Available online at http://scik.org
J. Math. Comput. Sci. 3 (2013), No. 1, 177-184
ISSN: 1927-5307

ON THE SET OF TRANSITIVE INDICES FOR REDUCIBLE TOURNAMENT MATRICES

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Abstract. We obtain the transitive indices set of the reducible tournament matrices of order n.

Keywords: Boolean matrix,Reducible tournament matrix,Primitive exponent, transitive index.

2000 AMS Subject Classification: 05C20, 15A18, 15B34, 15B48

1. Introduction

A Boolean matrix is a matrix over the binary Boolean algebra $\{0,1\}$, where the (Boolean)addition and (Boolean) multiplication in $\{0,1\}$ are defined as $a+b = max\{a,b\}$, $ab = min\{a,b\}$(we assume $0 < 1$). Let $\mathfrak{B}_n$ denote the set of all $n \times n$ matrices over the Boolean algebra $\{0,1\}$.

For $A, B \in \mathfrak{B}_n$, if there is a permutation matrix P such that $PBP^T = A$, then we say B is permutation similar to A (written $B \sim A$).

A matrix $B \in \mathfrak{B}_n$ is reducible if $B \sim \begin{pmatrix} B_1 & 0 \\ C & B_2 \end{pmatrix}$, where B_1 and B_2 are square(non-vacuous), and B is irreducible if it is not reducible.

Received November 21, 2012

A matrix $B \in \mathfrak{B}_n$ is primitive if there is a nonnegative integer k such that $B^k = J$,the all-ones matrix.The least such k is called the exponent of B, denoted by $\gamma(B)$.

A matrix $B \in \mathfrak{B}_n$ is called transitive if $B^2 \leq B$. Denote by $t(B)$ the least integer $s \geqq 1$ such that B^s is transitive, i.e.$B^{2s} \leq B^s$.

In 1970,$\breve{S}$.Schwarz[1] introduced a concept of the transitive index and gave some results.

A matrix $A = [a_{ij}] \in \mathfrak{B}_n$ is called tournament matrix if $a_{ii} = 0 (i = 1, 2, \ldots, n)$ and $a_{ij} + a_{ji} = 1 (1 \leq i < j \leq n)$. Let $\mathfrak{T}_n$ denote the set of all $n \times n$ tournament matrices. Notice that a matrix $T_n \in \mathfrak{T}_n$ satisfies the equation

$$A_n + A_n^T = J_n - I_n$$

where J_n is the matrix of all $1'$s and I_n is the identity matrix.

Certain properties of tournament matrix have been investigated in [2,3,5,6].

2. Preliminaries

The notation and terminology used in this paper will basically follow those in [4]. For convenience of the reader, we will include here the necessary definitions and basic results in [5,6].

Let $\bar{T}_n = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 1 & \cdots & 1 & 0 & 0 & 1 \\ 1 & \cdots & \cdots & 1 & 0 & 0 \end{pmatrix}_{n\times n}$ $(n \geq 3)$, $\mathbb{T}_l = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 1 & \cdots & 1 & 0 \end{pmatrix}_{l\times l}$,

$\mathcal{T}_{3m} = \begin{pmatrix} \bar{T}_3 & 0 & \cdots & 0 \\ J & \bar{T}_3 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ J & \cdots & J & \bar{T}_3 \end{pmatrix}$,$\mathcal{I}_{3m} = \begin{pmatrix} I_3 & 0 & \cdots & 0 \\ J & I_3 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ J & \cdots & J & I_3 \end{pmatrix}$,

where J is the matrix of all $1'$s, I_3 is the identity matrix of order 3.

Lemma 2.1 ([3]) Let $T_n \in \mathfrak{T}_n$.Then

$$T_n \sim \begin{pmatrix} A_1 & 0 & 0 & \cdots & 0 \\ J & A_2 & 0 & \cdots & 0 \\ J & J & A_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ J & J & J & \cdots & A_k \end{pmatrix},$$

where all the blocks J below the diagonal are matrices of 1's,and the diagonal blocks $A_1, \cdots, A_k$ are irreducible components of T_n. Let A_i be $n_i \times n_i$ matrix,$1 \le i \le k, 1 \le n_i \le n$. Then k and n_i are uniquely determined by T_n.

Lemma2.2([6]) If $T_n \in \mathfrak{T}_n$ and $n \ge 4$. Then T_n is primitive if and only if T_n is irreducible.

It is obvious that 3×3 tournament matrix is not primitive, the primitive exponent of 4×4 irreducible tournament matrix is 9. For $n > 4$, we have

Lemma 2.3([6]) If $T_n \in \mathfrak{T}_n$ and $n \ge 5$, then $\gamma(T_n) \le n + 2$.

Lemma2.4([5]) Let $n \ge 5$, then $\gamma(\bar{T}_n) = n + 2$.

Lemma2.5([5]) If $n \ge 5, T_n \in \mathfrak{T}_n$ is irreducible. Then $\gamma(T_n) = n + 2$ if and only if T_n is isomorphic to $\bar{T}_n$.

Lemma2.6([6]) If $3 \le e \le n + 2$ and $n \ge 6$, then there exists an irreducible $T_n \in \mathfrak{T}_n$ such that $\gamma(T_n) = e$.

3. Main results

It is evident that if $A \in \mathfrak{B}_n$ is primitive digraph, then $t(A) = \gamma(A)$. For primitive tournament matrix T_n, its primitive exponent is determined by Moon and Pullman in [6].

In this paper we obtain results on transitive index of reducible tournament matrices.

Theorem 3.1 Let $T_n \in \mathfrak{T}_n$ be reducible matrix and $n \geq 8$. Then there exists a positive integer $s \leq n+1$ such that

$$T_n^s \sim A^\star = \begin{pmatrix} B_1 & 0 & 0 & \cdots & 0 \\ J & B_2 & 0 & \cdots & 0 \\ J & J & B_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ J & J & J & \cdots & B_g \end{pmatrix}$$

where all the blocks J below the diagonal are matrices of 1's, and the diagonal blocks B_i is zero matrix of orderl_i, or $\mathcal{I}_{3q_i}$, or matrix of 1's of order $m_i(4 \leq m_i < n)$, $1 \leq i \leq g$. $0 \leq 3q_i, l_i \leq n$, and q_i, l_i, m_i, g are uniquely determined by T_n.

Proof. It is obvious that the irreducible tournament matrix of order 1 is zero matrix of order 1, the irreducible tournament matrix of order 2 is not exists,and the irreducible tournament matrix of order 3 is isomorphic to $\bar{T}_3$. Hence , in Lemma2.1, the diagonal blocks A_i is zero matrix of order 1, or $\bar{T}_3$,or irreducible tournament matrix of order $m_i(4 \leq m_i < n)$. Let $A_i \neq (0)_{1\times 1}$ (if there exists) ,$A_{i+1} = A_{i+2} = \ldots = A_{i+l_i} = (0)_{1\times 1}, A_{i+l_i+1} \neq (0)_{1\times 1}$(if there exists). Then

$$\begin{pmatrix} A_{i+1} & 0 & \cdots & 0 \\ J & A_{i+2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ J & J & \cdots & A_{i+l_i} \end{pmatrix} = \mathbb{T}_{l_i} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 1 & \cdots & 1 & 0 \end{pmatrix}_{l_i \times l_i}$$

Let $A_j \neq \bar{T}_3$ (if there exists), $A_{j+1} = A_{j+2} = \ldots = A_{j+q_i} = \bar{T}_3, A_{j+q_i+1} \neq \bar{T}_3$(if there exists). Then

$$\begin{pmatrix} A_{j+1} & 0 & \cdots & 0 \\ J & A_{j+2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ J & J & \cdots & A_{j+q_i} \end{pmatrix} = \mathcal{T}_{3q_i} = \begin{pmatrix} \bar{T}_3 & 0 & \cdots & 0 \\ J & \bar{T}_3 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ J & \cdots & J & \bar{T}_3 \end{pmatrix}_{3q_i \times 3q_i}$$

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There exists that one among $n-1, n, n+1$ is multiple of 3, and set such number be s. Since T_n is Boolean matrix of reducible tournament with order $n(\geq 8)$, ${\mathcal{T}_{3q_i}}^s = \mathcal{I}_{3q_i}, {\mathbb{T}_{l_i}}^s = (0)_{l_i \times l_i}$. If A_i is irreducible tournament matrix of order $m_i (4 \leq m_i < n)$ in Lemma2.1. Then ${A_i}^s = J$. By Lemma2.3, the conclusion established and we complete the proof.

Note that $t(\mathbb{T}_n) = 1$, $t(\mathcal{T}_{3n}) = 3$, $n > 1$.

Let $T_n \in \mathfrak{T}_n$ be reducible matrix.Then Hence $T_2 \sim \mathbb{T}_2$,$T_3 \sim \mathbb{T}_3$. We have $t(T_2) = t(T_3) = 1$.

For T_4. By Lemma 2.1, $T_4 \sim \mathbb{T}_4$ or $T_4 \sim \bar{A}_4 = \begin{pmatrix} 0 & 0 \\ J & \bar{T}_3 \end{pmatrix}$, or $T_4 \sim \tilde{A}_4 = \begin{pmatrix} \bar{T}_3 & 0 \\ J & 0 \end{pmatrix}$. Since $t(\mathbb{T}_4) = 1$, $t(\bar{A}_4) = t(\tilde{A}_4) = 3$, hence $t(T_4) \leq 3$.

For T_5. By Lemma 2.2, $T_5 \sim \mathbb{T}_5$, or $T_5 \sim \tilde{A}_5 = \begin{pmatrix} \mathbb{T}_2 & 0 \\ J & \bar{T}_3 \end{pmatrix}$, or $T_5 \sim \hat{A}_5 = \begin{pmatrix} \bar{T}_3 & 0 \\ J & \mathbb{T}_2 \end{pmatrix}$, or $T_5 \sim \hat{\hat{A}}_5 = \begin{pmatrix} \mathbb{T}_1 & 0 & \\ J & \bar{T}_3 & 0 \\ J & J & \mathbb{T}_1 \end{pmatrix}$, or $T_5 \sim \bar{A}_5 = \begin{pmatrix} 0 & 0 \\ J & B_4 \end{pmatrix}$, or $T_5 \sim \check{A}_5 = \begin{pmatrix} B_4 & 0 \\ J & 0 \end{pmatrix}$, where B_4 is primitive tournament matrix of order 4. Clearly, $t(\mathbb{T}_5) = 1$, $t(\tilde{A}_5) = t(\hat{A}_5) = t(\hat{\hat{A}}_5) = 3$, $t(\bar{A}_5) = t(\check{A}_5) = 9$. Hence $t(T_5) \leq 9$.

Similarly, $t(T_i) \leq 9, i = 6, 7$. Let $\bar{A}_6 = \begin{pmatrix} \mathbb{T}_2 & 0 \\ J & B_4 \end{pmatrix}$, $\bar{A}_7 = \begin{pmatrix} \mathbb{T}_3 & 0 \\ J & B_4 \end{pmatrix}$, where B_4 is primitive tournament matrix of order 4. It is easy to see that $t(\bar{T}_6) = t(\bar{T}_7) = 9$.

For $n \geq 8$, we have

Theorem 3.2 Let $T_n \in \mathfrak{T}_n$ be reducible matrix and $n \geq 8$. Then $t(T_n) \leq n+1$.

Proof. By Theorem 3.1, there exists a positive integer $s \leq n+1$ such that

$$T_n^s \sim A^\star = \begin{pmatrix} B_1 & 0 & 0 & \cdots & 0 \\ J & B_2 & 0 & \cdots & 0 \\ J & J & B_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ J & J & J & \cdots & B_g \end{pmatrix}$$

where all the blocks J below the diagonal are matrices of 1's,and the diagonal blocks B_i is zero matrix of orderl_i,or $\mathcal{I}_{3q_i}$, or matrices of 1's of order $m_i(4 \leq m_i < n)$,$1 \leq i \leq g$. $0 \leq 3q_i, l_i \leq n$, and q_i, l_i, m_i, g are uniquely determined by T_n. Obviously, $(A^\star)^2 \leq A^\star$. $A^\star$ is transitive matrix. Hence $t(T_n) = t(A^\star) \leq s \leq n+1$. This completes the proof.

Let $T_n^{(1)} = \begin{pmatrix} 0 & 0 \\ J & \bar{T}_{n-1} \end{pmatrix}$, $\tilde{T}_n^{(1)} = \begin{pmatrix} \bar{T}_{n-1} & 0 \\ J & 0 \end{pmatrix}$, $T_n^{(2)} = \begin{pmatrix} \bar{T}_3 & 0 \\ J & \bar{T}_{n-3} \end{pmatrix}$ and $\tilde{T}_n^{(2)} = \begin{pmatrix} \bar{T}_{n-3} & 0 \\ J & \bar{T}_3 \end{pmatrix}$. Then $T_n^{(1)}, \tilde{T}_n^{(1)}, T_n^{(2)}, \tilde{T}_n^{(2)} \in \mathfrak{T}_n$. By Lemma2.5, $t(\bar{T}_{n-1}) = n-1+2 = n+1 (n \geq 8)$. Hence
$t(T_n^{(1)}) = t(\tilde{T}_n^{(1)}) = t(\bar{T}_{n-1}) = n+1 (n \geq 8)$.

Theorem 3.3 Let $T_n \in \mathfrak{T}_n$ be reducible matrix and $n \geq 8$. Then

(1) If $n \equiv 0$ or $1 (mod\ 3)$. Then $t(T_n) = n+1$ if and only if T_n is isomorphic to $T_n^{(1)}$ or $\tilde{T}_n^{(1)}$.

(2) If $n \equiv 2 (mod\ 3)$. Then $t(T_n) = n+1$ if and only if T_n is isomorphic to $T_n^{(1)}$, or $\tilde{T}_n^{(1)}$, or $T_n^{(2)}$, or $\tilde{T}_n^{(2)}$.

Proof. (1) Suppose $n \equiv 0$ or $1 (mod\ 3)$. If T_n is isomorphic to $T_n^{(1)}$ or $\tilde{T}_n^{(1)}$, by Theorem 3.3, $t(T_n) = t(T_n^{(1)}) = t(\tilde{T}_n^{(1)}) = n+1$.

Conversely, suppose $t(T_n) = n+1$. If there exists B_i that is $\mathcal{I}_{3q_i}$, $1 \leq i \leq g, 1 \leq 3q_i$, then set $s = n$ if $n \equiv 0 (mod\ 3)$ and $s = n-1$, if $n \equiv 1 (mod\ 3)$,in Theorem 3.1. Hence s is multiple of 3. By Theorem 3.2, $s < n+1$, and $t(T_n) \le s < n+1$. This is impossible. By Lemma 2.5 and Theorem 3.1, $T_n \sim \begin{pmatrix} 0 & 0 \\ J & A_0 \end{pmatrix}$, or $T_n \sim \begin{pmatrix} A_0 & 0 \\ J & 0 \end{pmatrix}$, where J is matrices of 1's, A_0 is irreducible tournament matrix of order $n-1$. By Lemma 2.5, we have that T_n is isomorphic to $T_n^{(1)}$ or $\tilde{T}_n^{(1)}$.

(2) Suppose $n \equiv 2$ $(mod\ 3)$. If T_n is isomorphic to $T_n^{(1)}$, or $\tilde{T}_n^{(1)}$, or $T_n^{(2)}$, or $\tilde{T}_n^{(2)}$, then $t(T_n) = t(T_n^{(1)}) = t(\tilde{T}_n^{(1)}) = n+1$. It is easy to verify that $t(T_n) = t(T_n^{(2)}) = t(\tilde{T}_n^{(2)}) = n+1$. Conversely, suppose $t(T_n) = n+1$.

If there does not exist B_i that is $\mathcal{I}_{3q_i}$, $1 \le i \le g, 1 \le 3q_i$, in Theorem 3.1. By Lemma 2.5 and Theorem 3.1, $T_n \sim \begin{pmatrix} 0 & 0 \\ J & A_0 \end{pmatrix}$, or $T_n \sim \begin{pmatrix} A_0 & 0 \\ J & 0 \end{pmatrix}$, where J is matrices of 1's, A_0 is irreducible tournament matrix of order $n-1$. By Lemma 2.5, we have that T_n is isomorphic to $T_n^{(1)}$ or $\tilde{T}_n^{(1)}$.

If there exists B_i that is $\mathcal{I}_{3q_i}$, $1 \le i \le g, 1 \le 3q_i$, in Theorem 3.1. By Lemma 2.5 and Theorem 3.1, $T_n \sim \begin{pmatrix} \bar{T}_3 & 0 \\ J & A_0 \end{pmatrix}$, or $T_n \sim \begin{pmatrix} A_0 & 0 \\ J & \bar{T}_3 \end{pmatrix}$, where J is matrices of 1's, A_0 is irreducible tournament matrix of order $n-3$. By Lemma 2.5, we have that T_n is isomorphic to $T_n^{(2)}$ or $\tilde{T}_n^{(2)}$. This completes the proof.

Let $\mathfrak{STR}_n$ denote the set of transitive indices of all reducible tournament matrices of order n. It is easy to verify that

$\mathfrak{STR}_2 = \mathfrak{STR}_3 = \{1\}$,

$\mathfrak{STR}_4 = \{1, 3\}$,

$\mathfrak{STR}_5 = \{1, 3, 9\}$,

$\mathfrak{STR}_6 = \{1, 3, 4, 6, 7, 9\}$,

$\mathfrak{STR}_7 = \{1, 3, 4, 5, 7, 8, 9\}$.

For $n \ge 8$, we have

Theorem 3.4 $\mathfrak{STR}_n = \{1, 3, 4, \ldots, n, n+1\}$, where $n \ge 8$.

Proof. Obviously, $t(\mathbb{T}_n) = 1$. Let $T_n = \begin{pmatrix} \bar{T}_3 & 0 \\ J & \mathbb{T}_{n-3} \end{pmatrix}$. $t(T_n) = 3$, hence, $1, 3 \in \mathfrak{STR}_n$.

By Lemma2.6, there exists an irreducible tournament matrix $\hat{T}_{n-1}$ of order $n-1$ such that $\gamma(\hat{T}_{n-1}) = e$, where $4 \le e \le n+1, n \ge 8$. Let $T_n = \begin{pmatrix} 0 & 0 \\ J & \hat{T}_{n-1} \end{pmatrix}$. Then $T_n \in \mathfrak{T}_n$ and $t(T_n) = \gamma(\hat{T}_{n-1}) = e$. This completes the proof.

Acknowledgements. This work is supported by the National Science Foundation of Guangdong(No:S2012010009759).

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