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# ESTIMATION OF GROUNDWATER VARIATIONS IN AN ANISOTROPIC LEAKY AQUIFER: A DITCH DRAIN MODEL

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**Abstract:** This paper deals with the development of mathematical models for water table fluctuation in 2-D anisotropic aquifer. Time dependent recharge and withdrawal are taken into consideration. A Boussineq equation - nonlinear partial differential equation governs the flow. The PDE is solved using finite Fourier sine transforms and closed from expression are obtained. The sensitivity of the parameters is tested using hypothetical data. Effect of leaky nature of aquifer base on the water table has been analyzed.

**Keywords:** semi pervious base; Boussinesq equation; recharge; withdrawal; Fourier transform.

**2010 AMS Subject Classification:** 93A30.

## 1. INTRODUCTION

Groundwater is a significant source of drinking and domestic water in the world. For sustainable development of this natural resource the appropriate knowledge of water table fluctuation due to recharge and withdrawal is essential. In such complex geological systems mathematical models

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are one of the best tool which helps groundwater managers to understand the phenomenon better. Mathematical models have better ability to predict the water table fluctuation in transient as well as steady state condition. These models proved to be a powerful tools to understand the effect of recharge and withdrawal on aquifer and to get the idea of variations in geological systems.

The unsteady solution of a model developed by Theis (1941), his study on the interaction of river and aquifer inspires many researchers to worked on this field. Many techniques were discussed by researchers while developing such models for simulation of seepage flow in confined as well as unconfined aquifer. (Polubarinova-Kochina (1962), Marino (1973), Hunt (1999) etc). In these research a nonlinear partial differential equation parabolic in nature called Boussinesq equation is used as governing equation. Use of Boussinesq equation for approximation of subsurface seepage flow is still a preferred choice of researchers for its simple hydrologic approach (Brutsaert 1994; Moench and Barlow 2000; Verhoest and Troch 2000; Upadhyaya and Chauhan 2001; Verhoest et al. 2002; Rai and Manglik 2012; Bansal (2012, 2013, 2015, 2016, 2020).

Some fundamental study concerning water table fluctuations in a rectangular shaped homogeneous aquifer system due to localized recharge and withdrawal include Hunt (1971), Latinopoulos (1984), Latinopoulos (1986), Manglik et al (1997). These studies use constant or time-varying recharge rate to simulate 'field like conditions'. Zomorodi (1991) presented a numerical study for evaluation of groundwater mound by including the effects of unsaturated zone on the infiltration rate and the role of in-transit water in reducing the fillable pore space. The analysis presented by him revealed that if rate of recharge is considered as constant, the predicted values might seriously underestimate or overestimate the actual results. Some researchers (Rai and Manglik 1999, 2012) propose a method consisting of sequence of line segments to approximate the rate of recharge.

Almost all models have developed under consideration that the aquifer base is perfectly impervious. In geological formations aquifer are often multilayered. Many times aquifers are leaky too. These qualities of aquifers affect fluctuation in the water table. Various parameter affected on the water table is studied by (Zlotnik and Tartakovsky 2008), Bansal and Teloglou (2013). Tang (2015) proposed a general approximate method to predict the aquifer response subject to water level variations in a free water body. Shaikh et al (2018) analyzed the dynamic behavior of tide induced water table variations in an unconfined aquifer system. Similarly, Lande and Bansal (2016, 2020) developed a model which predict the transient behavior of water table, under recharge and withdrawal condition where vertical recharge is approximated by exponential decaying function. The analysis is done by considering 2-Dimensional rectangular shape aquifer with leaky base. Finite Fourier transform is used to solve partial differential equation and variation in water mound as well as cone of depression is observed.

In this paper l leaky aquifer anisotropic in nature connected is considered. Water flowing towards the aquifer from two pervious side and two sides of the aquifer are completely impervious. Aquifer is subjected to time dependent recharge through basins and withdrawal activity through wells. The Parabolic partial differential equation which is the governing equation, solved using finite Fourier transform to obtain the closed form expressions for water head distribution in the aquifer. Finally, the hypothetical examples are used to illustrate the applicability and validity of the new result. Sensitivity of the hydraulic head based on variation in aquifer parameters is also analyzed.

## 2. Development of Mathematical Model

As shown in the Figure 1. An unconfined aquifer with length A and width B, anisotropic in nature which is resting on semi pervious base. These type of leaky formation often connected with unconfined aquifer with adjacent confined aquifers. Here the aquifer considered to be anisotropic, the hydraulic conductivity is different in both x and y direction denoted as $K_x$ and $K_y$ respectively. Recharge and withdrawal activities are carried out using rectangular basins and extraction wells located in the domain of the aquifer. If $h(x, y, t)$ denotes the variable water table measured from horizontal datum, then the flow of groundwater in unconfined horizontal aquifer with semi pervious base is governed by the following 2-dimensional partial differential equation (1) as follows:

$$K_x \frac{\partial}{\partial x}\left( h \frac{\partial h}{\partial x} \right) + K_y \frac{\partial}{\partial y}\left( h \frac{\partial h}{\partial y} \right) + P(x, y, t) = S \frac{\partial h}{\partial t} + \frac{k}{b}(h - h_0) \tag{1}$$

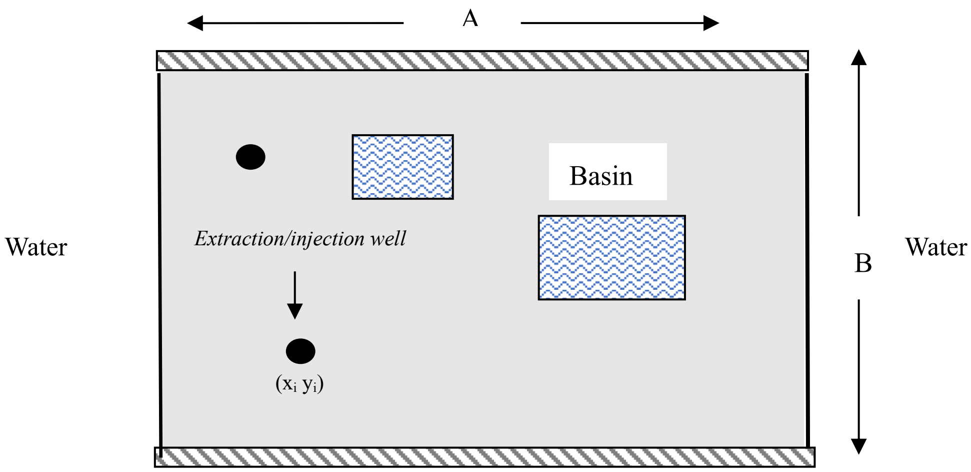



**Figure 1** Cross section view of a two-dimensional anisotropic aquifer with multiple recharge basins, injection and extraction wells.

where S is the specific yield of the aquifer, $k$ is the hydraulic conductivity of the semipervious base of the aquifer, $b$ is thickness of the of the leaky base, $h$ is the height of water head. The term $P(x,y,t)$ is the sum of recharge rate and withdrawal through the wells. In the present work, the number of basins and wells are considered to be $p_1$ and $p_2$ respectively. We assume that the $i^{th}$ basin is centered at $(x_i, y_i)$ and is of dimension $a_i$ x $b_i$; whereas the $j^{th}$ well is located at $(x_j, y_j)$. Dimension of wells are much small compared to that of basins. Recharge is considered at time-varying rate, whereas the extraction/injection is at constant rate. Thus, we define

$$P(x, y, t) = \left[ \sum_{i=1}^{p_1} R_i(x, y, t) + \sum_{j=1}^{p_2} \omega_j Q_j \delta(x - x_j) \delta(y - y_j) \right] \quad (2)$$

Where $R_i(x, y, t)$ denotes the transient recharge rate in the $i^{th}$ basin ($i = 1, 2, \ldots, p_1$) extending from $x_i \le x \le x_i + a_i$; $y_i \le y \le y_i + b_i$. The term $\omega_j$ is 1 or –1 according as the $j^{th}$ well corresponds to a injection or extraction well. $Q_j$ is the rate of injection/extraction in the $j^{th}$ well ( $j = 1, 2, \ldots, p_2$). $\delta$ is the Dirac delta function. While the extraction rate is usually constant during a complete cycle of pumping; the rate of recharge typically depends on several hydrologic parameters. It is noteworthy that the recession limb of a recharge hydrograph bears significant resemblance with an exponentially decaying function of time. Thus, we assume that

$$R_i(x,y,t)=\begin{cases} N(t) & x_i \le x \le x_i + a_i\,;\ y_i \le y \le y_i + b_i \\ 0 & \text{otherwise} \end{cases} \tag{3 a}$$

Where N(t)-time dependent recharge rate in a different time interval is define as follows:

$$N(t)=\begin{cases} N_0 & 0<x<t_1 \\ N_1 & t_1<x<t_2 \\ N_2 & t_2<x<t_3 \\ N_3 & t_3>t \end{cases} \tag{3 b}$$

The initial and the boundary conditions are prescribed as follows:

$$h\left(x,y,t=0\right)=h_0 \tag{4}$$

$$\left(\frac{\partial h}{\partial x}\right)_{x=0}=0;\qquad \left(\frac{\partial h}{\partial x}\right)_{x=A}=0 \tag{5}$$

$$h\left(x=0,y,t\right)=h_0;\quad h\left(x=A,y,t\right)=h_0 \tag{6}$$

Equation (1) is a second order partial differential equation of parabolic nature, often referred to as two-dimensional Boussinesq equation. Due to its nonlinearity, Boussinesq equation is analytically intractable. In order to find an approximate analytical solution of (1), we rewrite it in the form

$$K_x\frac{\partial^2 h^2}{\partial x^2}+K_y\frac{\partial^2 h^2}{\partial y^2}+2P\left(x,y,t\right)=S\left(\frac{1}{h}\frac{\partial h^2}{\partial t}\right)+\frac{2k}{b}\frac{\left(h^2-h_0^2\right)}{\left(h+h_0\right)} \tag{7}$$

Equation (7) is now linearized by replacing the term $h$ associated with $\partial h^2/\partial$t in the first bracket and the term $(h + h_0)/2$ of the right-hand side by the mean depth of the saturation $\hbar$. The value of $\hbar$ is obtained by successive application of the relation $\hbar = (h_0 + h_t)/2$ where $h_0$ is the initial water head and $h_t$ is the water head at the current moment (Marino 1973). The initial approximation of $\hbar$ is taken as $h_0$. We obtain

$$\frac{\partial^2 h^2}{\partial x^2}+\frac{K_y}{K_x}\frac{\partial^2 h^2}{\partial y^2}+\frac{2}{K_x}P\left(x,\ y,\ t\right)=\frac{S}{K_x\,\hbar}\left(\frac{\partial h^2}{\partial t}\right)+\frac{k}{K_x\,b\,\hbar}\left(h^2-{h_0}^2\right) \tag{8}$$

Now, define $H(x, y, t) = h^2 - h_0^2$, we get

$$\frac{\partial^2 H}{\partial x^2}+\frac{K_y}{K_x}\frac{\partial^2 H}{\partial y^2}+\frac{2}{K_x}P(x,\ y,\ t)=\frac{S}{K_x\,\hbar}\frac{\partial H}{\partial t}+\frac{2k}{K_x\,b\,\hbar}H \tag{9}$$

The initial and boundary conditions reads

$$H(x,\ y,\ 0)=0 \tag{10}$$

$$\left(\frac{\partial H}{\partial x}\right)_{x=0}=0;\qquad \left(\frac{\partial H}{\partial x}\right)_{x=A}=0 \tag{11}$$

$$H(x=0,\ y,\ t)=0;\quad H(x=A,\ y,\ t)=0 \tag{12}$$

Equation (9) along with the boundary conditions (10) to (12) is solved using Fourier Cosine transform. Fourier cosine transform is defining as follows:

$$\xi(m,\ n,\ t)=F_{sc}\{H(x,y,t);(x,\ y)\to(m,\ n)\}=\int_{x=0}^{A}\int_{y=0}^{B}H(x,\ y,\ t)\sin\left(\frac{m\pi x}{A}\right)\cos\left(\frac{n\pi y}{B}\right)dy\,dx \tag{13}$$

The finite Fourier cosine transform reduces equation (9) to the following form

$$-\beta_m{}^2\xi-\gamma_n{}^2\xi+\frac{2}{K_x}\overline{P}(m,\ n,\ t)=\frac{1}{\nu}\frac{d\xi}{dt}+c\,\xi \tag{14}$$

Where

$$\beta_m=\frac{m\pi}{A},\gamma_n=\frac{n\pi}{B}\sqrt{\frac{K_y}{K_x}},c=\frac{k}{K_x\,b\,\hbar}\text{ and }\ \nu=\frac{K_x\,\hbar}{S} \tag{15}$$

And

$$\overline{P}(m,n,t)=\left[\sum_{i=1}^{p_1}\Omega_i\,N(t)+\sum_{j=1}^{p_2}\omega_j\,\eta_j\,Q_j\right] \tag{16}$$

Where

$$\Omega_i=-\frac{1}{\beta_m\,\gamma_n}\left[\cos\{\beta_m(x_i+a_i)\}-\cos(\beta_m\,x_i)\right]\left[\sin\{\gamma_n(y_i+b_i)\}-\sin(\gamma_n\,y_i)\right]$$

(17)

and

$$\eta_j=\sin(\beta_m\,x_j)\cos\left(\gamma_n\sqrt{\frac{K_y}{K_x}}\,y_j\right) \tag{18}$$

equation (14) is solved using ordinary method, the solution obtained is

$$\xi(m,n,t)=\frac{2\nu}{K_x}\left[\sum_{i=1}^{p_1}\Omega_i\left\{\frac{N_{i0}}{\alpha+\nu c}\left(1-e^{-(\alpha+\nu c)t}\right)+\frac{N_{i1}}{\alpha+\nu c-\lambda_i}\left(e^{-\lambda_i t}-e^{-(\alpha+\nu c)t}\right)\right\}\right.$$
$$\left.+\sum_{j=1}^{p_2}\frac{\omega_j\,\eta_j\,Q_j}{\alpha+\nu c}\left(1-e^{-(\alpha+\nu c)t}\right)\right] \tag{19}$$

The term used is $\alpha=\nu\left(\beta_m{}^2+\gamma_n{}^2\right)$ and τ is the variable of integration.

$H(x, y, t)$ can now be obtained by using inverse Fourier cosine transform which is define as

$$H(m.n,t)=\frac{4}{AB}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\xi(m,n,t)\sin\left(\frac{m\pi x}{A}\right)\cos\left(\frac{n\pi y}{B}\right) \tag{20}$$

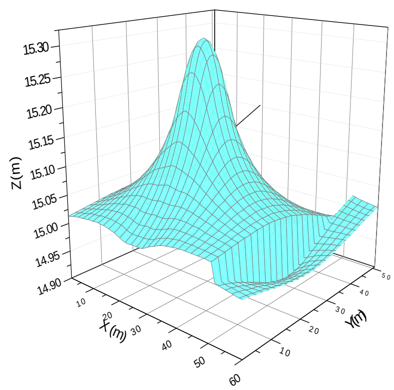
Thus the solution of the equation (8) is,

$$h^2=h_0{}^2+\frac{8\nu}{ABK_x}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sin\left(\beta_m x\right)\cos\left(\gamma_n\sqrt{\frac{K_y}{K_x}}y\right)\left[\sum_{j=1}^{p_2}\frac{\omega_j\,\eta_j\,Q_j}{\alpha+\nu c}\left(1-e^{-(\alpha+\nu c)t}\right)\right.$$
$$\left.+\sum_{i=1}^{p_1}\Omega_i\left\{\frac{1}{\alpha+\nu c}\left(N_0\left(1-e^{-(\alpha+\nu c)t_1}\right)\right)+N_1\left(e^{-(\alpha+\nu c)t_2}-e^{-(\alpha+\nu c)t_1}\right)+N_2\left(e^{-(\alpha+\nu c)t_3}-e^{-(\alpha+\nu c)t_2}\right)+N_3\left(-e^{-(\alpha+\nu c)t_3}\right)\right\}\right] \tag{21}$$

## 3. Discussion of Results

In order to implement the applicability of developed result in this study, two rectangular basins namely $B_1$ and $B_2$ are considered. The of dimension of each basin is 10 m x 10 m, and centered at (35 m, 35 m) and (115 m, 35 m) respectively. Two extractions wells $W_1$ and $W_2$ are also located in the domain at (35m, 75m) and (115m, 75m). Basins ate are subjected to have vertical time varying recharge. The recharge rates considered here are $N_o$=2.5, $N_1$=3, $N_2$=3.5 and $N_4$ = 4 for consecutive four days ($t_1$=1, $t_2$=2,$t_3$=3and $t_4$=4). Similarly, water is extracted from wells at a constant rate 40 and 30 $m^3$/day respectively. Average saturated depth of the aquifer is determined using an iterative relation $\hbar = (h_0 + h_t)/2$ where $h_0$ is the initial water head and $h_t$ is the water head at the current moment (Marino 1973). Initial approximation of $\hbar$ is taken as $h_0$.

Transient profiles of the water table fluctuations are determined. Distribution of water head along line $y = 35\ m$ (line passing through the centers of recharge basins $B_1$ and $B_2$). It is observed that the groundwater mounds forms are symmetrical about the centers of the basins. Figure 2 a & b shows the 3D view of the water mound form under recharge basin.



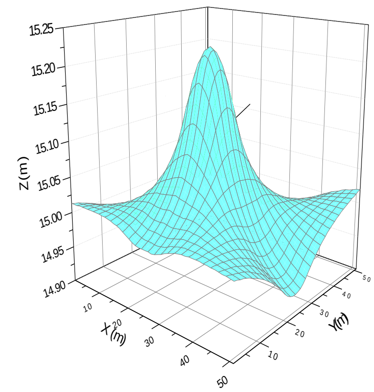
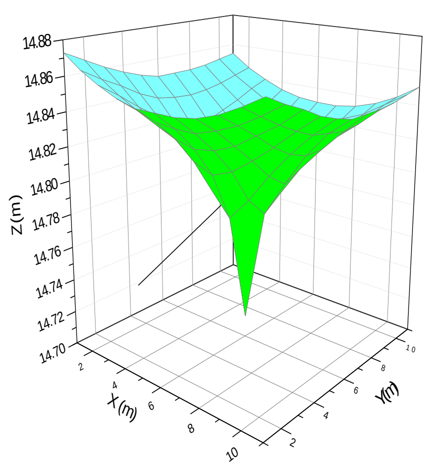




**Figure 2 (a)** Water mound for k = 0.25 **Figure 2 (b)** Water mound for k = 0

From the Figure 2 (a) and 2(b) it is clearly seen that the groundwater mound attains higher level in those aquifers which have comparatively higher values of hydraulic resistance. This is primarily due to vertical seepage loss through the aquifer's base, which decreases as the hydraulic resistance increases.

Lowering of water table due to continuous pumping from wells $W_1$ and $W_2$ are shown in Figure 3 a & b. The difference in depth of cone under $W_1$ and $W_2$ is primarily due to varying pumping rate ($Q_1 = 40$ m$^3$/d, $Q_2 = 30$ m$^3$/d).



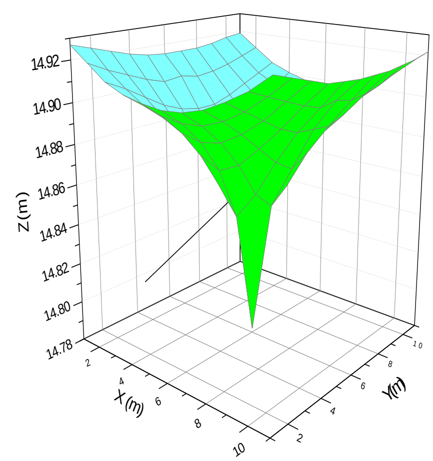




**Figure 3 (a)** Cone of depression when =0.25 **Figure 3 (b)** cone of depression when k=0

These profiles characterize cones of depression in the presence of semi pervious base with $k = 0.25$ m/d. It can be observed from these figures that the depth of cone increases with time. Moreover, water table depletion induced by pumping from wells is also affected by the hydraulic resistance of the aquifer's base. When the base is leaky, withdrawal from the wells is supplemented by the leakage induced vertical flow from hydraulically connected sources. Consequently, the depth of the cone of depression is mitigated.

Three dimensional view of the groundwater mound and the cone of depression along with cross sectional view is shown in Figure. 4 a & b and Figure 5 a & b for $k = 0.25$ m/d and fully impervious base (k=0) respectively. The changes in the water profile due to recharge and pumping is clearly seen in these figures. The effect of hydraulic resistance is also seen

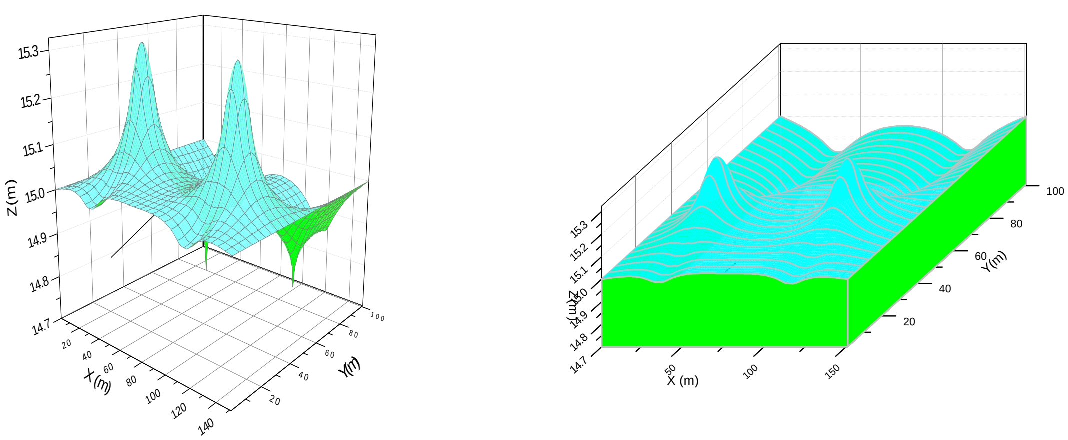



**Figure 4: (a)** 3D view of water profile for $k = 0.25$, **(b)** cross sectional view.

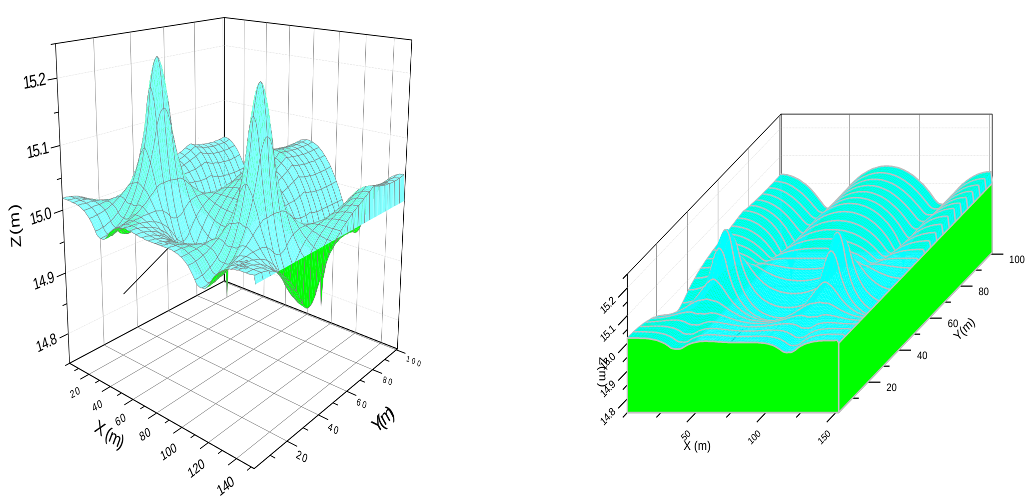



**Figure 5**: **(a)** 3D view of water profile for $k = 0$, **(b)** Cross sectional 3-D view.

## 4. Conclusion

In this study, a new analytical solution is developed to approximate 2-dimensional groundwater flow and variations in water table in a rectangular shaped unconfined leaky aquifer. The aquifer is subjected to multiple transient recharge and withdrawal through recharge basins and extraction wells located in the model domain. The water table fluctuation is simulated and its dependence on aquifer parameter is demonstrated. Finite Fourier sine transform is used to solve the linearized version of the flow equation. Formation and stabilization of groundwater mound beneth the recharge basins are observed in the simulated example. Similarly, cone of depression is observed under the extraction wells. Besides, the bed lekance has a direct impact on the height of the mound as well as depth of the cone. The results developed in this study can be used as test cases for experimental studies and guidelines for numerical modelling.

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## Conflict of Interests

The author(s) declare that there is no conflict of interests.

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