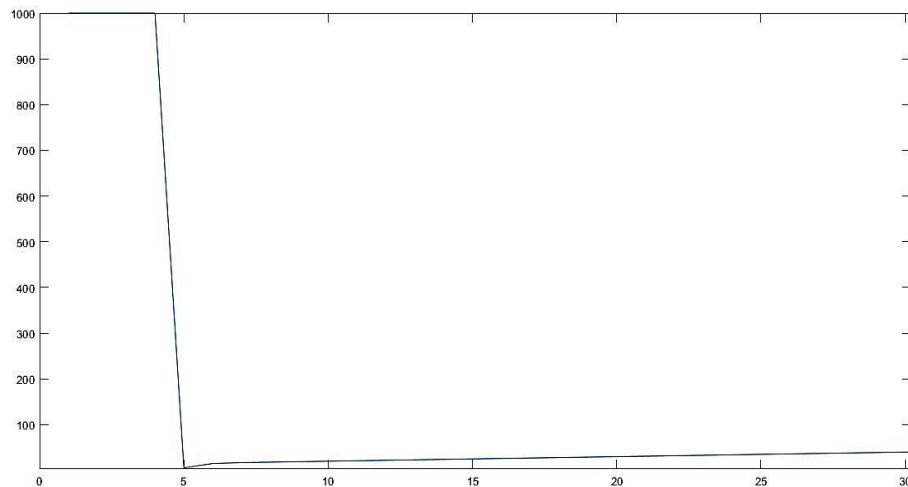
$\beta$	$\eta$	$\theta$	$\lambda$	$\delta$
1	0.5	0.65	1	0.01



**Figure 4:** Illustration of optimal strategy in Quota share reinsurance



**Figure 5:** Illustration of optimal strategy in XL reinsurance



**Figure 6:** Illustration of optimal strategy in stop loss reinsurance

## 5. CONCLUSION

The current paper discusses the optimal dynamic reinsurance based on the value function formulated with the criteria that involves the premium arrival through the risk theory. The theorems and characteristics of the formulated value function were deliberated along with its proof. The numerical illustration for the both the proportional and non-proportional reinsurance was estimated. The current work observes that the new premium arrival had some positive influence in the reinsurance strategies based on the value function that provide the economic benefits.

## CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

## REFERENCE

- [1] M. Taksar, C. Markussen, Optimal dynamic reinsurance policies for large insurance portfolios, *Finance Stoch.* 7 (2003), 97–121.
- [2] L. Bai, J. Guo, Optimal proportional reinsurance and investment with multiple risky assets and no-shorting constraint, *Insur. Math. Econ.* 42 (2008), 968–975.
- [3] S. Asmussen, B. Højgaard, M. Taksar, Optimal risk control and dividend distribution policies: example of excess-of-loss reinsurance for an insurance corporation, *Finance Stoch.* 4 (2000), 299–324.

## OPTIMIZATION OF THE CRAMER LUNDBERG MODEL

- [4] C. Irgens, J.Paulsen, Optimal control of risk exposure, reinsurance and investments for insurance portfolios, *Insur. Math. Econ.* 35 (2004), 21–51.
- [5] A. Cani, S. Thonhauser, An optimal reinsurance problem in the Cramér–Lundberg model, *Math. Meth. Oper. Res.* 85 (2017), 179–205.
- [6] K.H. Borch, *The mathematical theory of Insurance: an annotated selection of papers on insurance published 1960–1972*. Lexington Books, Lexington, (1974).
- [7] K.J. Arrow, Uncertainty and the welfare economics of medical care, *Amer. Econ. Rev.* 53 (1963), 941–973.
- [8] M. Kaluzka, Optimal reinsurance under convex principles of premium calculation, *Insurance: Math. Econ.* 36 (2005), 375–398.
- [9] M. L. Centeno, Dependent risks and excess of loss reinsurance, *Insurance: Math. Econ.* 37 (2005), 229–238.
- [10] P. Azcue, N. Muler, *Stochastic optimization in insurance a dynamic programming approach*. Springer, New York, (2014).
- [11] M. Mnif, A. Sulem, Optimal risk control and dividend policies under excess of loss reinsurance, *Stochastics* 77 (2005), 455–476
- [12] R.L. Wheeden, A. Zygmund, *Measure and Integral: an introduction to real analysis*. Marcel Dekker Inc., New York, (1977).
- [13] C. Hipp, M. Vogt, Optimal dynamic XL reinsurance. *ASTIN Bull.* 33 (2003), 193–207.
- [14] K.J. Arrow, Uncertainty and the welfare economics of medical care, *Amer. Econ. Rev.* 53 (1963), 941–973.
- [15] A. Balbs, B. Balbs, A. Heras, Optimal reinsurance with general risk measures. *Insur. Math. Econ.* 44 (2009), 374–384.
- [16] K. Borch, An attempt to determine the optimum amount of stop loss reinsurance. In: *Transactions of the 16th International Congress of Actuaries*, pp. 597–610 (1960).
- [17] K. Borch, The optimal reinsurance treaties. *ASTIN Bull.* 5 (1969), 293–297.
- [18] S. Najeema, K. Vasudevan, Optimization of the cramer lundberg model based value function of reinsurance with random claims using policy iteration method, *Adv. Appl. Math. Sci.* 20 (2021), 1743–1766.
- [19] J. Cai, Y. Fang, Z. Li, Optimal reciprocal reinsurance treaties under the joint survival probability and the joint profitable probability, *J. Risk Insur.* 80 (2013), 145–168.
- [20] J. Cai, C. Lemieux, F.D. Liu, Optimal reinsurance from the perspectives of both an insurer and a reinsurer, *ASTIN Bull.* 46 (2016), 815–849.
- [21] J. Cai, K.S. Tan, Optimal retention for a stop loss reinsurance under the VaR and CTE risk measures, *ASTIN Bull.* 37 (2007), 93–112.