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EFFICIENT DECOMPOSITION METHOD FOR INTEGRO-DIFFERENTIAL EQUATIONS

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Abstract. Different methods have been used in the solution of integro-differential equations. Many of these methods such as Standard Adomian Decomposition Method (SADM) take several iterations which might be difficult to solve and also consume time before getting an approximation. This present study developed a new Modified Adomian Decomposition Method (MADM) for Integro-Differential Equations. The modification was carried out by decomposing the source term function into series. The newly modified Adomian decomposition method (MADM) accelerates the convergence of the solution (MADM) faster the Standard Adomian Decomposition Method (SADM). This study recommends the use of MADM for solving Integro-Differential Equations.

Keywords: Adomian polynomials; integro-differential equations; Taylor series; convergence; source term.

2010 AMS Subject Classification: Primary 65-02, 65L10; Secondary 65L10.

1. Introduction

Several analytical solutions have been used to handle the problem of polynomial problems. These methods include Hirota's bilinear method, the Darboux transformation, the symmetry method, inverse scatting transformation, the Adomian decomposition method and other asymptotic methods. These methods have been used to solve nonlinear problems (Miura, 1978; Hirota,

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1980; Gu et al., 1999; Olver, 1986; Adomian, 1994).Amongst the method, the Adomian Decomposition method has been proved to be effective and reliable for handling different equations, linear or non-linear (Adomian, 1994; Adomian and Rack, 1986; Wazwaz, 2008).

The Adomian Decomposition Method (ADM) has been described as a method for solving both the linear and nonlinear differential equations and Boundary Value Problems (BVPs) seen in different fields of science engineering (Keskin, 2019). This method has been found that it is mainly depends upon the calculation of Adomian polynomials for nonlinear operators. The use of Adomian Decomposition method faces some problems which may arise from the nature of equations in consideration. Wazwaz (2008) introduced the modified Adomian decomposition method to solve some identified problems related to the nature of problems considered.

This present work introduces a new modification to Adomian Decomposition Method for integro-differential equations by using Adomian Polynomials. This new modification for integro-differential equations introduces a change in the formulation of Adomian polynomials; it provides a qualitative improvement over the standard Adomian method. The new modified Adomian Decomposition Method (MADM) can effectively improve the accuracy, speed of convergence and calculations.

2. METHOD

Consider the following integral equation:

$$y(x) = g(x) + \lambda \int_a^b k(x,t)[L(y(t)) + N(y(T))]dt, \; \lambda \neq 0. \tag{1}$$

where the kernel $k(x,t)$ and the function $g(x)$ are given real valued functions, λ is a numerical parameter, $L(y(t))$ and $N(y(t))$ are linear and non-linear operator of $y(x)$ respectively and the unknown function $y(x)$ is the solution to be determined. The solution of (1) is usually expressed in the form:

$$y(x) = \sum_{j=0}^{+\infty} y_j(x) \tag{2}$$

And the method identifies the non-linear term $N(y(t))$ by the decomposing series

$$N(y(x)) = \sum_{j=0}^{\infty} A_j(x) \tag{3}$$

where $A_j(x);\ j=1,2,3,\cdots$ is called the Adomian polynomials which are evaluated by

$$A_r = \frac{1}{r!}\frac{d^r}{dx^r}N\left[\sum_{j=0}^{r}\lambda^j y_j\right];\ r=0,1,2,\cdots \tag{4}$$

substituting (2) and (3) into (1), we have

$$\sum_{j=0}^{+\infty} y_j(x) = g(x) + \lambda\int_a^x k(x,t)\left[L(\sum_{j=0}^{+\infty} y_j(t)) + \sum_{j=0}^{+\infty} A_j(t)\right]dt. \tag{5}$$

By the SADM,

$$\begin{aligned} y_0(x) &= g(x), \\ y_{j+1}(x) &= \lambda \textstyle\int_a^x k(x,t)\left[L(y_j)+A_j\right]dt,\ j\geq 0 \end{aligned} \tag{6}$$

In what follows, equation in (6) is the standard ADM.

The Modified ADM by Wazwaz (1999). This modification is based on the assumption that the function $g(x)$ can be divided into two parts namely, $g_1(x)$ and $g_2(x)$ under the assumption that:

$$g(x) = g_1(x) + g_2(x) \tag{7}$$

thereby making a slight variation on the components $y_0(x)$ and $y_1(x)$. It says that only the part $g_1(x)$ will be assigned to the zeroth component $y_0(x)$ whereas the remaining part $g_2(x)$ will be combined with other terms given in (6). Consequently, the modified recursive relation is:

$$y_0(x) = g_1(x) \tag{8}$$

$$y_1(x) = g_2(x) + \lambda\int_a^x k(x,t)\left[L(y_0)+A_0\right]dt, \tag{9}$$

$$y_{j+1}(x) = \lambda\int_a^x k(x,t)\left[L(y_j)+A_j\right]dt,\ j\geq 1. \tag{10}$$

The following are some conclusions drawn from this approach:

(1) The slight variation in reducing the number of terms of y_0 will result in a reduction of the computational work and will accelerate the convergence.
(2) This slight variation in the definition of the component y_0 and y_1 may provide solution using two iterations only and sometimes may not need the computations of Adomian polynomial required for non-linear term.

However, how excellent as this powerful method is; it has the following drawbacks:

(1) It fails whenever the functions in the integral equation can not be evaluate analytically.

(2) It also fails whenever the source term $g(x)$ can not be split into divisions (say two).

The Modified ADM by Wazwaz & El-Sayed (2001). Here, the function $g(x)$ is expressed in Taylor series as

$$g(x) = \sum_{j=0}^{\infty} g_j(x) \tag{11}$$

thereby producing a new recursive relationship

$$y_0(x) = g_0(x), \tag{12}$$

$$y_1(x) = g_1(x) + \lambda \int_a^x k(x,t)\,[L(y_0) + A_0]\,dt, \tag{13}$$

$$y_{j+1}(x) = g_{j+1}(x) + \lambda \int_a^x k(x,t)\,\left[L(y_j) + A_j\right]dt. \tag{14}$$

The terms $y_0(x), y_1(x), y_2(x), \cdots$ of the solution $y(x)$ follow immediately and the solution $y(x)$ can be easily obtained using (3). It is evident that this algorithm reduces the number of terms involved in each component and hence the size of calculations is minimized compared to SADM. This reduction of terms in each component facilitates the construction of Adomiam polynomials for non-linear operators.

The Modified ADM by Lie-jun Xie (2013). The effectiveness of Wazwaz (1999) depends on the proper choice of $g_1(x)$ and $g_2(x)$ which may need quite a little computational work. Also the computations of $y_1(x)$ may be complicated to continue or analytically impossible, hence Xie (2013) suggests that $y_1(x)$ be expressed in Taylor series of the form

$$y_1(x) = \sum_{j=0}^{+\infty} y_{1j}(x) \tag{15}$$

and

$$y_2(x) = \lambda \int_a^x k(x,t)\,[L(y_1) + A_1]\,dt \tag{16}$$

where the Adomian polynomial A_1 can be evaluated by (4) with $y_0(x)$ and $y_1(x)$ obtained by

$$y_0(x) = g(x) \tag{17}$$

$$y_1(x) = \lambda \int_a^x k(x,t)\,[L(y_0) + A_0]\,dt \tag{18}$$

Also,

$$y_2(x) = \sum_{j=0}^{+\infty} y_{2j}(x) \tag{19}$$

and

$$y_{j+1}(x) = \lambda \int_a^x k(x,t)\,\left[L(y_j) + A_j\right]dt \tag{20}$$

and their Taylor series by

$$y_j(x) = \sum_{j=0}^{\infty} y_{ji}(x),\ j \geq 3. \tag{21}$$

In practice, some problems needs the determination of a few terms in the series such as

$$y(x) = \sum_{j=0}^{n} y_j(x),\ j \geq 3 \tag{22}$$

by truncating the series at certain terms. This produces a uniformly convergence in the infinite series, therefore a few terms will attain the maximum accuracy.

The New Approach to ADM (NADM). In recent times, several modifications of ADM have been proposed and duly applied to integral equations. However, these methods exist with their various drawbacks. For Wazwaz (1999), we have that it fails whenever the functions in the integral equation can not be evaluate analytically and whenever the source term $g(x)$ can not be split into divisions (say two). In Wazwaz & El-Sayed (2001), the reduction in the computation reduces also its accuracy and convergence. Also, in Xie (2013), the size of the computations increases as a result of series expansions of all components of $y(x)$. To improve on the accuracies and subsequently the convergence of these approaches, we shall based our assumption on the decomposition of the source term $g(x)$ in series of the form

$$g(x) = \sum_{j=0}^{+\infty} g_i(x) \tag{23}$$

and the new recursive relation obtained as:

$$y_0(x) = g_0(x), \tag{24}$$

$$y_1(x) = g_1(x) + g_2(x) + \lambda \int_a^x k(x,t)\left(L(y_0(x)) + A_0\right) dt, \tag{25}$$

$$y_2(x) = g_3(x) + g_4(x) + \lambda \int_a^x k(x,t)\left(L(y_1(x)) + A_1\right) dt, \tag{26}$$

$$\vdots$$

$$y_{j+1}(x) = g_{2(j+1)}(x) + g_{2(j+1)-1}(x) + \lambda \int_a^x k(x,t)\left(L(y_j(x)) + A_1\right) dt. \tag{27}$$

And subsequently the function $y(x)$ is obtained as

$$y(x) = \sum_{j=0}^{+\infty} y_j(x). \tag{28}$$

Assuming that the nonlinear function is $F(y(x))$ therefore, below are few of the Adomian polynomials.

$$A_0 = F(y_0), \tag{29}$$

$$A_1 = y_1 F'(y_0), \tag{30}$$

$$A_2 = y_2 F'(y_0) + \frac{1}{2!} y_1^2 F''(y_0) \tag{31}$$

$$A_3 = y_3 F'(y_0) + y_1 y_2 F''(y_0) + \frac{1}{3!} y_1^3 F'''(y_0), \tag{32}$$

$$A_4 = y_4 F'(y_0) + \left(\frac{1}{2!} y_2^2 + y_1 y_3\right) F''(y_0) + \frac{1}{2!} y_1^2 y_2 F'''(y_0) + \frac{1}{4} y_1^4 F^{(iv)}(y_0). \tag{33}$$

Two important observations can be made here. First, A_0 depends only on y_0, A_1 depends only on y_0 and y_1, A_2 depends on y_0, y_1 and y_2, and so on. Secondly, substituting these $A_j's$ in (3) gives:

$$\begin{aligned} F(y) &= A_0 + A_1 + A_2 + A_3 + \cdots \\ &= F(y_0) + (y_1 + y_2 + y_3 + \cdots)F'(y_0) + \frac{1}{2!}(y_1^2 + 2y_1y_2 + 2y_1y_3 + y_2^2)F''(y_0) \\ &\quad + \frac{1}{3!}(y_1^3 + 3y_1^2 y_3 + 6y_1y_2y_3 + \cdots)F'''(y_0) + \cdots \\ &= F(y_0) + (y - y_0)F'(y_0) + \frac{1}{2!}(y - y_0)^2 F''(y_0) + \cdots \end{aligned}$$

In the following, we will calculate Adomian polynomials for several linear terms that may arise in nonlinear integral equations.

Case 1.

The first four Adomian polynomials for $F(y) = y^2$ are given by

$$A_0 = y_0^2, \tag{34}$$

$$A_1 = 2y_0y_1, \tag{35}$$

$$A_2 = 2y_0y_2 + y_1^2 \tag{36}$$

$$A_3 = 2y_0y_3 + 2y_1y_2 \tag{37}$$

Case 2.

The first four Adomian polynomials for $F(y) = y^3$ are given by

$$A_0 = y_0^3, \tag{38}$$

$$A_1 = 3y_0^2y_1, \tag{39}$$

$$A_2 = 3y_0^2y_2 + 3y_0y_1^2, \tag{40}$$

$$A_3 = 3y_0^2y_3 + 6y_0y_1y_2 + y_1^3. \tag{41}$$

Case 3.

The first four Adomian polynomials for $F(y) = y^4$ are given by

$$A_0 = y_0^4, \tag{42}$$

$$A_1 = 4y_0^3y_1, \tag{43}$$

$$A_2 = 4y_0^3y_2 + 6y_0^2y_1^2, \tag{44}$$

$$A_3 = 4y_0^3y_3 + 4y_1^3y_0 + 12y_0^2y_1 + y_2. \tag{45}$$

Case 4.

The first four Adomian polynomials for $F(y) = siny$ are given by

$$A_0 = siny_0, \tag{46}$$

$$A_1 = y_1 cosy_0, \tag{47}$$

$$A_2 = y_2 cosy_0 - \frac{1}{2!} y_1^2 siny_0, \tag{48}$$

$$A_3 = y_3 cosy_0 - y_1 y_2 siny_0 - \frac{1}{3!} y_1^3 cosy_0. \tag{49}$$

Case 5.

The first four Adomian polynomials for $F(y) = cosy$ are given by

$$A_0 = cosy_0, \tag{50}$$

$$A_1 = -y_1 siny_0, \tag{51}$$

$$A_2 = -y_2 siny_0 - \frac{1}{2!} y_1^2 cosy_0, \tag{52}$$

$$A_3 = -y_3 siny_0 - y_1 y_2 cosy_0 + \frac{1}{3!} y_1^3 siny_0. \tag{53}$$

Case 6.

The first four Adomian polynomials for $F(y) = \exp(y)$ are given by

$$A_0 = \exp(y_0), \tag{54}$$

$$A_1 = y_1 \exp(y_0), \tag{55}$$

$$A_2 = \left(y_2 + \frac{1}{2!} y_1^2 \right) \exp(y_0), \tag{56}$$

$$A_3 = \left(y_3 + y_1 y_2 + \frac{1}{3!} y_1^3 \right) \exp(y_0). \tag{57}$$

Features of the New Approach

(1) It converges faster that SADM

(2) It could be observed that the expansion of the source term needs to be as long as possible. This is for availability of the series decomposition terms needed in calculating the current points.

(3) The slight increase in selection of the terms of decomposed source term is to improve its convergence.

(4) The increase in terms used in the integral sign is to improve the accuracy and subsequently the Adomian polynomials. It is worthy to note here that the Adomian polynomial is only needed in nonlinear problems.

3. Numerical Examples

Example 1. Consider the standard integro-differential equation

$$y(x) = 1 + sinhx - coshx + \int_0^x y(t)dt. \tag{58}$$

Using the Standard Adomian Decomposition Method (SADM). Let

$$y_0 = 1 + sinh(x) - cosh(x) \tag{59}$$

then

$$\begin{aligned}
y_0(x) &= 1 + sinhx - coshx \\
y_1(x) &= \int_0^x y_0(t) = x - 1 + cosh(x) - sinh(x) \\
y_2(x) &= \int_0^x y_1(t) = 1 + \frac{1}{2}x^2 - x + sinh(x) - cosh(x) \\
y_3(x) &= \int_0^x y_2(t) = -1 + x + \frac{1}{6}x^3 - \frac{1}{2}x^2 + cosh(x) - sinh(x) \\
y_4(x) &= \int_0^x y_3(t) = 1 - x + \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{6}x^3 + sinh(x) - cosh(x) \\
y_5(x) &= \int_0^x y_4(t) = -1 + x - \frac{1}{2}x^2 - \frac{1}{24}x^4 + \frac{1}{6}x^3 + \frac{1}{120}x^5 - sinh(x) + cosh(x) \\
&\vdots
\end{aligned}$$

Using (2),

$$y(x) = x + \frac{1}{6}x^3 + \frac{1}{120}x^5 + \cdots \tag{60}$$

This obviously discovered that the solution tends to exact solution

$$y(x) = sinhx \tag{61}$$

Using the Modified Adomian Decomposition Method (MADM). The source term $1 + sinh(x) - cosh(x)$ can be expanded in Taylor series in the form:

$$x - \frac{1}{2}x^2 + \frac{1}{6}x^3 - \frac{1}{24}x^4 + \frac{1}{120}x^5 - \frac{1}{720}x^6 + \frac{1}{5040}x^7 - \frac{1}{40320}x^8 + \frac{1}{362880}x^9 + \cdots \tag{62}$$

After two (02) iterations,

$$\begin{aligned} y_0(x) &= x \\ y_1(x) = -\frac{1}{2}x^2 + \frac{1}{6}x^3 + \int_0^x y_0(t)dt &= -\frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{2}x^2 = \frac{1}{6}x^3 \\ y_2(x) = -\frac{1}{24}x^4 + \frac{1}{120}x^5 + \int_0^x y_1(t)dt &= \frac{1}{120}x^5 \\ &\vdots \end{aligned}$$

Thus

$$y(x) = \sum_{j=0}^{2} y_j(x) = x + \frac{1}{6}x^3 + \frac{1}{120}x^5 \tag{63}$$

Obviously we discovered that the solution tends to exact solution in (61). Also, we have that the two iterations of the NADM produces the result of SADM at the fifth computation.

Example 2. Consider the following problem:

$$y'(x) = 1 - \frac{1}{3}x \int_0^1 xty(t)dt \; y(0) = 0, \; y(x) = x. \tag{64}$$

Applying a one fold integral linear operator defined by

$$L^{-1} = \int_0^x (.)dx, \tag{65}$$

the differential equation is transformed to

$$y(x) = x - \frac{1}{6}x^2 + L^{-1}\left(\int_0^1 xty(t)dt\right)dx \tag{66}$$

Using SADM.

$$\begin{aligned}
y_0(x) &= x - \frac{1}{6}x^2 \\
y_1(x) &= \frac{7}{48}x^2 \\
y_2(x) &= \frac{7}{384}x^2 \\
y_3(x) &= \frac{7}{3072}x^2 \\
y_4(x) &= \frac{7}{24576}x^2 \\
&\vdots
\end{aligned}$$

Thus

$$y(x) = \sum_{j=0}^{4} y_j(x) + \cdots \tag{67}$$

This obviously discovered that the solution tends to exact solution

$$y(x) = x \tag{68}$$

Using the Modified ADM. Expanding $x - \frac{1}{6}x^2$ in series, we have:

$$x - \frac{1}{6}x^2 + \cdots \tag{69}$$

After two (02) iterations,

$$\begin{aligned}
y_0(x) &= x \\
y_1(x) &= 0 \\
&\vdots
\end{aligned}$$

Clearly, every value of $y_j,\ j \geq 1 = 0$ Thus

$$y(x) = \sum_{j=0}^{1} y_j(x) + \cdots = x \tag{70}$$

Obviously we discovered that the solution tends to exact solution in just one iteration.

Example 3. Consider the following problem:

$$y'(x) = \exp(x) + \frac{1}{16}(3+\exp)x + \frac{1}{4}\int_0^1 xt(1+y(t)-y^2(t))dt;\; y(0)=2. \tag{71}$$

Applying a one fold integral linear operator defined by

$$L^{-1} = \int_0^x (.)dx, \tag{72}$$

the differential equation is transformed to

$$y(x) = 1+\exp(x) + \frac{1}{32}(3+\exp)x^2 + \frac{1}{4}L^{-1}\left(\int_0^1 xt(1+y(t)-y^2(t))dt\right)dx. \tag{73}$$

or

$$y(x) = 1+\exp(x) + 0.324658003x^2 + \frac{1}{4}L^{-1}\left(\int_0^1 xt(1+y(t)-y^2(t))dt\right)dx. \tag{74}$$

Using the relation as obtained in Wazwaz & El-Sayed (2001) and NADM on (74), we have the following table of results at $j=3$.

TABLE 1. Table of Absolute Errors for Example 3

x	Exact(x)	Wazwaz & El-Sayed (2001)(x)	NADM(x)
x	$2+\exp(x)$	$2+x+\frac{2235972463x^2}{2946170880}+\frac{1}{6}x^3+\frac{1}{24}x^4$	$2+x+\frac{2955595512151x^2}{4120514592768}+\frac{1}{6}x^3+\frac{1}{24}x^4+\frac{x^5}{120}+\frac{x^6}{720}+\frac{x^7}{5040}+\frac{x^8}{40320}$
0.0	2.000000000	2.000000000	2.000000000
0.1	2.105170918	2.5893 -03	2.1729 -03
0.2	2.221402759	1.0355 -02	8.6915 -03
0.3	2.349858807	2.3283 -02	1.9556 -02
0.4	2.491824697	4.1339 -02	3.4766 -03
0.5	2.648721265	6.4452 -02	5.4321 -03
0.6	2.822118771	9.2500 -02	7.8224 -02
0.7	3.013752588	1.2530 -01	1.0647 -02
0.8	3.225540527	1.6258 -01	1.3906 -01
0.9	3.459601938	2.0398 -01	1.7600 -01
1.0	3.718278771	2.4900 -01	2.1729 -01

Example 4. Consider the following problem:

$$y''(x) = \frac{1}{2}\exp(x) + \frac{1}{2}\int_0^1 \exp(x-2t)y^2(t)dt;\ y(0) = y'(0) = 1. \tag{75}$$

Applying two fold integral linear operator defined by

$$L^{-1} = \int_0^x \int_0^x (.)dxdx, \tag{76}$$

the differential equation is transformed to

$$y(x) = \frac{1}{2} + \frac{1}{2}x + \frac{1}{2}\exp(x) + \frac{1}{2}L^{-1}\left(\int_0^1 \exp(x-2t)y^2(t)dt\right)dxdx. \tag{77}$$

Using the relation as obtained in SADM and NADM on (77), we have the following table of results at $j = 3$.

TABLE 2. Table of Absolute Errors for Example 4

x	Exact(x)	SADM(x)	NADM(x)
x	$\frac{1}{2}+\frac{1}{2}x+\frac{1}{2}\mathrm{e}^x$	$1.016213001+\frac{1}{2}x-0.0162130007\,\mathrm{e}^x$ $+0.5162130007\,\mathrm{e}^x x$	$1.504983400+x+\frac{1}{4}x^2$ $+0.5049834007\,\mathrm{e}^x x-0.504983400\,\mathrm{e}^x$ $+\frac{1}{12}x^3+\frac{1}{48}x^4+\frac{x^5}{240}+\frac{x^6}{1440}+\frac{x^7}{1080}$ $+\frac{x^8}{80640}$
0.0	1.000000000	1.000000000	1.000000000
0.1	1.102585459	2.7560 -03	2.7000 -03
0.2	1.210701379	1.1811 -02	1.1155 -02
0.3	1.324929404	2.8443 -02	2.7824 -02
0.4	1.445912349	5.4153 -02	5.2977 -02
0.5	1.574360636	9.0667 -02	8.8870 -02
0.6	1.711059400	1.3997 -01	1.3670 -01
0.7	1.856876354	2.0436 -01	1.9998 -01
0.8	2.012770464	2.8644 -01	2.8038 -01
0.9	2.179801556	3.8925 -01	3.8117 -01
1.0	2.359140914	5.1621 -01	5.0581 -01

4. CONCLUSION

This work has introduced the new modified Adomian Decomposition Method (MADM) for Integro-Differential Equation. This new method converges faster than Standard Adomian Decomposition Method (SADM). It could be observed that the expansion of the source term needs to be as long as possible. The slight increase in selection of the terms of decomposed source terms of decomposed source term is to improve its convergence. The increase in terms used in the integral sign is to improve the accuracy and subsequently the Adomian polynomials. It is worthy to note here that the Adomian polynomial is only needed in nonlinear problems.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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