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MITTAG-LEFFLER-HYERS-ULAM STABILITY OF A LINEAR DIFFERENTIAL EQUATIONS OF SECOND ORDER USING LAPLACE TRANSFORM

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Abstract. In this paper, we investigate the Mittag-Leffler-Hyers-Ulam stability and Mittag-Leffler-Hyers-Ulam-Rassias stability of a homogeneous and non-homogeneous linear differential equations of second order by using Laplace transforms.

Keywords: Mittag-Leffler-Hyers-Ulam stability; Mittag-Leffler-Hyers-Ulam-Rassias stability; linear differential equations; Laplace transform..

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1. INTRODUCTION

A simulating and famous talk presented by Ulam [38] in 1940, motivated the study of stability problems for various functional equations. He gave a wide range of talk before a Mathematical

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Colloquium at the University of Wisconsin in which he presented a list of unsolved problems. One of his question was that when is it true that a mapping that approximately satisfies a functional equation must be close to an exact solution of the equation? If the answer is affirmative, we say that the functional equation for homomorphisms is stable. In 1941, Hyers [8] was the first Mathematician to present the result concerning the stability of functional equations. He brilliantly answered the question of Ulam, the problem for the case of approximately additive mappings on Banach spaces. In course of time, the Theorem formulated by Hyers was generalized by Rassias [34], Aoki [3] and Bourgin [4] for additive mappings (see also [32, 39]).

The generalization of Ulam's question has been relatively recently proposed by replacing functional equations with differential equations: Let I be a subinterval of \mathbb{R} , let \mathbb{K} denote either \mathbb{R} or \mathbb{C} , and let n be a positive integer. The differential equation $\psi(f, x, x', x'', \dots, x^{(n)}) = 0$ has the Hyers-Ulam stability if there exists a constant $K > 0$ such that the following statement is true for any $\varepsilon > 0$: If an n times continuously differentiable function $z : I \rightarrow \mathbb{K}$ satisfies the inequality $|\psi(f, z, z', z'', \dots, z^{(n)})| \leq \varepsilon$ for all $t \in I$, then there exists a solution $y : I \rightarrow \mathbb{K}$ of the differential equation that satisfies the inequality $|z(t) - y(t)| \leq K\varepsilon$ for all $t \in I$.

Obłozza seems to be the first author who has investigated the Hyers-Ulam stability of linear differential equations (see [28, 29]). Then, in 1998, Alsina and Ger continued the study of Obłozza's Hyers-Ulam stability of differential equations. Indeed, they proved in [2] the following theorem.

Theorem 1. *Let $I \neq \emptyset$ be an open subinterval of \mathbb{R} . If a differentiable function $x : I \rightarrow \mathbb{R}$ satisfies the differential inequality $\|x'(t) - x(t)\| \leq \varepsilon$ for any $t \in I$ and for some $\varepsilon > 0$, then there exists a differentiable function $y : I \rightarrow \mathbb{R}$ satisfying $y'(t) = y(t)$ and $\|x(t) - y(t)\| \leq 3\varepsilon$ for any $t \in I$.*

This result of Alsina and Ger has been generalized by Takahashi *et al.* They proved in [37] that the Hyers-Ulam stability holds true for the Banach space valued differential equation $x'(t) = \lambda x(t)$. Indeed, the Hyers-Ulam stability has been proved for the first-order linear differential equations in more general settings (see [9, 10, 11, 12, 17]).

In 2006, Jung [12] investigated the Hyers-Ulam stability of a system of first-order linear differential equations with constant coefficients by using matrix method. Then, in 2008, Wang *et al.* [40] studied the Hyers-Ulam stability of linear differential equations of first order using the

integral factor method. Meanwhile, Rus [36] discussed various types of Hyers-Ulam stability of the ordinary differential equation $x'(t) = Ax(t) + f(t, x(t))$. In 2014, Alqifiary and Jung [1] proved the generalized Hyers-Ulam stability of linear differential equation of the form

$$x^{(n)}(t) + \sum_{k=0}^{n-1} \alpha_k x^{(k)}(t) = f(t)$$

by using the Laplace transform method, where α_k are scalars and $x(t)$ is an n times continuously differentiable function and of the exponential order (see also [35]).

In recent years, many authors are studying the Hyers-Ulam stability of differential equations, and a number of mathematicians are paying attention to the new results of the Hyers-Ulam stability of differential equations (see [5, 6, 16, 19, 20, 24, 30, 31]).

Note that, in particular, during these days most of the mathematicians are studied only the Hyers-Ulam stability of the second order differential equations by various directions (See [7, 14, 15, 18, 21, 22]).

In recent days, few authors have investigated the Ulam stability of the linear differential equations using various integral transform techniques, like, Fourier transform, Mahgoub transform and Aboodh transform in [21, 25, 26, 27, 33].

Based on the above discussions, by applying the Laplace Transform Method, we study the Mittag-Leffler-Hyers-Ulam stability and Mittag-Leffler-Hyers-Ulam-Rassias stability of a general homogeneous and non-homogeneous linear differential equations of second order

$$(1) \quad u''(t) + l u'(t) + m u(t) = 0$$

and

$$(2) \quad u''(t) + l u'(t) + m u(t) = r(t)$$

for all $t \in I$, l, m are constants in \mathbb{F} , $u(t) \in C^2(I)$ and $r(t) \in C(I)$ where $I = [a, b] \subset \mathbb{R}$.

2. PRELIMINARIES

In this section, we introduce some notations, definitions and preliminaries which are used throughout this paper.

Throughout this paper, \mathbb{F} denotes the real field \mathbb{R} or the complex field \mathbb{C} . A function $f : (0, \infty) \rightarrow \mathbb{F}$ of exponential order if there exists a constants $M(> 0) \in \mathbb{R}$ such that $|f(t)| \leq Me^{at}$ for all $t > 0$. For each function $f : (0, \infty) \rightarrow \mathbb{F}$ of exponential order, we define the Laplace Transform of f by

$$F(s) = \int_0^{\infty} f(t) e^{-st} dt.$$

The Laplace transform of f is sometimes denoted by $\mathcal{L}(f)$. It is also well known that \mathcal{L} is linear and one-to-one. Then, at points of continuity of f , we have

$$f(t) = \frac{1}{2\pi i} \lim_{n \rightarrow \infty} \int_{\alpha - iT}^{\alpha + iT} F(s) e^{st} ds = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{\alpha + iy} F(\alpha + iy) dy,$$

this is called the inverse Laplace transforms.

Definition 2. (Convolution). Given two functions f and g , both Lebesgue integrable on $(-\infty, +\infty)$. Let S denote the set of x for which the Lebesgue integral

$$h(x) = \int_{-\infty}^{\infty} f(t) g(x-t) dt$$

exists. This integral defines a function h on S called the convolution of f and g . We also write $h = f * g$ to denote this function.

Theorem 3. The Laplace transform of the convolution of $f(x)$ and $g(x)$ is the product of the Laplace transform of $f(x)$ and $g(x)$. That is,

$$\mathcal{L}\{f(x) * g(x)\} = \mathcal{L}\{f(x)\} \mathcal{L}\{g(x)\} = F(s) G(s)$$

or

$$\mathcal{L} \left\{ \int_0^{\infty} f(t) g(x-t) dt \right\} = \mathcal{L}(f) \mathcal{L}(g) = F(s) G(s),$$

where $F(s)$ and $G(s)$ are Laplace transform of $f(x)$ and $g(x)$ respectively.

Definition 4. [13] The Mittag-Leffler function of one parameter is denoted by $E_{\alpha}(z)$ and defined as

$$E_{\alpha}(z) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(\alpha k + 1)} z^k$$

where $z, \alpha \in \mathbb{C}$ and $\text{Re}(\alpha) > 0$.

Definition 5. [13] *The generalization of $E_\alpha(z)$ is defined as a function*

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(\alpha k + \beta)} z^k$$

where $z, \alpha, \beta \in \mathbb{C}$, $\operatorname{Re}(\alpha) > 0$ and $\operatorname{Re}(\beta) > 0$.

Let $I, J \subseteq \mathbb{R}$. Throughout this paper, we denote the space of k continuously differentiable functions from I to J by $C^k(I, J)$ and denote $C^k(I, I)$ by $C^k(I)$. Furthermore, $C(I, J) = C^0(I, J)$ denotes the space of continuous functions from I to J . In addition, $\mathbb{R}_+ := [0, \infty)$. From now on, we assume that $I = [a, b]$, where $-\infty < a < b < \infty$.

We firstly give some definitions of various forms of Mittag-Leffler-Hyers-Ulam stability of the second order differential equations (1) and (2).

Definition 6. *We say that the differential equation (1) has the Mittag-Leffler-Hyers-Ulam stability, if there exists a positive constant K satisfies the following conditions: For every $\varepsilon > 0$ and there exists $u(t) \in C^2(I)$ satisfying the inequality*

$$|u''(t) + l u'(t) + m u(t)| \leq \varepsilon E_\alpha(t),$$

for all $t \in I$. Then there exists a solution $v \in C^2(I)$ satisfying $v''(t) + l v'(t) + m v(t) = 0$ such that $|u(t) - v(t)| \leq K \varepsilon E_\alpha(t)$, for all $t \in I$. We call such K as the Mittag-Leffler-Hyers-Ulam stability constant for (1).

Definition 7. *We say that the differential equation (2) has the Mittag-Leffler-Hyers-Ulam stability, if there exists a positive constant K satisfies the following conditions: For every $\varepsilon > 0$ and there exists $u(t) \in C^2(I)$ satisfying the inequality*

$$|u''(t) + l u'(t) + m u(t) - r(t)| \leq \varepsilon E_\alpha(t),$$

for all $t \in I$. Then there exists a solution $v \in C^2(I)$ satisfying the linear differential equation $v''(t) + l v'(t) + m v(t) = r(t)$ such that $|u(t) - v(t)| \leq K \varepsilon E_\alpha(t)$, for all $t \in I$. We call such K as the Mittag-Leffler-Hyers-Ulam stability constant for (2).

Definition 8. *We say that the differential equation (1) has the Mittag-Leffler-Hyers-Ulam-Rassias stability with respect to $\phi : (0, \infty) \rightarrow (0, \infty)$, if there exists a positive constant K satisfies*

the following conditions: For every $\varepsilon > 0$ and there exists $u(t) \in C^2(I)$ satisfying the inequality

$$|u''(t) + l u'(t) + m u(t)| \leq \phi(t) \varepsilon E_\alpha(t),$$

for all $t \in I$. Then there exists a solution $v \in C^2(I)$ satisfies $v''(t) + l v'(t) + m v(t)$ such that

$$|u(t) - v(t)| \leq K \phi(t) \varepsilon E_\alpha(t),$$

for all $t \in I$. We call such K as the Mittag-Leffler-Hyers-Ulam-Rassias stability constant for (1).

Definition 9. We say that the differential equation (2) has the Mittag-Leffler-Hyers-Ulam-Rassias stability with respect to $\phi : (0, \infty) \rightarrow (0, \infty)$, if there exists a positive constant K satisfies the following conditions: For every $\varepsilon > 0$ and there exists $u(t) \in C^2(I)$ satisfying the inequality

$$|u''(t) + l u'(t) + m u(t) - r(t)| \leq \phi(t) \varepsilon E_\alpha(t),$$

for all $t \in I$. Then there exists a solution $v \in C^2(I)$ satisfies the linear differential equation $v''(t) + l v'(t) + m v(t) = r(t)$ such that $|u(t) - v(t)| \leq K \phi(t) \varepsilon E_\alpha(t)$, for all $t \in I$. We call such K as the Mittag-Leffler-Hyers-Ulam-Rassias stability constant for (2).

3. MITTAG-LEFFLER-HYERS-ULAM STABILITY FOR HOMOGENEOUS DIFFERENTIAL EQUATION (1)

In this section, we prove the Mittag-Leffler-Hyers-Ulam stability and Mittag-Leffler-Hyers-Ulam-Rassias stability of the second order homogeneous differential equation (1) by using the Laplace transform.

Theorem 10. The differential equation (1) is Mittag-Leffler-Hyers-Ulam stable.

Proof. Given $\varepsilon > 0$. Suppose that $u(t) \in C^2(I)$ satisfying the inequality

$$(3) \quad |u''(t) + l u'(t) + m u(t)| \leq \varepsilon E_\alpha(t),$$

for all $t \in I$. We wish to prove that there exists real number $K > 0$ which is independent of ε and u such that $|u(t) - v(t)| \leq K \varepsilon E_\alpha(t)$, for some $v \in C^2(I)$ satisfies the differential equation $v''(t) + l v'(t) + m v(t) = 0$ for all $t \in I$. Define a function $p : (0, \infty) \rightarrow \mathbb{R}$ such that

$$p(t) =: u''(t) + l u'(t) + m u(t)$$

for all $t > 0$. In view of (3), we have $|p(t)| \leq \varepsilon E_\alpha(t)$. Taking Laplace transform to $p(t)$, then

$$(4) \quad \mathcal{L}\{p\} = (s^2 + l s + m)\mathcal{L}\{u\} - [s u(0) + l u(0) + u'(0)],$$

and thus

$$(5) \quad \mathcal{L}\{u\} = \frac{\mathcal{L}\{p\} + s u(0) + l u(0) + u'(0)}{s^2 + l s + m}.$$

In view of the (4), a function $u_0 : (0, \infty) \rightarrow \mathbb{R}$ is a solution of (1) if and only if

$$(s^2 + l s + m)\mathcal{L}\{u_0\} - [s u_0(0) + l u_0(0) + u_0'(0)] = 0.$$

Let l and m are constants in \mathbb{F} such that there exists μ and ν are in \mathbb{F} with $\mu + \nu = -l$, $\mu \nu = m$ and $\mu \neq \nu$. Then we have $(s^2 + l s + m) = (s - \mu)(s - \nu)$, (5) implies that

$$(6) \quad \mathcal{L}\{u\} = \frac{\mathcal{L}\{p\} + s u(0) + l u(0) + u'(0)}{(s - \mu)(s - \nu)}.$$

If we set

$$v(t) = u(0) \left(\frac{\mu e^{\mu t} - \nu e^{\nu t}}{\mu - \nu} \right) + [l u(0) + u'(0)] \left(\frac{e^{\mu t} - e^{\nu t}}{\mu - \nu} \right)$$

then one can easily have $v(0) = u(0)$ and $v'(0) = u'(0)$. Taking Laplace transform to $v(t)$, then

$$(7) \quad \mathcal{L}\{v\} = \frac{s u(0) + l u(0) + u'(0)}{(s - \mu)(s - \nu)}.$$

On the other hand,

$$\mathcal{L}\{v''(t) + l v'(t) + m v(t)\} = (s^2 + l s + m)\mathcal{L}\{v\} - [s v(0) + l v(0) + v'(0)].$$

Using (7), we get $\mathcal{L}\{v''(t) + l v'(t) + m v(t)\} = 0$. Since \mathcal{L} is one-to-one operator and linear, then we get $v''(t) + l v'(t) + m v(t) = 0$. This means that $v(t)$ is a solution of (1). It follows from (5) and (7) that

$$\begin{aligned} \mathcal{L}\{u\} - \mathcal{L}\{v\} &= \frac{\mathcal{L}\{p\}}{(s - \mu)(s - \nu)} \\ \mathcal{L}\{u(t) - v(t)\} &= \mathcal{L}\{p(t) * q(t)\}. \end{aligned}$$

where

$$q(t) = \frac{e^{\mu t} - e^{\nu t}}{\mu - \nu}.$$

The above equalities show that

$$u(t) - v(t) = p(t) * q(t).$$

Taking modulus on both sides and using $|p(t)| \leq \varepsilon E_\alpha(t)$, we get

$$\begin{aligned} |u(t) - v(t)| &= |p(t) * q(t)| \leq \left| \int_0^t p(t) q(t-x) dx \right| \\ &\leq |p(t)| \left| \int_0^t q(t-x) dx \right| \leq \varepsilon E_\alpha(t) \left| \int_0^t q(t-x) dx \right| = K \varepsilon E_\alpha(t) \end{aligned}$$

for all $t > 0$, where

$$\begin{aligned} K &= \left| \int_0^t q(t-x) dx \right| = \left| \int_0^t \left(\frac{e^{\mu(t-x)} - e^{\nu(t-x)}}{\mu - \nu} \right) dx \right| \\ &\leq \frac{1}{|\mu - \nu|} \left\{ e^{\Re(\mu)t} \int_0^t e^{-\Re(\mu)x} dx + e^{\Re(\nu)t} \int_0^t e^{-\Re(\nu)x} dx \right\} \leq \frac{\mathcal{C}}{|\mu - \nu|}, \end{aligned}$$

where $\int_0^t e^{-\Re(\mu)x} dx$ and $\int_0^t e^{-\Re(\nu)x} dx$ exists. Hence, $|u(t) - v(t)| \leq K \varepsilon E_\alpha(t)$. By the virtue of Definition 6, the linear differential equation (1) has the Mittag-Leffler-Hyers-Ulam stability.

This finishes the proof. \square

By applying the same idea as Theorem 10, we can prove the following corollary which shows the Hyers-Ulam stability of the differential equation (1).

Corollary 11. *Given $\varepsilon > 0$. Suppose that $u(t) \in C^2(I)$ satisfying the inequality*

$$|u''(t) + l u'(t) + m u(t)| \leq \varepsilon$$

for all $t \in I$. Then there exists real number $K > 0$ which is independent of ε and u such that $|u(t) - v(t)| \leq K \varepsilon$, for some $v \in C^2(I)$ satisfies $v''(t) + l v'(t) + m v(t) = 0$ for all $t \in I$.

Proof. Substituting $\varepsilon E_\alpha(t)$ as ε in the inequality (3) and by applying same methodology of the Theorem 10, we can arrive that the corresponding differential equation (1) has the Hyers-Ulam stability. \square

By using the same technique in Theorem 10, we can also prove that the following theorem which shows the Mittag-Leffler-Hyers-Ulam-Rassias stability of the differential equation (1). The method of proof is similar, but we include it for the sake of completeness.

Theorem 12. *The linear differential equation (1) has the Mittag-Leffler-Hyers-Ulam-Rassias stability.*

Proof. Given $\varepsilon > 0$. Suppose that $u(t) \in C^2(I)$ and $\phi(t) : (0, \infty) \rightarrow (0, \infty)$ satisfying

$$(8) \quad |u''(t) + l u'(t) + m u(t)| \leq \phi(t) \varepsilon E_\alpha(t),$$

for all $t \in I$. We wish to prove that there exists real number $K > 0$ which is independent of ε and u such that $|u(t) - v(t)| \leq K \phi(t) \varepsilon E_\alpha(t)$, for some $v \in C^2(I)$ satisfies

$$v''(t) + l v'(t) + m v(t) = 0$$

for all $t \in I$. Define a function $p : (0, \infty) \rightarrow \mathbb{R}$ such that $p(t) =: u''(t) + l u'(t) + m u(t)$ for all $t > 0$. In view of (8), we have $|p(t)| \leq \phi(t) \varepsilon E_\alpha(t)$. Taking Laplace transform to $p(t)$, we have

$$(9) \quad \mathcal{L}\{p\} = (s^2 + l s + m) \mathcal{L}\{u\} - [s u(0) + l u(0) + u'(0)],$$

and thus

$$(10) \quad \mathcal{L}\{u\} = \frac{\mathcal{L}\{p\} + s u(0) + l u(0) + u'(0)}{s^2 + l s + m}.$$

In view of the (9), a function $u_0 : (0, \infty) \rightarrow \mathbb{R}$ is a solution of (1) if and only if

$$(s^2 + l s + m) \mathcal{L}\{u_0\} - [s u_0(0) + l u_0(0) + u_0'(0)] = 0.$$

Let l and m are constants in \mathbb{F} such that there exists μ and ν are in \mathbb{F} with $\mu + \nu = -l$, $\mu \nu = m$ and $\mu \neq \nu$. Then we have $(s^2 + l s + m) = (s - \mu)(s - \nu)$, (10) implies that

$$(11) \quad \mathcal{L}\{u\} = \frac{\mathcal{L}\{p\} + s u(0) + l u(0) + u'(0)}{(s - \mu)(s - \nu)}.$$

If we set

$$v(t) = u(0) \left(\frac{\mu e^{\mu t} - \nu e^{\nu t}}{\mu - \nu} \right) + [l u(0) + u'(0)] \left(\frac{e^{\mu t} - e^{\nu t}}{\mu - \nu} \right)$$

then one can easily have $v(0) = u(0)$ and $v'(0) = u'(0)$. Taking Laplace transform to $v(t)$, we obtain

$$(12) \quad \mathcal{L}\{v\} = \frac{s u(0) + l u(0) + u'(0)}{(s - \mu)(s - \nu)}.$$

On the other hand,

$$\mathcal{L}\{v''(t) + l v'(t) + m v(t)\} = (s^2 + l s + m)\mathcal{L}\{v\} - [s v(0) + l v(0) + v'(0)].$$

Using (12), we get $\mathcal{L}\{v''(t) + l v'(t) + m v(t)\} = 0$. Since \mathcal{L} is one-to-one operator and linear, then we get $v''(t) + l v'(t) + m v(t) = 0$. This means that $v(t)$ is a solution of (1). It follows from (10) and (12) that

$$\begin{aligned}\mathcal{L}\{u\} - \mathcal{L}\{v\} &= \frac{\mathcal{L}\{p\}}{(s - \mu)(s - \nu)} \\ \mathcal{L}\{u(t) - v(t)\} &= \mathcal{L}\{p(t) * q(t)\}.\end{aligned}$$

where

$$q(t) = \frac{e^{\mu t} - e^{\nu t}}{\mu - \nu}.$$

The above equalities show that $u(t) - v(t) = p(t) * q(t)$. Taking modulus on both sides and using $|p(t)| \leq \phi(t)\varepsilon E_\alpha(t)$, we get

$$\begin{aligned}|u(t) - v(t)| &= |p(t) * q(t)| \leq \left| \int_0^t p(t-x) q(x) dx \right| \\ &\leq |p(t)| \left| \int_0^t q(t-x) dx \right| \\ &\leq \phi(t)\varepsilon E_\alpha(t) \left| \int_0^t q(t-x) dx \right| = K\phi(t)\varepsilon E_\alpha(t)\end{aligned}$$

for all $t > 0$, where

$$\begin{aligned}K &= \left| \int_0^t q(t-x) dx \right| = \left| \int_0^t \left(\frac{e^{\mu(t-x)} - e^{\nu(t-x)}}{\mu - \nu} \right) dx \right| \\ &\leq \frac{1}{|\mu - \nu|} \left\{ e^{\Re(\mu)t} \int_0^t e^{-\Re(\mu)x} dx + e^{\Re(\nu)t} \int_0^t e^{-\Re(\nu)x} dx \right\} \leq \frac{\mathcal{C}}{|\mu - \nu|},\end{aligned}$$

where the integrals $\int_0^t e^{-\Re(\mu)x} dx$ and $\int_0^t e^{-\Re(\nu)x} dx$ exists. Then

$$|u(t) - v(t)| \leq K\phi(t)\varepsilon E_\alpha(t).$$

Hence, by the virtue of Definition 8, the linear differential equation (1) has the Mittag-Leffler-Hyers-Ulam-Rassias stability. This completes the proof. \square

By using the same technique as applied in the Theorem 12, we can prove the following corollary which shows the Hyers-Ulam-Rassias stability of the differential equation (1).

Corollary 13. *Given $\varepsilon > 0$, there exists real number $K > 0$ which is independent of ε and u such that $u(t) \in C^2(I)$ satisfying the inequality*

$$|u''(t) + l u'(t) + m u(t)| \leq \varepsilon \phi(t)$$

for all $t \in I$. Then for some $v \in C^2(I)$ satisfies the differential equation

$$v''(t) + l v'(t) + m v(t) = 0$$

such that $|u(t) - v(t)| \leq K\varepsilon\phi(t)$, for all $t \in I$.

Proof. Setting $\phi(t)\varepsilon E_\alpha(t)$ as $\varepsilon\phi(t)$ in the inequality (8) and by applying same terminology as used in the Theorem 12, we can easily prove that the differential equation (1) has the Hyers-Ulam-Rassias stability. \square

4. MITTAG-LEFFLER-HYERS-ULAM STABILITY FOR NON-HOMOGENEOUS DIFFERENTIAL EQUATION (2)

In this section, we investigate the Mittag-Leffler-Hyers-Ulam stability and Mittag-Leffler-Hyers-Ulam-Rassias stability of the non-homogeneous differential equation (2). Firstly, we prove the Mittag-Leffler-Hyers-Ulam stability of the non-homogeneous linear differential equation (2).

Theorem 14. *The differential equation (2) has Mittag-Leffler-Hyers-Ulam stability.*

Proof. Given $\varepsilon > 0$. Suppose that a twice continuously differentiable function $u(t)$ satisfying the inequality

$$(13) \quad |u''(t) + l u'(t) + m u(t) - r(t)| \leq \varepsilon E_\alpha(t),$$

for all $t \in I$. We have to prove that there exists real number $K > 0$ which is independent of ε and $u(t)$ such that $|u(t) - v(t)| \leq K\varepsilon E_\alpha(t)$, for some $v \in C^2(I)$ satisfies $v''(t) + l v'(t) + m v(t) = r(t)$

for all $t \in I$. Define a function $p : (0, \infty) \rightarrow \mathbb{R}$ such that $p(t) =: u''(t) + l u'(t) + m u(t) - r(t)$ for all $t > 0$. In view of (13), we have $|p(t)| \leq \varepsilon E_\alpha(t)$. Taking Laplace transform to $p(t)$, we have

$$(14) \quad \mathcal{L}\{p\} = (s^2 + l s + m)\mathcal{L}\{u\} - [s u(0) + l u(0) + u'(0)] - \mathcal{L}(r),$$

and thus

$$(15) \quad \mathcal{L}\{u\} = \frac{\mathcal{L}\{p\} + s u(0) + l u(0) + u'(0) + \mathcal{L}(r)}{s^2 + l s + m}.$$

In view of the (14), a function $u_0 : (0, \infty) \rightarrow \mathbb{R}$ is a solution of (2) if and only if

$$(s^2 + l s + m)\mathcal{L}\{u_0\} - [s u_0(0) + l u_0(0) + u'_0(0)] = \mathcal{L}(r).$$

Assume that l and m are constants in \mathbb{F} such that there exists μ and ν are in \mathbb{F} with $\mu + \nu = -l$, $\mu\nu = m$ and $\mu \neq \nu$. Then we have $(s^2 + l s + m) = (s - \mu)(s - \nu)$, (15) implies that

$$(16) \quad \mathcal{L}\{u\} = \frac{\mathcal{L}\{p\} + s u(0) + l u(0) + u'(0) + \mathcal{L}(r)}{(s - \mu)(s - \nu)}.$$

Now, set

$$v(t) = u(0) \left(\frac{\mu e^{\mu t} - \nu e^{\nu t}}{\mu - \nu} \right) + [l u(0) + u'(0)]q(t) + (q * r)(t),$$

where $q(t) = \frac{e^{\mu t} - e^{\nu t}}{\mu - \nu}$ then we have $v(0) = u(0)$ and $v'(0) = u'(0)$. Taking Laplace transform to $v(t)$, we obtain

$$(17) \quad \mathcal{L}\{v\} = \frac{s u(0) + l u(0) + u'(0) + \mathcal{L}(r)}{(s - \mu)(s - \nu)}.$$

On the other hand,

$$\mathcal{L}\{v''(t) + l v'(t) + m v(t) - r(t)\} = (s^2 + l s + m)\mathcal{L}\{v\} - [s v(0) + l v(0) + v'(0)] - \mathcal{L}(r).$$

Using (17), we get $\mathcal{L}\{v''(t) + l v'(t) + m v(t) - r(t)\} = 0$. Since \mathcal{L} is one-to-one operator and linear, then we get $v''(t) + l v'(t) + m v(t) = r(t)$. This means that $v(t)$ is a solution of (2). It follows from (15) and (17) that

$$\begin{aligned} \mathcal{L}\{u\} - \mathcal{L}\{v\} &= \frac{\mathcal{L}\{p\}}{(s - \mu)(s - \nu)} \\ \mathcal{L}\{u(t) - v(t)\} &= \mathcal{L}\{p(t) * q(t)\}. \end{aligned}$$

The above equalities show that

$$u(t) - v(t) = (p * q)(t).$$

Taking modulus on both sides and using $|p(t)| \leq \varepsilon E_\alpha(t)$, we get

$$\begin{aligned} |u(t) - v(t)| &= |(p * q)(t)| \leq \left| \int_0^t p(t-x) q(x) dx \right| \\ &\leq |p(t)| \left| \int_0^t q(t-x) dx \right| \leq \varepsilon E_\alpha(t) \left| \int_0^t q(t-x) dx \right| \end{aligned}$$

for all $t > 0$ and

$$\begin{aligned} \left| \int_0^t q(t-x) dx \right| &= \left| \int_0^t \left(\frac{e^{\mu(t-x)} - e^{\nu(t-x)}}{\mu - \nu} \right) dx \right| \\ &\leq \frac{1}{|\mu - \nu|} \left\{ e^{\Re(\mu)t} \int_0^t e^{-\Re(\mu)x} dx + e^{\Re(\nu)t} \int_0^t e^{-\Re(\nu)x} dx \right\} \leq \frac{\mathcal{C}}{|\mu - \nu|}, \end{aligned}$$

where the integrals $\int_0^t e^{-\Re(\mu)x} dx$ and $\int_0^t e^{-\Re(\nu)x} dx$ exists. Therefore,

$$|u(t) - v(t)| \leq K\varepsilon E_\alpha(t).$$

Then by the virtue of Definition 7, the linear differential equation (2) has the Mittag-Leffler-Hyers-Ulam stability. This completes the proof. \square

By using the methodology as applied in Theorem 14, we can establish the following corollary which shows the Hyers-Ulam stability of the non-homogeneous differential equation (2).

Corollary 15. *Let $\varepsilon > 0$ and for each $u(t) \in C^2(I)$ satisfying the inequality*

$$|u''(t) + l u'(t) + m u(t) - r(t)| \leq \varepsilon$$

for all $t \in I$. Then there exists real number $K > 0$ which is independent of ε and u such that $|u(t) - v(t)| \leq K\varepsilon$, for some $v \in C^2(I)$ satisfies the differential equation $v''(t) + l v'(t) + m v(t) = r(t)$ for all $t \in I$.

Proof. Replacing $\varepsilon E_\alpha(t)$ as ε in the inequality (13) and by using same terminology of the Theorem 14, we can prove that the differential equation (2) has the Hyers-Ulam stability. \square

In analogous to Theorem 14, we also have the following result which shows the Mittag-Leffler-Hyers-Ulam-Rassias stability of the differential equation (2).

Theorem 16. *The non-homogeneous linear differential equation (2) has Mittag-Leffler-Hyers-Ulam-Rassias stability.*

Proof. Given $\varepsilon > 0$. Suppose that $u(t) \in C^2(I)$ and $\phi(t) : (0, \infty) \rightarrow (0, \infty)$ satisfying

$$(18) \quad |u''(t) + l u'(t) + m u(t) - r(t)| \leq \phi(t) \varepsilon E_\alpha(t),$$

for all $t \in I$. We wish to prove that there exists real number $K > 0$ which is independent of ε and u such that $|u(t) - v(t)| \leq K\phi(t)\varepsilon E_\alpha(t)$, for some twice continuously differentiable function v satisfies the differential equation $v''(t) + l v'(t) + m v(t) = r(t)$ for all $t \in I$. Define a function $p : (0, \infty) \rightarrow \mathbb{R}$ such that $p(t) =: u''(t) + l u'(t) + m u(t) - r(t)$ for all $t > 0$. In view of (18), we have $|p(t)| \leq \phi(t)\varepsilon E_\alpha(t)$. Taking Laplace transform from $p(t)$, we have

$$(19) \quad \mathcal{L}\{p\} = (s^2 + l s + m)\mathcal{L}\{u\} - [s u(0) + l u(0) + u'(0)] - \mathcal{L}(r),$$

and thus

$$(20) \quad \mathcal{L}\{u\} = \frac{\mathcal{L}\{p\} + s u(0) + l u(0) + u'(0) + \mathcal{L}(r)}{s^2 + l s + m}.$$

In view of the (19), a function $u_0 : (0, \infty) \rightarrow \mathbb{R}$ is a solution of (2) if and only if

$$(s^2 + l s + m)\mathcal{L}\{u_0\} - [s u_0(0) + l u_0(0) + u_0'(0)] = \mathcal{L}(r).$$

Assume that l and m are constants in \mathbb{F} such that there exists μ and ν are in \mathbb{F} with $\mu + \nu = -l$, $\mu\nu = m$ and $\mu \neq \nu$. Then we have $(s^2 + l s + m) = (s - \mu)(s - \nu)$, (20) implies that

$$(21) \quad \mathcal{L}\{u\} = \frac{\mathcal{L}\{p\} + s u(0) + l u(0) + u'(0) + \mathcal{L}(r)}{(s - \mu)(s - \nu)}.$$

Now, set

$$v(t) = u(0) \left(\frac{\mu e^{\mu t} - \nu e^{\nu t}}{\mu - \nu} \right) + [l u(0) + u'(0)]q(t) + (q * r)(t),$$

where $q(t) = \frac{e^{\mu t} - e^{\nu t}}{\mu - \nu}$ then we have $v(0) = u(0)$ and $v'(0) = u'(0)$. Taking Laplace transform to $v(t)$, we obtain

$$(22) \quad \mathcal{L}\{v\} = \frac{s u(0) + l u(0) + u'(0) + \mathcal{L}(r)}{(s - \mu)(s - \nu)}.$$

On the other hand,

$$\mathcal{L}\{v''(t) + l v'(t) + m v(t) - r(t)\} = (s^2 + l s + m)\mathcal{L}\{v\} - [s v(0) + l v(0) + v'(0)] - \mathcal{L}(r).$$

Using (22), we get $\mathcal{L}\{v''(t) + l v'(t) + m v(t) - r(t)\} = 0$. Since \mathcal{L} is one-to-one operator and linear, then we get $v''(t) + l v'(t) + m v(t) = r(t)$. This means that $v(t)$ is a solution of (2). It follows from (20) and (22) that

$$\begin{aligned}\mathcal{L}\{u\} - \mathcal{L}\{v\} &= \frac{\mathcal{L}\{p\}}{(s - \mu)(s - \nu)} \\ \mathcal{L}\{u(t) - v(t)\} &= \mathcal{L}\{p(t) * q(t)\}.\end{aligned}$$

The above equalities show that $u(t) - v(t) = (p * q)(t)$. Taking modulus on both sides and using $|p(t)| \leq \phi(t)\varepsilon E_\alpha(t)$, we get

$$\begin{aligned}|u(t) - v(t)| &= |(p * q)(t)| \leq \left| \int_0^t p(t-x) q(x) dx \right| \\ &\leq |p(t)| \left| \int_0^t q(t-x) dx \right| \leq \phi(t)\varepsilon E_\alpha(t) \left| \int_0^t q(t-x) dx \right|\end{aligned}$$

for all $t > 0$, where

$$\begin{aligned}\left| \int_0^t q(t-x) dx \right| &= \left| \int_0^t \left(\frac{e^{\mu(t-x)} - e^{\nu(t-x)}}{\mu - \nu} \right) dx \right| \\ &\leq \frac{1}{|\mu - \nu|} \left\{ e^{\Re(\mu)t} \int_0^t e^{-\Re(\mu)x} dx + e^{\Re(\nu)t} \int_0^t e^{-\Re(\nu)x} dx \right\} \leq \frac{\mathcal{C}}{|\mu - \nu|},\end{aligned}$$

where the integrals $\int_0^t e^{-\Re(\mu)x} dx$ and $\int_0^t e^{-\Re(\nu)x} dx$ exists.

Thus, $|u(t) - v(t)| \leq K\phi(t)\varepsilon E_\alpha(t)$. By the virtue of Definition 9, the linear differential equation (2) has the Mittag-Leffler-Hyers-Ulam-Rassias stability. This finishes the proof. \square

Corollary 17. *For every $\varepsilon > 0$, there exists positive real number K which is independent of ε and u such that $u(t) \in C^2(I)$ satisfying the inequality*

$$|u''(t) + l u'(t) + m u(t)| \leq \varepsilon \phi(t)$$

for all $t \in I$. Then for some $v \in C^2(I)$ satisfies the differential equation

$$v''(t) + l v'(t) + m v(t) = r(t)$$

such that $|u(t) - v(t)| \leq K\varepsilon\phi(t)$, for all $t \in I$.

Proof. Setting $\phi(t)\varepsilon E_\alpha(t)$ as $\varepsilon\phi(t)$ in the inequality (18) and by applying same terminology as used in the Theorem 16, we can easily prove that the differential equation (2) has the Hyers-Ulam-Rassias stability. \square

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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