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CUBIC IDEAL IN Γ-SEMIRINGS

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Abstract. In this paper we introduce the notion cubic ideal in Γ-semiring and we study basic properties of cubic ideal.

Keywords: cubic set; cubic ideal; fuzzy set; interval-value fuzzy set; Γ-semiring.

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1. Introduction

Zadeh initiated the concept of fuzzy sets in 1965. In 1975, Zadeh made an extension concept of a fuzzy set by an interval-valued fuzzy set. A semigroup is an algebraic structure consisting of a non-empty sets together with an associative binary operation.Semirings which is a common generalization of rings and distributive lattices, was introduced by Vandiver [8]. It has been found very useful for solving problems in different areas of pure and applied mathematics, information sciences, etc., since the structure of a semiring provides an algebraic framework for modelling and studying the key factors in these applied areas. Ideals of semiring play a central role in the structure theory and useful for many purposes. The theory of Γ-semirings was introduced by [2]. Since then many researchers enriched this field. Many authors have studied semigroups in terms of fuzzy sets. Kuroki is the main contributor of this study. Kuroki introduced

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the notion of fuzzy ideals and fuzzy bi-ideals in semigroups. Atanassov introduced intuitionistic fuzzy set is characterized by a membership function and a non-membership function for each element in the Universe. In 2010, K. Hur and H.W. Kang introduced interval-valued fuzzy subgroups and rings. Jun et al. introduced the new concept called cubic sets. These structures encompass interval-valued fuzzy set and fuzzy set. Also Jun et al. introduced the notion of cubic subgroups. Vijayabalaji et al. introduced the notion of cubic linear space. V. Chinnadurai et al. introduced cubic ring. The purpose of this paper to introduce the notion of cubic ideals of Γ-semigroups and we provide some results on it.

In this paper we studied properties of cubic ideals of Γ-semirings. Furthermore we can show that the images or inverse images of a cubic ideal of an Γ-semiring become a cubic ideal.

2. PRELIMINARIES

2.1. Γ-Semiring.

Now we review definition of some types Γ-semiring, which we use in the next section.

Let S and Γ be two additive commutative semigroups. We called S is an Γ-*semiring* if there exist mappings from $S \times \Gamma \times S$ ot S written as $(a, \alpha, b) \to a\alpha b$, and satisfying the following conditions:

(1) $a\alpha(b+c) = a\alpha b + a\alpha c$ and $(a+b)\alpha c = a\alpha c + b\alpha c$

(2) $a(\alpha+\beta)b = a\alpha b + a\beta b$

(3) $(a\alpha b\beta)c = a\alpha(b\beta)c$ for all $a, b, c \in S$ and $\alpha, \beta \in \Gamma$.

An Γ-semiring S is said to be *regular* if for each element $a \in S$, there exists an element $x \in S$ and $\alpha, \beta \in \Gamma$ such that $a = a\alpha x\beta a$. An Γ-semigroup S is called *intra-regular* if for every $a \in S$ there exist $x, y \in S$ and $\alpha, \beta, \gamma \in \Gamma$ such that $a = x\alpha a\beta a\gamma y$. A non-empty subset A of an Γ-semiring S is an Γ-subsemiring of S if A is a subsemigroup of $(S, +)$ and $A\Gamma A \subseteq A$. A non-empty subset A of an Γ-semiring S is called a *left*(right)ideal of S if A is a additive subsemigroup of S and $S\Gamma A \subseteq A (A\Gamma S \subseteq A)$. An ideal of S is a non-empty subset which is both a left ideal and a right ideal of S. [2]

2.2. Fuzzy Γ-semiring and interval valued fuzzy set. Now we will give definition of a fuzzy subset and types of fuzzy Γ-subsemigroups. Let X be a non-empty set. A mapping $\omega : X \to [0,1]$ is a *fuzzy subset* of S.

Definition 2.1. [5] Let S be an Γ-semiring. A fuzzy subset ω of S is said to be a *fuzzy* Γ-*subsemiring* of S if $\omega(x+y) \geq \min\{\omega(x), \omega(y)\}$ and $\omega(x\alpha y) \geq \min\{\omega(x), \omega(y)\}$, for all $x, y \in S$ and $\alpha \in \Gamma$.

Definition 2.2. [5] Let S be a Γ-semiring. A fuzzy subset ω of S is said to be a *fuzzy left* (right) ideal of S if $\omega(x+y) \geq \min\{\omega(x), \omega(y)\}$ and $\omega(x\alpha y) \geq \omega(y)(\omega(x\alpha y) \geq \omega(x))$, for all $x, y \in S$ and $\alpha \in \Gamma$. A non-empty fuzzy subset of an Γ-semigroup S is a *fuzzy ideal* of S if it is a fuzzy left ideal and fuzzy right ideal of S.

Definition 2.3. For a family $\{\omega_i \mid i \in I\}$ of fuzzy sets in X, we define the join $(\vee)$ and meet $(\wedge)$ operations as follows:

$$\left(\bigvee_{i\in I} \omega_i\right)(x) = \sup\{\omega_i(x) \mid i \in I\} \quad \text{and} \quad \left(\bigwedge_{i\in I} \omega_i\right)(x) = \inf\{\omega_i(x) \mid i \in I\}$$

respectively, for all $x \in X$.

Now we will introduce a new relation of an interval.

Definition 2.4. An interval number on $[0,1]$, say $\overline{a}$ is a closed subinterval of $[0,1]$, that is $\overline{a} = [a^-, a^+]$, where $0 \leq a^- \leq a^+ \leq 1$. Let $D[0,1]$ denoted the family of all closed subinterval of $[0,1]$, i.e.,

$$D[0,1] = \{\overline{a} = [a^-, a^+] \mid 0 \leq a^- \leq a^+ \leq 1\}.$$

The interval $[a,a]$ is identified with the number $a \in [0.1]$.

Definition 2.5. Let $\overline{a_i} = [a_i^-, a_i^+] \in D[0,1]$ for all $i \in I$ where I is an index set. We define

$$r\inf_{i\in I} \overline{a_i} = \left[\inf_{i\in I} a_i^-, \inf_{i\in I} a_i^+\right] \quad \text{and} \quad r\sup_{i\in I} \overline{a_i} = \left[\sup_{i\in I} a_i^-, \sup_{i\in I} a_i^+\right].$$

We define the operations "$\succeq$", "$\preceq$", "$=$", "$r\min$" "$r\max$" in case of two element in $D[0,1]$. We consider two interval numbers $\overline{a} := [a^-, a^+]$ and $\overline{b} := [b^-, b^+]$ in $D[0,1]$. Then

(1) $\overline{a} \succeq \overline{b}$ if and only if $a^- \geq b^-$ and $a^+ \geq b^+$

(2) $\overline{a} \preceq \overline{b}$ if and only if $a^- \leq b^-$ and $a^+ \leq b^+$

(3) $\overline{a} = \overline{b}$ if and only if $a^- = b^-$ and $a^+ = b^+$

(4) $r\min\{\overline{a},\overline{b}\} = [\min\{a^-,b^-\},\min\{a^+,b^+\}]$

(5) $r\max\{\overline{a},\overline{b}\} = [\max\{a^-,b^-\},\max\{a^+,b^+\}]$.

Definition 2.6. Let X be a set. An interval valued fuzzy set A on X is defined as

$$A = \{(x,[\mu^-(x),\mu^+(x)] \mid x \in X\},$$

where μ^- and μ^+ are two fuzzy sets of X such that $\mu^-(x) \leq \mu^+(x)$ for all $x \in X$. Putting $\overline{\mu}(x) = [\mu^-(x),\mu^+(x)]$, we see that $A = \{x,\overline{\mu}(x) \mid x \in X\}$, where $\overline{\mu} : X \to D[0,1]$.

Definition 2.7. For a family $\{\overline{\mu}_i \mid i \in I\}$ of interval valued fuzzy sets in X, we define the $(\sqcup_{i\in I}\overline{\mu}_i)$ and $(\sqcap_{i\in I}\mu_i)$ are defined as follows:

$$(\sqcup_{i\in I}\overline{\mu}_i)(x) = r\sup\overline{\mu}_i(x) \quad \text{and} \quad (\sqcap_{i\in I}\overline{\mu}_i)(x) = r\inf\overline{\mu}_i(x)$$

respectively, for all $x \in X$ where $\overline{\mu} : X \to D[0,1]$.

2.3. Cubic Γ-semiring.

Definition 2.8. Let X be a non-empty set. A *cubic set* $\mathscr{A}$ in X is a structure of the form

$$\mathscr{A} = \{\langle x,\overline{\mu}(x),\omega(x)\rangle : x \in X\}$$

and denoted by $\mathscr{A} = \langle\overline{\mu},\omega\rangle$ where $\overline{\mu}$ is an interval valued fuzzy set (briefly. IVF) in X and ω is a fuzzy set in X. In this case we will use

$$\mathscr{A}(x) = \langle\overline{\mu}(x),\omega(x)\rangle = \langle[\mu^-(x),\mu^+(x)],\omega(x)\rangle$$

For all $x \in X$. Note that a cubic set is a generalization of an intuitionistic fuzzy set.

Definition 2.9. Let A be a subset of a non-empty set X. Then cubic set characteristic function $\chi_A = \langle\overline{\mu}_{\chi_A},\omega_{\chi_A}\rangle$ of is defined as

$$\overline{\mu}_{\chi_A}(x) = \begin{cases} [1,1], & \text{if} \quad x \in A, \\ [0,0], & \text{if} \quad x \notin A. \end{cases} \quad \text{and} \quad \omega_{\chi_A}(x) = \begin{cases} 0, & \text{if} \quad x \in A, \\ 1, & \text{if} \quad x \notin A. \end{cases}$$

Definition 2.10. Let $\mathscr{A} = \langle \overline{\mu}, \omega \rangle$ is cubic set in X. For any $k \in [0,1]$ and $[s,t] \in D[0,1]$, we define $U(A,[s,t],k)$ as follows:

$$U(\mathscr{A},[s,t],k) = \{x \in X \mid \overline{\mu}(x) \succeq [s,t], \omega(x) \leq k\},$$

and we say it is a cubic level set of $\mathscr{A} = \langle \overline{\mu}, \omega \rangle$.

Definition 2.11. For two cubic set $\mathscr{A} = \langle \overline{\mu}_1, \omega_1 \rangle$ and $\mathscr{B} = \langle \overline{\mu}_2, \omega_2 \rangle$ in an Γ-semiring S, we define

$$\mathscr{A} \sqsubseteq \mathscr{B} \Leftrightarrow \overline{\mu}_1 \preceq \overline{\mu}_2 \quad \text{and} \quad \omega_1 \geq \omega_2$$

Definition 2.12. Let $\mathscr{A} = \langle \overline{\mu}_1, \omega_1 \rangle$ and $\mathscr{B} = \langle \overline{\mu}_2, \omega_2 \rangle$ be two cubic set in an Γ-semiring S. Then

$$\mathscr{A} \circledcirc \mathscr{B} = \{\langle x, (\overline{\mu}_1 \tilde{\circ} \overline{\mu}_2)(x), (\omega_1 \cdot \omega_2)(x) \rangle : x \in S\}$$

which is briefly denoted by $\mathscr{A} \circledcirc \mathscr{B} = \langle (\overline{\mu}_1 \tilde{\circ} \overline{\mu}_2), (\omega_1 \cdot \omega_2) \rangle$ where $\overline{\mu}_1 \tilde{\circ} \overline{\mu}_2$ and $\omega_1 \cdot \omega_2$ are defined as follows, respectively:

$$(\overline{\mu}_1 \tilde{\circ} \overline{\mu}_2)(x) = \begin{cases} r\sup_{x=y\beta z} \{r\min\{\overline{\mu}_1(y), \overline{\mu}_2(z)\}\} & \text{if} \quad x = y\beta z, \\ [0,0], & \text{otherwise} \end{cases}$$

and

$$(\omega_1 \cdot \omega_2)(x) = \begin{cases} \inf\limits_{x=y\beta z} \max\{\omega_1(y), \omega_2(z)\} & \text{if} \quad x = y\beta z, \\ 1, & \text{otherwise} \end{cases}$$

for all $x, y, z \in S$ and $\beta \in \Gamma$. And

$$\mathscr{A} \circledast \mathscr{B} = \{\langle x, (\overline{\mu}_1 \tilde{*} \overline{\mu}_2)(x), (\omega_1 * \omega_2)(x) \rangle : x \in S\}$$

which is briefly denoted by $\mathscr{A} \circledcirc \mathscr{B} = \langle (\overline{\mu}_1 \tilde{*} \overline{\mu}_2), (\omega_1 * \omega_2) \rangle$ where $\overline{\mu}_1 \tilde{*} \overline{\mu}_2$ and $\omega_1 * \omega_2$ are defined as follows, respectively:

$$(\overline{\mu}_1 \tilde{*} \overline{\mu}_2)(x) = \begin{cases} r\sup_{x=y\beta z} \{r\min\{\overline{\mu}_1(y), \overline{\mu}_2(z)\}\} & \text{if} \quad x = y\beta z, \\ [0,0], & \text{otherwise} \end{cases}$$

and

$$(\omega_1 * \omega_2)(x) = \begin{cases} \inf\limits_{x=y\beta z} \max\{\omega_1(y), \omega_2(z)\} & \text{if} \quad x = y\beta z, \\ 1, & \text{otherwise} \end{cases}$$

for all $x, y, z \in S$ and $\beta \in \Gamma$.

Definition 2.13. Let $\mathscr{A} = \langle \overline{\mu}_A, f_A \rangle$ and $\mathscr{B} = \langle \overline{\mu}_B, f_B \rangle$ be two cubic set in a semiring S. Then the intersection of $\mathscr{A}$ and $\mathscr{B}$ denoted by $\mathscr{A} \sqcap \mathscr{B}$ is the cubic set

$$\mathscr{A} \sqcap \mathscr{B} = \langle \overline{\mu}_A \sqcap \overline{\mu}_B, f_A \vee f_B \rangle$$

where $(\overline{\mu}_A \sqcap \overline{\mu}_B)(x) = r\min\{\overline{\mu}_A(x), \overline{\mu}_B(x)\}$ and $(f_A \vee f_B)(x) = \max\{f_A(x), f_B(x)\}$ for all $x \in S$.

3. Cubic Ideal in Γ-Semiring.

Definition 3.1. A cubic set $\mathscr{A} = \langle \overline{\mu}, \omega \rangle$ in an Γ-semiring S is called a *cubic* Γ-*subsemiring* of S if it satisfies:

(1) $\overline{\mu}(x+y) \succeq r\min\{\mu(x), \mu(y)\}$ and $\omega(x+y) \leq \max\{\omega(x), \omega(y)\}$
(2) $\overline{\mu}(x\alpha y) \succeq r\min\{\overline{\mu}(x), \overline{\mu}(y)\}$ and $\omega(x\alpha y) \leq \max\{\omega(x), \omega(y)\}$,
for all $x, y \in S$ and $\alpha \in \Gamma$.

Definition 3.2. A cubic set $\mathscr{A} = \langle \overline{\mu}, \omega \rangle$ in an Γ-semiring S is called a *cubic left* (right) ideal of S if it satisfies:

(1) $\overline{\mu}(x+y) \succeq r\min\{\mu(x), \mu(y)\}$ and $\omega(x+y) \leq \max\{\omega(x), \omega(y)\}$
(2) $\overline{\mu}(x\alpha y) \succeq \overline{\mu}(y)$ ($\overline{\mu}(x\alpha y) \succeq \overline{\mu}(x)$) and $\omega(x\alpha y) \leq \omega(y)$, ($\omega(x\alpha y) \leq \omega(x)$)
for all $x, y \in S$ and $\alpha \in \Gamma$.

A non-empty cubic set $\mathscr{A} = \langle \overline{\mu}, \omega \rangle$ of S is called *cubic ideal* of S if it is a cubic left ideal and a cubic right ideal of S.

The following theorems we will study basic properties of a cubic ideal.

Theorem 3.1. Let $\mathscr{A} = \langle \overline{\mu}, \omega \rangle$ be cubic left (right) ideal and $\mathscr{B} = \langle \overline{\mu}_1, \omega_1 \rangle$ be cubic left (right) ideal of Γ-semiring S. Then $\mathscr{A} \sqcap \mathscr{B}$ is a cubic left (right) ideal of Γ-semiring S.

Proof. Let $x,y\in S$, $\gamma\in\Gamma$. Then

$$\begin{aligned}(\overline{\mu}\sqcap\overline{\mu}_1)(x+y) &= r\min\{\overline{\mu}(x+y),\overline{\mu}_1(x+y)\}\\ &\succeq r\min\{r\min\{\overline{\mu}(x),\overline{\mu}(y),r\min\{\overline{\mu}_1(x),\overline{\mu}_1(y)\}\}\\ &= r\min\{r\min\{\overline{\mu}(x),\overline{\mu}(y),\overline{\mu}_1(x),\overline{\mu}_1(y)\}\}\\ &= r\min\{r\min\{\overline{\mu}(x),\overline{\mu}_1(x),\overline{\mu}(y),\overline{\mu}_1(y)\}\}\\ &= r\min\{\overline{\mu}\sqcap\overline{\mu}_1(x),\overline{\mu}\sqcap\overline{\mu}_1(y)\}\end{aligned}$$

and

$$\begin{aligned}(\omega\vee\omega_1)(x+y) &= \max\{\omega(x+y),\omega_1(x+y)\}\\ &\leq \max\{\max\{\omega(x),\omega(y)\},\max\{\omega_1(x),\omega_1(y)\}\}\\ &= \max\{\max\{\omega(x),\omega(y),\omega_1(x),\omega_1(y)\}\}\\ &= \max\{\max\{\omega(x),\omega_1(x),\omega(y),\omega_1(y)\}\}\\ &= \max\{\omega\vee\omega_1(x),\omega\sqcap\omega_1(y)\}\end{aligned}$$

Thus the condition of (1) in Defintion 3.2 is true. Consider

$$(\overline{\mu}\sqcap\overline{\mu}_1)(x\alpha y)=r\min\{\overline{\mu}(x\alpha y),\overline{\mu}_1(x\alpha y)\}\succeq r\min\{\overline{\mu}(y),\overline{\mu}_1(y)\}=(\overline{\mu}\sqcap\overline{\mu}_1)(y).$$

and

$$(\omega\cup\omega_1)(x\alpha y)=\max\{\omega(x\alpha y),\omega_1(x\alpha y)\}\leq\max\{\omega(y),\omega_1(y)\}=(\omega\cup\omega_1)(y).$$

Thus, $\mathscr{A}\sqcap\mathscr{B}$ is a cubic left ideal of S. □

Theorem 3.2. The intersection of any family of cubic left (right) ideals of Γ-semiring S is a cubic left (right) ideal of Γ-semiring S.

Proof. Let $\{\mathscr{A}_i\}_{i\in I}$ be a family of cubic left ideal of S and $x,y\in S$, $\alpha\in\Gamma$. Then

$$(\sqcap_{i\in I}\overline{\mu}_i)(x+y)=\inf\{\overline{\mu}_i(x+y)\}\succeq r\inf\{\overline{\mu}_i(x)\curlywedge\overline{\mu}_i(y)\}$$

and

$$(\cup_{i\in I}\omega_i)(x+y)=\sup\{\omega_i(x+y)\}\leq\sup\{\omega_i(x)\vee\omega_i(y)\}.$$

Thus the condition of (1) in Defintion 3.2 is true. Consider

$$(\sqcap_{i\in I}\overline{\mu}_i)(x\alpha y)=r\inf\{\overline{\mu}_i(x\alpha y)\}\succeq r\inf\{\overline{\mu}_i(y)\}=\overline{\mu}(y)$$

and

$$(\cup_{i\in I}\omega_i)(x\alpha y) = \sup\{\omega_i(x\alpha y)\} \leq \sup\{\omega_i(y)\} = \omega(y).$$

Thus, $\overline{\sqcap}_{i\in I}\mathscr{A}_i$ is a cubic left ideal of S. □

Theorem 3.3. Let S be an Γ-semiring and let A be non-empty subset of S. Then A is a left (right) ideal of S if and only if the characteristic cubic set $\chi_A = \langle \overline{\mu}_{\chi_A}, \omega_{\chi_A}\rangle$ is a cubic left (right) ideal of S.

Proof. Suppose that A is a left ideal of S and let $x, y \in S$ with $y \in A$ and $\alpha \in \Gamma$. Then $x+y \in A$ and $x\alpha y \in A$. Thus $\overline{\mu}_{\chi_A}(x+y) = \overline{\mu}_{\chi_A}(x\alpha y) = [1,1]$ and $\omega_{\chi_A}(x+y) = \omega_{\chi_A}(x\alpha y) = 0$. So $\overline{\mu}_{\chi_A}(x+y) \succeq r\min\{\overline{\mu}_{\chi_A}(x), \overline{\mu}_{\chi_A}(y)\}$ and $\omega_{\chi_A}(x+y) \leq \max\{\omega_{\chi_A}(x)\omega_{\chi_A}(y)\}$. Thus the condition of (1) in Defintion 3.2 is true. And $\overline{\mu}_{\chi_A}(x\alpha y) \succeq \overline{\mu}_{\chi_A}(y)$, $\omega_{\chi_A}(x\alpha y) \leq \omega_{\chi_A}(y)$.

If $y \notin A$, then $\overline{\mu}_{\chi_A}(y) = [0,0]$ and $\omega_{\chi_A}(y) = 1$. Since A is a left ideal of S we have $x+y \in A$ and $x\alpha y \in A$. Thus $\overline{\mu}_{\chi_A}(x+y) = \overline{\mu}_{\chi_A}(x\alpha y) = [1,1]$ and $\omega_{\chi_A}(x+y) = \omega_{\chi_A}(x\alpha y) = 0$. So $\overline{\mu}_{\chi_A}(x+y) \succeq r\min\{\overline{\mu}_{\chi_A}(x), \overline{\mu}_{\chi_A}(y)\}$, $\omega_{\chi_A}(x+y) \leq \max\{\omega_{\chi_A}(x), \omega_{\chi_A}(y)\}$ and $\overline{\mu}_{\chi_A}(x\alpha y) \succeq \overline{\mu}_{\chi_A}(y)$, $\omega_{\chi_A}(x\alpha y) \leq \omega_{\chi_A}(y)$. Thus $\chi_A = \langle \overline{\mu}_{\chi_A}, \omega_{\chi_A}\rangle$ is a cubic left ideal of S.

Conversely, suppose that $\chi_A = \langle \overline{\mu}_{\chi_A}, \omega_{\chi_A}\rangle$ is a cubic left ideal of S.
We will show that A is a left ideal of S. Let $x, y \in S$ with $y \in A$. Since $\chi_A = \langle \overline{\mu}_{\chi_A}, \omega_{\chi_A}\rangle$ is a cubic left ideal of S we have

$$(1) \quad \left\{ \overline{\mu}_{\chi_A}(x+y) \succeq r\min\{\overline{\mu}_{\chi_A}(x), \overline{\mu}_{\chi_A}(y)\} \text{ and } \omega_{\chi_A}(x\alpha y) \leq \max\{\omega_{\chi_A}(x), \omega_{\chi_A}(y)\} \right.$$

$$(2) \quad \left\{ \overline{\mu}_{\chi_A}(x\alpha y) \succeq \overline{\mu}_{\chi_A}(y) \text{ and } \omega_{\chi_A}(x\alpha y) \leq \omega_{\chi_A}(y). \right.$$

If $x+y \notin A$ and $x\alpha y \notin A$, then by (1) $\overline{\mu}_{\chi_A}(x+y) = r\min\{\overline{\mu}_{\chi_A}(x), \overline{\mu}_{\chi_A}(y)\}$ and $\omega_{\chi_A}(x\alpha y) = \max\{\omega_{\chi_A}(x), \omega_{\chi_A}(y)$. By (2), $\overline{\mu}_{\chi_A}(x\alpha y) \prec \overline{\mu}_{\chi_A}(y)$ and $\omega_{\chi_A}(x\alpha y) > \omega_{\chi_A}(y)$. It is a contradiction. Hence $x\alpha y \in A$. Therefore A is a left ideal of S. □

Theorem 3.4. Let S be an Γ-semiring. Then $\mathscr{A} = \langle \overline{\mu}, \omega\rangle$ is a cubic left (right) ideal of S if and only if the level set $U(\mathscr{A}, [s,t], k)$ is a left (right) ideal of S.

Proof. Suppose that $\mathscr{A} = \langle\overline{\mu},\omega\rangle$ is a cubic left ideal of S. Then $\overline{\mu}(x+y) \succeq r\min\{\overline{\mu}(x),\overline{\mu}(y)\} \succeq [s,t]$ and $\omega(x+y) \leq \max\{\omega(x),\omega(y)\} \leq k$. Thus $x+y \in U(\mathscr{A},[s,t],k)$ for all $x,y \in U(\mathscr{A},[s,t],k)$. And $\overline{\mu}(x\alpha y) \succeq \overline{\mu}(y)$ and $\omega(x\alpha y) \leq \omega(y)$. Let $x \in S$ with $y \in U(\mathscr{A},[s,t],k)$ and $\alpha \in \Gamma$. Then $\overline{\mu}(y) \succeq [s,t]$ and $\omega(y) \leq k$. Thus

$$\overline{\mu}(x\alpha y) \succeq \overline{\mu}(y) \succeq [s,t]$$

and

$$\omega(x\alpha y) \leq \omega(y) \leq k.$$

So $x\alpha y \in U(\mathscr{A},[s,t],k)$. Hence $U(\mathscr{A},[s,t],k)$ is a left ideal of S.
Therefore $U(\mathscr{A},[s,t],k)$ is a cubic level set is a left ideal of S.

Conversely, suppose that $U(\mathscr{A},[s,t],k)$ is a cubic level set is a left ideal of S. Let $x,y \in S$ and $\alpha \in \Gamma$. By assumption, $x+y$ and $x\alpha y$ are elements of $U(\mathscr{A},[s,t],k)$. Then $\overline{\mu}(x+y) \succeq [s,t]$, $\overline{\mu}(x\alpha y) \succeq [s,t]$ and $\omega(x+y) \leq k$, $\omega(x\alpha y) \leq k$. Since $U(\mathscr{A},[s,t],k)$ is a cubic level set is a left ideal of S we have $x+y$ and $x\alpha y \in U(\mathscr{A},[s,t],k)$. Thus $r\min\{\overline{\mu}(x),\overline{\mu}(y)\} \succeq [s,t]$ and $\max\{\omega(x),\omega(y)\} \leq k$, $\overline{\mu}(y) \succeq [s,t]$ and $\omega(y) \leq k$. So, $\overline{\mu}(x+y) \succeq r\min\{\overline{\mu}(x),\overline{\mu}(y)\} \succeq [s,t]$ and $\omega(x+y) \leq \max\{\omega(x),\omega(y)\} \leq k$, $\overline{\mu}(x\alpha y) \succeq \overline{\mu}(y) \succeq [s,t]$ and $\omega(x\alpha y) \leq \omega(y) \leq k$. Hence $\mathscr{A} = \langle\overline{\mu},\omega\rangle$ is a cubic left ideal of S. □

Lemma 3.5. Let $\mathscr{A}_1 = \langle\overline{\mu}_1,\omega_1\rangle$ and $\mathscr{A}_2 = \langle\overline{\mu}_2,\omega_2\rangle$ be a cubic right ideal and a cubic left ideal of an Γ-semiring S respectively. Then $\mathscr{A}_1 \circledcirc \mathscr{A}_2 \sqsubseteq \mathscr{A}_1 \sqcap \mathscr{A}_2$.

Proof. Let $x \in S$ and $\gamma \in \Gamma$. Suppose that x cannot be expressed as $x \neq p\gamma q$, then $(\overline{\mu}_1\tilde{\circ}\overline{\mu}_2)(x) = [0,0]$ and $(\omega_1 \cdot \omega_2)(x) = 1$. Thus $(\overline{\mu}_1\tilde{\circ}\overline{\mu}_2)(x) = [0,0] \preceq (\overline{\mu}_1 \sqcap \overline{\mu}_2)(x)$ and $(\omega_1 \cdot \omega_2)(x) = 1 \geq (\omega_1 \cup \omega_2)(x)$. Suppose that $x = p\gamma q$. Then,

$$\begin{aligned}
(\overline{\mu}_1\tilde{\circ}\overline{\mu}_2)(x) &= r\sup_{x=p\gamma q}\{r\min\{(\overline{\mu}_1)(p),\overline{\mu}_2(q)\}\} \\
&\preceq r\sup_{x=p\gamma q}\{r\min\{(\overline{\mu}_1)(p\gamma q),\overline{\mu}_2(p\gamma q)\}\} \\
&\preceq r\min\{(\overline{\mu}_1)(x),\overline{\mu}_2(x)\} \\
&= (\overline{\mu}_1 \sqcap \overline{\mu}_2)(x).
\end{aligned}$$

And

$$\begin{aligned}(\omega_1\cdot\omega_2)(x) &= \inf_{x=p\gamma q}\{\max\{(\omega_1)(p),\omega_2(q)\}\}\\ &\geq \inf_{x=p\gamma q}\{\max\{(\omega_1)(p\gamma q),\omega_2(p\gamma q)\}\}\\ &\geq \max\{(\omega_1)(x),\omega_2(x)\}\\ &= (\omega_1\cup\omega_2)(x).\end{aligned}$$

Thus, $(\overline{\mu}_1\tilde{\circ}\overline{\mu}_2)(x)\preceq(\overline{\mu}_1\sqcap\overline{\mu}_2)(x)$ and $(\omega_1\cdot\omega_2)(x)\geq(\omega_1\cup\omega_2)(x)$.

Hence $\mathscr{A}_1\circledcirc\mathscr{A}_2\sqsubseteq\mathscr{A}_1\sqcap\mathscr{A}_2$. □

Lemma 3.6. Let S be a regular Γ-semiring and let $\mathscr{A}_1=\langle\overline{\mu}_1,\omega_1\rangle$ and $\mathscr{A}_2=\langle\overline{\mu}_2,\omega_2\rangle$ be a cubic right ideal and a cubic left ideal of an Γ-semiring S respectively. Then $\mathscr{A}_1\circledcirc\mathscr{A}_2\sqsupseteq\mathscr{A}_1\sqcap\mathscr{A}_2$.

Proof. Let $x\in S$ and $\gamma\in\Gamma$. Since S is regular, there exist $p\in S$ and $\alpha,\beta\in\Gamma$ such that $x=x\alpha p\beta x=x\gamma x$ where $\gamma=\alpha p\beta\in\Gamma$. Thus,

$$\begin{aligned}(\overline{\mu}_1\tilde{\circ}\overline{\mu}_2)(x) &= r\sup_{x=x\gamma x}\{r\min\{(\overline{\mu}_1)(x),\overline{\mu}_2(x)\}\}\\ &\succeq r\min\{(\overline{\mu}_1)(x),\overline{\mu}_2(x)\}\\ &= (\overline{\mu}_1\sqcap\overline{\mu}_2)(x).\end{aligned}$$

And

$$\begin{aligned}(\omega_1\cdot\omega_2)(x) &= \inf_{x=x\gamma x}\{\max\{(\omega_1)(x),\omega_2(x)\}\}\\ &\leq \max\{(\omega_1)(x),\omega_2(x)\}\\ &= (\omega_1\cup\omega_2)(x).\end{aligned}$$

Thus $(\overline{\mu}_1\tilde{\circ}\overline{\mu}_2)(x)\succeq(\overline{\mu}_1\sqcap\overline{\mu}_2)(x)$ and $(\omega_1\cdot\omega_2)(x)\leq(\omega_1\cup\omega_2)(x)$.

Hence $\mathscr{A}_1\circledcirc\mathscr{A}_2\sqsupseteq\mathscr{A}_1\sqcap\mathscr{A}_2$. □

Theorem 3.7. Let $\mathscr{A}_1=\langle\overline{\mu}_1,\omega_1\rangle$ and $\mathscr{A}_2=\langle\overline{\mu}_2,\omega_2\rangle$ be a cubic right ideal and a cubic left ideal an of regular Γ-semiring. respectively. Then $\mathscr{A}_1\circledcirc\mathscr{A}_2=\mathscr{A}_1\sqcap\mathscr{A}_2$.

Proof. It follows from Lemma 3.5 and 3.6. □

4. Characterizations of Regular and Intra-Regualr Γ-Semirings by Cubic Ideals

In this section, we study characterizations of regular and intra-regualr Γ-semirings by cubic ideals.

Lemma 4.1. Let A and B be non-empty subsets of a semigroup S and Q be any non-empty subset. Then the following statements hold

(1) $(\chi_A) \sqcap (\chi_B) = (\chi_{A\cap B})$.

(2) $(\chi_A \circ \chi_B) = (\chi_{AB})$.

Lemma 4.2. [3] In Γ-semiring S following statements are equivalent.

(1) S is regular

(2) For every left ideal L and a right ideal R of S, $R \cap L = RL$.

Theorem 4.3. Let $\mathscr{A}_1 = \langle \overline{\mu}_1, \omega_1 \rangle$ and $\mathscr{A}_2 = \langle \overline{\mu}_2, \omega_2 \rangle$ be a cubic right ideal and a cubic left ideal an of Γ-semiring. respectively. Then S is a regular semiring if and only if $\mathscr{A}_1 \circledcirc \mathscr{A}_2 = \mathscr{A}_1 \sqcap \mathscr{A}_2$.

Proof. Suppose that S is regular. Then Lemma 3.5 and 3.6. Thus $\mathscr{A}_1 \circledcirc \mathscr{A}_2 = \mathscr{A}_1 \sqcap \mathscr{A}_2$.

Conversely, let R and L be a right ideal and left ideal of S respectively. Then by Theorem 3.3, $\chi_R = \langle \overline{\mu}_{\chi_R}, \omega_{\chi_R} \rangle$ is a cubic right ideal and $\chi_L = \langle \overline{\mu}_{\chi_L}, \omega_{\chi_L} \rangle$ is a cubic left ideal of S respectively. By supposition and Lemma 4.1, we have

$$(\overline{\mu}_{\chi_{RL}})(u) = (\overline{\mu}_{\chi_R} \circ \overline{\mu}_{\chi_L})(u) = (\overline{\mu}_{\chi_R} \sqcap \overline{\mu}_{\chi_L})(u) = (\overline{\mu}_{\chi_{R\cap L}})(u) = [1,1]$$

and

$$(\omega_{\chi_{RL}})(u) = (\omega_{\chi_R} \circ \omega_{\chi_L})(u) = (\omega_{\chi_R} \vee \omega_{\chi_L})(u) = (\omega_{\chi_{R\cap L}})(u) = 0$$

Thus $u \in RL$, and so $RL = R \cap L$. It follows that by Lemma 4.2, S is a regular. □

Lemma 4.4. [3] In Γ-semiring S following statements are equivalent.

(1) S is intra-regular

(2) For every left ideal L and a right ideal R of S, $R \cap L \subseteq RL$.

Theorem 4.5. Let $\mathscr{A}_1 = \langle \overline{\mu}_1, \omega_1 \rangle$ and $\mathscr{A}_2 = \langle \overline{\mu}_2, \omega_2 \rangle$ be a cubic right ideal and a cubic left ideal an of Γ-semiring. respectively. Then S is an intra-regular semiring if and only if $\mathscr{A}_1 \sqcap \mathscr{A}_2 \sqsubseteq \mathscr{A}_1 \circledcirc \mathscr{A}_2$.

Proof. Suppose that S is intra-regular. Let $u \in S$. Then there exist $x, y \in S$ and $\alpha, \beta, \gamma \in \Gamma$ such that $u = x\alpha u\beta u\gamma y$. Thus,

$$\begin{aligned}(\overline{\mu}_1 \tilde{\circ} \overline{\mu}_2)(x) &= r\sup_{u=x\alpha u\beta u\gamma y}\{r\min\{(\overline{\mu}_1)(x), \overline{\mu}_2(x)\}\} \\ &\succeq r\min\{(\overline{\mu}_1)(x\alpha u), \overline{\mu}_2(u\gamma y)\} \succeq r\min\{(\overline{\mu}_1)(u), \overline{\mu}_2(u)\} \\ &= (\overline{\mu}_1 \sqcap \overline{\mu}_2)(u).\end{aligned}$$

And

$$\begin{aligned}(\omega_1 \cdot \omega_2)(x) &= \inf_{u=x\alpha u\beta u\gamma y}\{\max\{(\omega_1)(x), \omega_2(x)\}\} \\ &\leq \max\{(\omega_1)(x\alpha u), \omega_2(u\gamma y)\} \leq \max\{(\omega_1)(u), \omega_2(u)\} \\ &= (\omega_1 \cup \omega_2)(u).\end{aligned}$$

Thus $(\overline{\mu}_1 \tilde{\circ} \overline{\mu}_2)(u) \succeq (\overline{\mu}_1 \sqcap \overline{\mu}_2)(u)$ and $(\omega_1 \cdot \omega_2)(u) \leq (\omega_1 \cup \omega_2)(u)$.

Hence $\mathscr{A}_1 \sqcap \mathscr{A}_2 \sqsubseteq \mathscr{A}_1 \circledcirc \mathscr{A}_2$.

Conversely, let R and L be a right ideal and left ideal of S respectively. Then by Theorem 3.3, $\chi_R = \langle \overline{\mu}_{\chi_R}, \omega_{\chi_R} \rangle$ is a cubic rignt ideal and $\chi_L = \langle \overline{\mu}_{\chi_L}, \omega_{\chi_L} \rangle$ is a cubic left ideal of S respectively. By supposition and Lemma 4.1, we have

$$(\overline{\mu}_{\chi_{RL}})(u) = (\overline{\mu}_{\chi_R} \circ \overline{\mu}_{\chi_L})(u) \succeq (\overline{\mu}_{\chi_R} \sqcap \overline{\mu}_{\chi_L})(u) = (\overline{\mu}_{\chi_{R\cap L}})(u) = [1, 1]$$

and

$$(\omega_{\chi_{RL}})(u) = (\omega_{\chi_R} \circ \omega_{\chi_L})(u) \leq (\omega_{\chi_R} \vee \omega_{\chi_L})(u) = (\omega_{\chi_{R\cap L}})(u) = 0$$

Thus $u \in RL$, and so $R \cap L \subseteq RL$. It follows that by Lemma 4.2, S is a intra-regular. □

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CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

REFERENCES

[1] A. Ali, Y.B.Jun, M. Khan, F.G. Shi and S. Anis, Genralized cubic soft sets and their applications to algebraic structres. Italian J. Pure Appl. Math. 35 (2015), 393-414.

[2] H. Durna and D. Yilmaz, Generalized centroid of Γ-semirings. Proyeciones J. Math. 37 (2018), 805-817.

[3] R.D. Jagatap and Y.S. Pawar, Regular, intra-regular and duo Γ-semiring. Asia Pac. J. Math. 4 (2017), 62-74.

[4] V. Chinnadurai and K. Bharathivelan, Cubic ideals of PO-Γ-semigroups. Int. J. Appl. Adv. Sci. Res. 1 (2016), 127-135.

[5] M. Murali Krishna Rao, Fuzzy bi-interior ideal of Γ-semirings. J. Int. Math. Virt. Inst. (accepted).

[6] Y.B. Jun and A. Khan, Cubic ideal in semigroups, Honam Math. J. 35 (2013), 607-623.

[7] Y.B. Jun, C.S. Kim and K.O. Yang, Cubic sets. Ann. Fuzzy Math. Inform. 6 (2012), 83-98.

[8] L.A. Zadeh, Fuzzy sets. Inform. Control, 8 (1965), 338-353.