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WELL-POSEDNESS OF RIEMANN-LIOUVILLE FRACTIONAL DEGENERATE EQUATIONS WITH FINITE DELAY IN BANACH SPACES

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Abstract. We study the Existence and uniqueness of solutions of the Riemann-Liouville fractional integro-differential degenerate equations

$\frac{d}{dt}(B\frac{1}{\Gamma(1-\alpha)}\int_{-\infty}^{t}(t-s)^{-\alpha}x(s)ds) = Ax(t)+\int_{-\infty}^{t}a(t-s)x(s)ds+L(x_t)+\frac{1}{\Gamma(\beta)}\int_{-\infty}^{t}(t-s)^{\beta-1}x(s)ds+f(t).$

where A and B are a linear closed operators in a Banach space.

Keywords: Riemann-Liouville fractional; integro-differential equations; L^p-multipliers; UMD-spaces.

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1. Introduction

Differential equations play an important role in describing many real-world processes. For many years the models are successfully used to study a number of physical, biological. A particular interest is in differential equations with many variables such as partial differential equations and/or integral differential equations in the case when one of the variables is times. In this work, we study the existence of periodic solutions for the following Riemann-Liouville fractional integro-differential degenerate equations.

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$$\frac{d}{dt}\Big(B\frac{1}{\Gamma(1-\alpha)}\int_{-\infty}^{t}(t-s)^{-\alpha}x(s)ds\Big)=Ax(t)+\int_{-\infty}^{t}a(t-s)x(s)ds$$
$$+L(x_t)+\frac{1}{\Gamma(\beta)}\int_{-\infty}^{t}(t-s)^{\beta-1}x(s)ds+f(t);\quad 0\leq t\leq 2\pi \tag{1.1}$$

where $\Gamma(.)$ is the Euler gamma function, $\alpha,\beta\in\mathbb{R}^{+},0\leq\beta\leq\alpha$ and $A:D(A)\subseteq X\to X$ and B are a linear closed operators on Banach space $(X,\|.\|)$ such that $D(A)\subset D(B)$, $f\in L^{p}([-r_{2\pi},0],X)$ for all $p\geq 1$ and $r_{2\pi}:=2\pi N$ (some $N\in\mathbb{N}$), $a\in L^{1}(\mathbb{R}_{+})$, L is a linear operator and x_t is an element of $L^{p}([-r_{2\pi}\ ,0],\ X)$ which is defined as follows

$$x_t(\theta)=x(t+\theta)\ \text{ for }\ \theta\in[-r_{2\pi},\ 0].$$

The operator-valued Fourier multiplier Theorems 2.8 have been used by Keyantuo and Lizama in [19] to establish maximal regularity results for an integro-differential equation in Banach space. The authors consider the following problem

$$x'(t)=Ax(t)+\int_{-\infty}^{t}a(t-s)Ax(s)ds+f(t);\ \ x(0)=x(2\pi)$$

Maximal regularity for the evolution problem in L^p was treated earlier by Weis [30, 31] (see also [12] for a different proof of the operator-valued Mikhlin multiplier theorem using a transference principle). The study in the L^p framework (when $1<p<\infty$) was made possible thanks to the introduction of the concept of randomized boundedness (hereafter R-boundedness, also known as Riesz-boundedness or Rademacher-boundedness). With this, necessary conditions for operator-valued Fourier multipliers were found in this context. In addition, the space X must have the UMD property. This was done initially by L. Weis [30, 31] for the evolutionary problem and then by Arendt-Bu [2] for periodic boundary conditions. For non-degenerate integro-differential equations both in the periodic and non periodic cases, operator-valued Fourier multipliers have been used by various authors to obtain well-posedness in various scales of function spaces: [7, 9, 10, 19, 25, 20, 21, 27] and the corresponding references. The well-posedness or maximal regularity results are important in that they allow for the treatment of nonlinear problems. Earlier results on the application of operator-valued Fourier multiplier theorems to evolutionary integral equations can be found in [12]. More recent examples of second order integro-differential equations with frictional damping and memory terms have been studied in the paper [11]

In [8] Bu et al studied the well-posedness of thethird-order integro-differential equations

$$\alpha u'''(t)+u''(t)=\beta Au(t)+\beta\int_{-\infty}^{t}a(t-s)Ax(s)ds+\gamma Bu'(t)+f(t),$$

with periodic boundaryconditions $u(0)=u(2\pi),u'(0)=u'(2\pi),u''(0)=u''(2\pi)$.

In [22], S.Koumla, Kh.Ezzinbi, R.Bahloul established mild solutions for some partial functional integrodifferential equations with finite delay

$$\frac{d}{dt}x(t)=Ax(t)+\int_{0}^{t}B(t-s)x(s)ds+f(t,x_t)+h(t,x_t)$$

where $A:D(A)X\to X$ is the infinitesimal generator of a C_0-semigroup $(T(t))_{t\geq 0}$ on a Banach space X, for $t\geq 0,B(t)$ is a closed linear operator with domain $D(B)\supset D(A)$.

This work is organized as follows : In Section 2 we collect some preliminary results and definitions. In section 3, we study the existence and uniqueness of strong L^p-solution of the Eq. (1.1) solely in terms of a property of R-boundedness for the sequence of operators $(ik)^{\alpha}((ik)^{\alpha}I-A-L_k-\tilde{a}(ik)-(ik)^{-\beta}I)^{-1}$. We optain that the following assertion are equivalent in UMD space :

(1): $((ik)^{\alpha}B-A-L_k-\tilde{a}(ik)-(ik)^{-\beta}I)$ is invertible and $\{((ik)^{\alpha}((ik)^{\alpha}B-A-L_k-\tilde{a}(ik)-(ik)^{-\beta}I)^{-1},k\in\mathbb{Z}\}$ is R-bounded.

(2): For every $f\in L^p(\mathbb{T};X)$ there exist a unique function $u\in H^{\alpha,p}(\mathbb{T};X)$ such that $u\in D(A)$ and equation (1.1) holds for a.e $t\in[0,2\pi]$.

2. Preliminaries

In this section, we collect some results and definitions that will be used in the sequel. Let X be a complex Banach space. We denote as usual by $L^1(0,2\pi,X)$ the space of Bochner integrable functions with values in X. For a function $f\in L^1(0,2\pi;X)$, we denote by $\hat{f}(k),k\in\mathbb{Z}$ the kth Fourier coefficient of f:

$$\hat{f}(k)=\frac{1}{2\pi}\int_{0}^{2\pi}e_{-k}(t)f(t)dt,$$

where $e_k(t) = e^{ikt}, t \in \mathbb{R}$.

Lemma 2.1. [24]

Let $L : L^p(\mathbb{T}, X) \to X$ be a bounded linear operateur. Then

$$\widehat{L(u_.)}(k) = L(e_k \hat{u}(k)) := L_k \hat{u}(k) \text{ for all } k \in \mathbb{Z}.$$

Let $a \in L^1(\mathbb{R}_+)$. We consider the the function

$$F(t) = \int_{-\infty}^{t} a(t-s)u(s)ds, \quad t \in \mathbb{R}.$$

Since

$$F(t) = \int_{-\infty}^{t} a(t-s)u(s)ds = \int_{0}^{\infty} a(s)u(t-s)ds, \tag{2.1}$$

we have $\|F\|_{L^1} \leq \|a\|_1 \|u\|_{L^1} = \|a\|_{L^1(\mathbb{R}_+)} \|u\|_{L^1(0,2\pi;X)}$ and F is periodic of period $T = 2\pi$ as u. Now using Fubini's theorem and (2.1) we obtain, for $k \in \mathbb{Z}$, that

$$\hat{F}(k) = \tilde{a}(ik)\hat{u}(k), k \in \mathbb{Z} \tag{2.2}$$

where $\tilde{a}(\lambda) = \int_0^\infty e^{-\lambda t} a(t)dt$ denotes the Laplace transform of a. This identity plays a crucial role in the paper.

Let X, Y be Banach spaces. We denote by $\mathscr{L}(X,Y)$ the set of all bounded linear operators from X to Y. When $X = Y$, we write simply $\mathscr{L}(X)$.

Proposition 2.2 ([2, Fejer's Theorem])**.** *Let $f \in L^p(0, 2\pi; X))$, then one has*

$$f = \lim_{n\to\infty} \frac{1}{n+1} \sum_{m=0}^{n} \sum_{k=-m}^{m} e_k \hat{f}(k)$$

with convergence in $L^p(0,2\pi;Y)$.

R-boundedness-UMD space,L^p-multiplier and Riemann-Liouville fractional integral. For $j \in \mathbb{N}$, denote by r_j the j-th Rademacher function on $[0,1]$, i.e. $r_j(t) = sgn(\sin(2^j \pi t))$. For $x \in X$ we denote by $r_j \otimes x$ the vector valued function $t \to r_j(t)x$.

The important concept of R-bounded for a given family of bounded linear operators is defined as follows.

Definition 2.3. A family $\mathbf{T} \subset \mathscr{L}(X,Y)$ is called R-bounded if there exists $c_q \geq 0$ such that

$$\|\sum_{j=1}^{n} r_j \otimes T_j x_j\|_{L^q(0,1;X)} \leq c_q \|\sum_{j=1}^{n} r_j \otimes x_j\|_{L^q(0,1;X)} \tag{2.3}$$

for all $T_1,\dots,T_n \in \mathbf{T}, x_1,\dots,x_n \in X$ and $n \in \mathbb{N}$, where $1 \leq q < \infty$. We denote by $R_q(\mathbf{T})$ the smallest constant c_q such that (2.3) holds.

Definition 2.4. Let $\varepsilon \in]0,1[$ and $1 < p < \infty$. Define the operator H_ε by: for all $f \in L^p(\mathbb{R};X)$

$$(H_\varepsilon f)(t) := \frac{1}{\pi} \int_{\varepsilon < |s| < \frac{1}{\varepsilon}} \frac{f(t-s)}{s} ds$$

if $\lim_{\varepsilon \to 0} H_\varepsilon f := Hf$ exists in $L^p(\mathbb{R};X)$ Then Hf is called the Hilbert transform of f on $L^p(\mathbb{R},X)$.

Definition 2.5. A Banach space X is said to be UMD space if the Hilbert transform is bounded on $L^p(\mathbb{R};\ X)$ for all $1 < p < \infty$.

Definition 2.6. For $1 \leq p < \infty$, a sequence $\{M_k\}_{k\in\mathbb{Z}} \subset \mathbf{B}(X,Y)$ is said to be an L^p-multiplier if for each $f \in L^p(\mathbb{T},X)$, there exists $u \in L^p(\mathbb{T},Y)$ such that $\hat{u}(k) = M_k \hat{f}(k)$ for all $k \in \mathbb{Z}$.

Proposition 2.7. Let X be a Banach space and $\{M_k\}_{k\in\mathbb{Z}}$ be an L^p-multiplier, where $1 \leq p < \infty$. Then the set $\{M_k\}_{k\in\mathbb{Z}}$ is R-bounded.

Theorem 2.8. (Marcinkiewicz operator-valued multiplier Theorem).
Let X, Y be UMD spaces and $\{M_k\}_{k\in\mathbb{Z}} \subset B(X,Y)$. If the sets $\{M_k\}_{k\in\mathbb{Z}}$ and $\{k(M_{k+1}-M_k)\}_{k\in\mathbb{Z}}$ are R-bounded, then $\{M_k\}_{k\in\mathbb{Z}}$ is an L^p-multiplier for $1 < p < \infty$.

Definition 2.9. The Riemann-Liouville fractional integral operator of order $\alpha > 0$ is defined by

$$\mathscr{I}^{\alpha}_{-\infty} f(t) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{t} (t-s)^{\alpha-1} f(s) ds$$

where $\Gamma(\alpha) = \int_0^{+\infty} e^{-t} t^{\alpha-1} dt$, is the Euler gamma function.

Definition 2.10. The Riemann-Liouville fractional integral derivative operator of order $\alpha > 0$ is defined by

$$\mathscr{D}^{\alpha}_{-\infty} f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} (\int_{-\infty}^{t} (t-s)^{-\alpha} f(s) ds)$$

Those familiar with the Fourier transform know that the Fourier transform of a derivative can be expressed by the following:

$$\widehat{\frac{dx}{dt}}(k) = ik\hat{x}(k), \forall k \in \mathbb{Z}$$

and more generally,

$$\widehat{\frac{d^n x}{dt^n}}(k) = (ik)^n \hat{x}(k), \forall k \in \mathbb{Z}$$

A similar identity holds for anti-derivatives

$$\widehat{\mathscr{I}^s_{-\infty} f}(k) = (ik)^{-s} \hat{x}(k), \forall k \in \mathbb{Z}$$

$$\widehat{\mathscr{D}^s_{-\infty} f}(k) = (ik)^{s} \hat{x}(k), \forall k \in \mathbb{Z}$$

Remark 2.11. *If we set $u(x) = e^{ikx}$ for $k \in \mathbb{Z}$ we have*

$$1) \mathscr{D}^{\alpha}_{-\infty} u(t) = (ik)^{\alpha} e^{ikx}$$

$$2) \mathscr{I}^{\alpha}_{-\infty} u(t) = (ik)^{-\alpha} e^{ikx}.$$

3. Periodic Solutions in UMD space

For $a \in L^1(\mathbb{R}_+)$, we denote by $a * x$ the function

$$(a * x)(t) := \int_{-\infty}^{t} a(t-s)x(s)ds$$

with this notation we may rewrite Eq. (1.1) in the following was:

$$\mathscr{D}^{\alpha}_{-\infty} Bx(t) = Ax(t) + L(x_t) + (a*x)(t) + \mathscr{I}^{\beta}_{-\infty} x(t) + f(t) \ for \ t \in \mathbb{R}. \tag{3.1}$$

we have $\widehat{a*x}(k) = \tilde{a}(ik)\hat{x}(k)$. We define

$$\Delta_k = ((ik)^{\alpha} B - A - L_k - \tilde{a}(ik)I - (ik)^{-\beta} I)$$

and

$$\sigma_{\mathbb{Z}}(\Delta) = \{k \in \mathbb{Z} : \Delta_k \ is \ not \ bijective\}$$

the periodic vector-valued space is defined by

$$H^{\alpha,p}(\mathbb{T};X)=\{u\in L^p(\mathbb{T},X):\exists v\in L^p(\mathbb{T},X),\hat{v}(k)=(ik)^{\alpha}\hat{u}(k) \text{ for all } k\in\mathbb{Z}\}$$

Definition 3.1. For $1\le p<\infty$, we say that a sequence $\{M_k\}_{k\in\mathbb{Z}}\subset\mathbf{B}(X,Y)$ is an $(L^p,H^{1,p})$-multiplier, if for each $f\in L^p(\mathbb{T},X)$ there exists $u\in H^{1,p}(\mathbb{T},Y)$ such that

$$\hat{u}(k)=M_k\hat{f}(k) \quad \text{for all } k\in\mathbb{Z}.$$

Lemma 3.2. *Let $1\le p<\infty$ and $(M_k)_{k\in\mathbb{Z}}\subset\mathbf{B}(X)$ ($\mathbf{B}(X)$ is the set of all bounded linear operators from X to X). Then the following assertions are equivalent:*
(i) $(M_k)_{k\in\mathbb{Z}}$ is an $(L^p,H^{\alpha,p})$-multiplier.
(ii) $((ik)^{\alpha}M_k)_{k\in\mathbb{Z}}$ is an (L^p,L^p)-multiplier.

We begin by establishing our concept of strong solution for Eq. (3.1)

Definition 3.3. Let $f\in L^p(\mathbb{T};X)$. A function $x\in H^{\alpha,p}(\mathbb{T};X)$ is said to be a 2π-periodic strong L^p-solution of Eq.(3.1) if $x(t)\in D(A)$ for all $t\ge 0$ and Eq. (3.1) holds almost every where.

Proposition 3.4. *Let A be a closed linear operator defined on an UMD space X. Suppose that $\sigma_{\mathbb{Z}}(\Delta)=\phi$ .Then the following assertions are equivalent :*

(i): $\left((ik)^{\alpha}((ik)^{\alpha}B-A-L_k-\tilde{a}(ik)I-(ik)^{-\beta}I)^{-1}\right)_{k\in\mathbb{Z}}$ *is an L^p-multiplier for $1<p<\infty$*

(ii): $\left((ik)^{\alpha}((ik)^{\alpha}B-A-L_k-\tilde{a}(ik)I-(ik)^{-\beta}I)^{-1}\right)_{k\in\mathbb{Z}}$ *is R-bounded.*

Proof. (i) $\Rightarrow$ (ii) As a consequence of Proposition (2.7)
(ii) $\Rightarrow$ (i) Let $a_{s,k}=(ik)^{-s},s\in\mathbb{R},k\neq 0$
Define $M_k=(ik)^{\alpha}(C_k-A)^{-1}$, where $C_k:=(ik)^{\alpha}B-L_k-\tilde{a}(ik)I-(ik)^{-\beta}I$. By Theorem (2.8) it is sufficient to prove that the set $\{k(M_{k+1}-M_k)\}_{k\in\mathbb{Z}}$ is R-bounded. Since

$$\begin{aligned}
&k\left[M_{k+1}-M_k\right]\\
&=k\left[(i(k+1))^{\alpha}(C_{k+1}-A)^{-1}-(ik)^{\alpha}(C_k-A)^{-1}\right]\\
&=k(C_{k+1}-A)^{-1}\left[(i(k+1))^{\alpha}(C_k-A)-(ik)^{\alpha}(C_{k+1}-A)\right](C_k-A)^{-1}\\
&=kM_{k+1}\left[a_{\alpha,k}(C_k-A)-a_{\alpha,k+1}(C_{k+1}-A)\right]M_k\\
&=kM_{k+1}\left[a_{\alpha,k}C_k-a_{\alpha,k+1}C_{k+1}+(a_{\alpha,k+1}-a_{\alpha,k})A\right]M_k
\end{aligned}$$

$$= ka_{\alpha,k}M_{k+1}C_kM_k - ka_{\alpha,k+1}M_{k+1}C_{k+1}M_k + k(a_{\alpha,k+1} - a_{\alpha,k})M_{k+1}AM_k$$

$$= ka_{\alpha,k}M_{k+1}C_kM_k - ka_{\alpha,k+1}M_{k+1}C_{k+1}M_k$$

$$+ k(\frac{a_{\alpha,k+1} - a_{\alpha,k}}{a_{\alpha,k}})M_{k+1}(a_{\alpha,k}M_kC_k - I).$$

Observe that for $\alpha > 0$ we have that $|(i(k+1))^{\alpha} - (ik)^{\alpha}|$ can be estimated by $(ik)^{\alpha-1}$ uniformly in k according to the definition of $|(ik)^{\alpha}|$ and the mean value theorem. This implies that $\frac{k(a_{\alpha,k+1}-a_{\alpha,k})}{a_{\alpha,k}}$ is bounded sequence. Since $ka_{\alpha,k}$ also is bounded for $\alpha > 0$. Since products and sums of R-bounded sequences is R-bounded [24, Remark 2.2]. Then the proof is complete. □

Lemma 3.5. *Let $1 \leq p < \infty$. Suppose that $\sigma_{\mathbb{Z}}(\Delta) = \phi$ and that for every $f \in L^p(\mathbb{T};X)$ there exists a 2π-periodic strong L^p-solution x of Eq.* (3.1). *Then x is the unique 2π-periodic strong L^p-solution.*

Proof. Suppose that x_1 and x_2 two strong L^p-solution of Eq. (3.1) then $x = x_1 - x_2$ is a strong L^p-solution of Eq. (3.1) corresponding to $f = 0$. Taking Fourier transform in (3.1), we obtain that

$$(ik)^{\alpha}B\hat{x}(k) = A\hat{x}(k) + L_k\hat{x}(k) + \tilde{a}(ik)\hat{x}(k) + (ik)^{-\beta}\hat{x}(k), k \in \mathbb{Z}.$$

Then

$$((ik)^{\alpha}B - A - L_k - \tilde{a}(ik)I - (ik)^{-\beta}I)\hat{x}(k) = 0$$

It follows that $\hat{x}(k) = 0$ for every $k \in \mathbb{Z}$ and therefore $x = 0$. Then $x_1 = x_2$. □

Theorem 3.6. *Let X be a Banach space. Suppose that for every $f \in L^p(\mathbb{T};X)$ there exists a unique strong solution of Eq.* (3.1) *for $1 \leq p < \infty$. Then*

(1) for every $k \in \mathbb{Z}$ the operator $\Delta_k = ((ik)^{\alpha}B - A - L_k - \tilde{a}(ik)I - (ik)^{-\beta}I)$ has bounded inverse

(2) $\{(ik)^{\alpha}\Delta_k^{-1}\}_{k\in\mathbb{Z}}$ is R-bounded.

Before to give the proof of Theorem 3.6, we need the following Lemma.

Lemma 3.7. *if $((ik)^{\alpha}B - A - L_k - \tilde{a}(ik)I - (ik)^{-\beta}I)(x) = 0$ for all $k \in \mathbb{Z}$, then $u(t) = e^{ikt}x$ is a 2π-periodic strong L^p-solution of the following equation*

$$\mathscr{D}^{\alpha}_{-\infty}(Bu)(t) = Au(t) + (a * u)(t) + \mathscr{I}^{\beta}_{-\infty}(u)(t).$$

Proof. We have $((ik)^{\alpha}B - A - L_k - \tilde{a}(ik)I - (ik)^{-\beta}I)x = 0.$

Then

$$(ik)^{\alpha}Bx = Ax + L_k x + \tilde{a}(ik)x + (ik)^{-\beta}x$$

We have $u(t) = e^{ikt}x$. In fact, since $u_t(\theta) = e^{ik\theta}u(t)$ we obtain $u_t = e_k u(t)$. By Remark 2.11 (2),

$$\begin{aligned}\mathscr{D}^{\alpha}_{-\infty}(Bu)(t) &= (ik)^{\alpha}Be^{ikt}x = e^{ikt}((ik)^{\alpha}Bx)\\ &= e^{ikt}[Ax + L_k x + \tilde{a}(ik)x + (ik)^{-\beta}x]\\ &= Ae^{ikt}x + L_k(e^{ikt}x) + \tilde{a}(ik)e^{ikt}x + (ik)^{-\beta}e^{ikt}x]\\ &= Au(t) + L(e_k u(t)) + \tilde{a}(ik)u(t) + (ik)^{-\beta}u(t)]\\ &= Au(t) + L(u_t) + (a*u)(t) + \mathscr{I}^{\alpha}_{-\infty}u(t)\end{aligned}$$

Proof of Theorem 3.6: 1) Let $k \in \mathbb{Z}$ and $y \in X$. Then for $f(t) = e^{ikt}y$, there exists $x \in H^{\alpha,p}(\mathbb{T};X)$ such that:

$$\mathscr{D}^{\alpha}_{-\infty}(Bx)(t) = Ax(t) + L(x_t) + (a*x)(t) + \mathscr{I}^{\beta}_{-\infty}(x)(t) + f(t)$$

Taking Fourier transform. We have $\widehat{\mathscr{D}^{\alpha}_{-\infty}Bx}(k) = (ik)^{\alpha}B\hat{x}(k)$ and $\widehat{\mathscr{I}^{\beta}_{-\infty}x}(k) = (ik)^{-\beta}\hat{x}(k)$

Consequently, we have

$$(ik)^{\alpha}B\hat{x}(k) = A\hat{x}(k) + L_k\hat{x}(k) + \tilde{a}(ik)\hat{x}(k) + (ik)^{-\beta}\hat{x}(k) + \hat{f}(k)$$

$[(ik)^{\alpha}B - A - L_k - \tilde{a}(ik) - (ik)^{-\beta}]\hat{x}(k) = \hat{f}(k) = y \Rightarrow ((ik)^{\alpha}B - A - L_k - \tilde{a}(ik) - (ik)^{-\beta})$ is surjective.

if $((ik)^{\alpha}B - A - L_k - \tilde{a}(ik) - (ik)^{-\beta})(u) = 0$, then by Lemma 3.7, $x(t) = e^{ikt}u$ is a 2π-periodic strong L^p-solution of Eq.(3.1) corresponing to the function $f(t) = 0$ Hence $x(t) = 0$ and $u = 0$ then $((ik)^{\alpha}B - A - L_k - \tilde{a}(ik) - (ik)^{-\beta})$ is injective.

2) Let $f \in L^p(\mathbb{T};X)$. By hypothesis, there exists a unique $x \in H^{\alpha,p}(\mathbb{T},X)$ such that the Eq. (3.1) is valid. Taking Fourier transforms, we deduce that

$$\hat{x}(k) = ((ik)^{\alpha}B - A - L_k - \tilde{a}(ik) - (ik)^{-\beta})^{-1}\hat{f}(k) \quad \text{for all } k \in \mathbb{Z}.$$

Hence

$$(ik)^{\alpha}\hat{x}(k) = (ik)^{\alpha}((ik)^{\alpha}B - A - L_k - \tilde{a}(ik) - (ik)^{-\beta})^{-1}\hat{f}(k) \ \text{ for all } k \in \mathbb{Z}$$

Since $x \in H^{\alpha,p}(\mathbb{T};X)$, then there exists $v \in L^p(\mathbb{T};X)$ such that

$$\hat{v}(k) = (ik)^{\alpha}\hat{x}(k) = (ik)^{\alpha}((ik)^{\alpha}B - A - L_k - \tilde{a}(ik) - (ik)^{-\beta})^{-1}\hat{f}(k).$$

Then $\left\{(ik)^{\alpha}\Delta_k^{-1}\right\}_{k\in\mathbb{Z}}$ is an L^p-multiplier and $\left\{(ik)^{\alpha}\Delta_k^{-1}\right\}_{k\in\mathbb{Z}}$ is R-bounded. □

4. Main Result

Our main result in this work is to establish that the converse of Theorem 3.6, are true, provided X is an UMD space.

Theorem 4.1. *Let X be an UMD space and $A : D(A) \subset X \to X$ be an closed linear operator. Then the following assertions are equivalent for $1 < p < \infty$.*

- **(1):** *for every $f \in L^p(\mathbb{T};X)$ there exists a unique 2π-periodic strong L^p-solution of Eq.* (3.1).
- **(2):** $\sigma_{\mathbb{Z}}(\Delta) = \phi$ *and* $\left\{(ik)^{\alpha}\Delta_k^{-1}\right\}_{k\in\mathbb{Z}}$ *is R-bounded.*

Lemma 4.2. [2]. *Let $f,g \in L^p(\mathbb{T};X)$. If $\hat{f}(k) \in D(A)$ and $A\hat{f}(k) = \hat{g}(k)$ for all $k \in \mathbb{Z}$ Then*

$$f(t) \in D(A) \;\; and \;\; Af(t) = g(t) \;\; for\ all \;\;\; t \in [0,2\pi].$$

Proof. $1) \Rightarrow 2)$ see Theorem 3.6

$1) \Leftarrow 2)$ Let $f \in L^p(\mathbb{T};X)$. Define

$$\Delta_k = ((ik)^{\alpha}B - A - L_k - \tilde{a}(ik)I - (ik)^{-\beta}I)$$

By Lemma 3.2, the family $\left\{(ik)^{\alpha}\Delta_k^{-1}\right\}_{k\in\mathbb{Z}}$ is an L^p-multiplier it is equivalent to
the family $\left\{\Delta_k^{-1}\right\}_{k\in\mathbb{Z}}$ is an L^p-multiplier that maps $L^p(\mathbb{T};X)$ into $H^{\alpha,p}(\mathbb{T};X)$,
namely there exists $x \in H^{1,p}(\mathbb{T},X)$ such that

$$\hat{x}(k) = \Delta_k^{-1}\hat{f}(k) = ((ik)^{\alpha}B - A - L_k - \tilde{a}(ik)I - (ik)^{-\beta}I)^{-1}\hat{f}(k) \tag{4.1}$$

In particular, $x \in L^p(\mathbb{T};X)$ and there exists $v \in L^p(\mathbb{T};X)$ such that $\hat{v}(k) = (ik)^{\alpha}\hat{x}(k)$

$$\widehat{\mathscr{D}^{\alpha}_{-\infty}Bx}(k) := \hat{v}(k) = (ik)^{\alpha}B\hat{x}(k) \tag{4.2}$$

Using now (4.1) and (4.2) we have:

$$\widehat{\mathscr{D}^{\alpha}_{-\infty}Bx}(k) = (ik)^{\alpha}B\hat{x}(k) = A\hat{x}(k) + \widehat{L(x_{.})}(k) + \widehat{a*x}(k) + \widehat{\mathscr{I}^{\beta}_{-\infty}x}(k) + \hat{f}(k)$$

for all $k \in \mathbb{Z}$. Since A is closed, then $x(t) \in D(A)$ [Lemma 4.2]
and from the uniqueness theorem of Fourier coefficients, that Eq. (3.1) is valid . □

Conflict of Interests

The author(s) declare that there is no conflict of interests.

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