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CUBIC INTERIOR IDEAL IN Γ-SEMIRINGS

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Abstract. In this paper we introduce the notion cubic interior ideal in Γ-semiring and we study basic properties of cubic interior ideal.

Keywords: cubic set; cubic interior ideal; fuzzy set; interval-value fuzzy set; Γ-semiring.

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1. Introduction

Zadeh initiated the concept of fuzzy sets in 1965. In 1975, Zadeh made an extension concept of a fuzzy set by an interval-valued fuzzy set. A semigroup is an algebraic structure consisting of a non-empty sets together with an associative binary operation.Semirings which is a common generalization of rings and distributive lattices, was introduced by Vandiver [8]. It has been found very useful for solving problems in different areas of pure and applied mathematics, information sciences, etc., since the structure of a semiring provides an algebraic framework for modelling and studying the key factors in these applied areas. Ideals of semiring play a central

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role in the structure theory and useful for many purposes. The theory of Γ-semirings was introduced by [2]. Since then many researchers enriched this field. Many authors have studied semigroups in terms of fuzzy sets. Kuroki is the main contributor of this study. Kuroki introduced the notion of fuzzy ideals and fuzzy bi-ideals in semigroups. Atanassov introduced intuitionistic fuzzy set is characterized by a membership function and a non-membership function for each element in the Universe. In 2010, K. Hur and H.W. Kang introduced interval-valued fuzzy subgroups and rings. Jun et al. introduced the new concept called cubic sets. These structures encompass interval-valued fuzzy set and fuzzy set. Also Jun et al. introduced the notion of cubic subgroups. Vijayabalaji et al. introduced the notion of cubic linear space. V. Chinnadurai et al. introduced cubic ring. The purpose of this paper to introduce the notion of cubic ideals of Γ-semigroups and we provide some results on it.

In this paper we studied properties of cubic interior ideals of Γ-semirings. Furthermore we can show that the images or inverse images of a cubic interor ideal of an Γ-semiring become a cubic interior ideal.

2. Preliminaries

2.1. Γ-Semiring.

Now we review definition of some types Γ-semiring, which we use in the next section.

Let S and Γ be two additive commutative semigroups. We called S is an Γ-*semiring* if there exist mappings from $S \times \Gamma \times S$ ot S written as $(a,\alpha,b) \to a\alpha b$, and satisfying the following conditions:

(1) $a\alpha(b+c) = a\alpha b + a\alpha c$ and $(a+b)\alpha c = a\alpha c + b\alpha c$

(2) $a(\alpha+\beta)b = a\alpha b + a\beta b$

(3) $(a\alpha b\beta)c = a\alpha(b\beta)c$ for all $a,b,c \in S$ and $\alpha,\beta \in \Gamma$.

An Γ-semiring S is said to be *regular* if for each element $a \in S$, there exists an element $x \in S$ and $\alpha,\beta \in \Gamma$ such that $a = a\alpha x\beta a$. An Γ-semigroup S is called *intra-regular* if for every $a \in S$ there exist $x,y \in S$ and $\alpha,\beta,\gamma \in \Gamma$ such that $a = x\alpha a\beta a\gamma y$. A non-empty subset A of an Γ-ring S is an Γ-subsemiring of S if A is a subsemigroup of $(S,+)$ and $A\Gamma A \subseteq A$. A non-empty subset A of an Γ-semiring S is called a *left*(right)ideal of S if A is a additive subsemigroup of S and

$S\Gamma A \subseteq A(A\Gamma S \subseteq A)$. An ideal of S is a non-empty subset which is both a left ideal and a right ideal of S. An Γ-subsemiring A of a semiring S is called a *interior ideal* of S if $S\Gamma A\Gamma S \subseteq A$ [6].

2.2. Fuzzy Γ-semiring and interval valued fuzzy set. Now we will give definition of a fuzzy subset and types of fuzzy Γ-subsemigroups. Let X be a non-empty set. A mapping $\omega : X \to [0,1]$ is a *fuzzy subset* of S.

Definition 2.1. [6] Let S be an Γ-semiring. A fuzzy subset ω of S is said to be a *fuzzy Γ-subsemiring* of S if $\omega(x+y) \geq \min\{\omega(x), \omega(y)\}$ and $\omega(x\alpha y) \geq \min\{\omega(x), \omega(y)\}$, for all $x, y \in S$ and $\alpha \in \Gamma$.

Definition 2.2. [6] Let S be a Γ-semiring. A fuzzy subset ω of S is said to be a *fuzzy left* (right) ideal of S if $\omega(x+y) \geq \min\{\omega(x), \omega(y)\}$ and $\omega(x\alpha y) \geq \omega(y)(\omega(x\alpha y) \geq \omega(x))$, for all $x, y \in S$ and $\alpha \in \Gamma$. A non-empty fuzzy subset of an Γ-semigroup S is a *fuzzy ideal* of S if it is a fuzzy left ideal and fuzzy right ideal of S.

Definition 2.3. For a family $\{\omega_i \mid i \in I\}$ of fuzzy sets in X, we define the join $(\vee)$ and meet $(\wedge)$ operations as follows:

$$\left(\bigvee_{i\in I} \omega_i\right)(x) = \sup\{\omega_i(x) \mid i \in I\} \quad \text{and} \quad \left(\bigwedge_{i\in I} \omega_i\right)(x) = \inf\{\omega_i(x) \mid i \in I\}$$

respectively, for all $x \in X$.

Now we will introduce a new relation of an interval.

Definition 2.4. An interval number on $[0,1]$, say $\overline{a}$ is a closed subinterval of $[0,1]$, that is $\overline{a} = [a^-, a^+]$, where $0 \leq a^- \leq a^+ \leq 1$. Let $D[0,1]$ denoted the family of all closed subinterval of $[0,1]$, i.e.,

$$D[0,1] = \{\overline{a} = [a^-, a^+] \mid 0 \leq a^- \leq a^+ \leq 1\}.$$

The interval $[a,a]$ is identified with the number $a \in [0.1]$.

Definition 2.5. Let $\overline{a_i} = [a_i^-, a_i^+] \in D[0,1]$ for all $i \in I$ where I is an index set. We define

$$r\inf_{i\in I} \overline{a_i} = \left[\inf_{i\in I} a_i^-, \inf_{i\in I} a_i^+\right] \quad \text{and} \quad r\sup_{i\in I} \overline{a_i} = \left[\sup_{i\in I} a_i^-, \sup_{i\in I} a_i^+\right].$$

We define the operations "$\succeq$", "$\preceq$", "$=$", "$r\min$" "$r\max$" in case of two element in $D[0,1]$. We consider two interval numbers $\overline{a} := [a^-, a^+]$ and
$\overline{b} := [b^-, b^+]$ in $D[0,1]$. Then

(1) $\overline{a} \succeq \overline{b}$ if and only if $a^- \geq b^-$ and $a^+ \geq b^+$
(2) $\overline{a} \preceq \overline{b}$ if and only if $a^- \leq b^-$ and $a^+ \leq b^+$
(3) $\overline{a} = \overline{b}$ if and only if $a^- = b^-$ and $a^+ = b^+$
(4) $r\min\{\overline{a}, \overline{b}\} = [\min\{a^-, b^-\}, \min\{a^+, b^+\}]$
(5) $r\max\{\overline{a}, \overline{b}\} = [\max\{a^-, b^-\}, \max\{a^+, b^+\}]$.

Definition 2.6. Let X be a set. An interval valued fuzzy set A on X is defined as

$$A = \{(x, [\mu^-(x), \mu^+(x)] \mid x \in X\},$$

where μ^- and μ^+ are two fuzzy sets of X such that $\mu^-(x) \leq \mu^+(x)$ for all $x \in X$. Putting $\overline{\mu}(x) = [\mu^-(x), \mu^+(x)]$, we see that $A = \{x, \overline{\mu}(x) \mid x \in X\}$, where $\overline{\mu} : X \to D[0,1]$.

Definition 2.7. For a family $\{\overline{\mu}_i \mid i \in I\}$ of interval valued fuzzy sets in X, we define the $(\sqcup_{i\in I}\overline{\mu}_i)$ and $(\sqcap_{i\in I}\mu_i)$ are defined as follows:

$$(\sqcup_{i\in I}\overline{\mu}_i)(x) = r\sup\overline{\mu}_i(x) \quad \text{and} \quad (\sqcap_{i\in I}\overline{\mu}_i)(x) = r\inf\overline{\mu}_i(x)$$

respectively, for all $x \in X$ where $\overline{\mu} : X \to D[0,1]$.

2.3. Cubic Γ-semiring.

Definition 2.8. Let X be a non-empty set. A *cubic set* $\mathscr{A}$ in X is a structure of the form

$$\mathscr{A} = \{\langle x, \overline{\mu}(x), \omega(x)\rangle : x \in X\}$$

and denoted by $\mathscr{A} = \langle\overline{\mu}, \omega\rangle$ where $\overline{\mu}$ is an interval valued fuzzy set (briefly. IVF) in X and ω is a fuzzy set in X. In this case we will use

$$\mathscr{A}(x) = \langle\overline{\mu}(x), \omega(x)\rangle = \langle[\mu^-(x), \mu^+(x)], \omega(x)\rangle$$

For all $x \in X$. Note that a cubic set is a generalization of an intuitionistic fuzzy set.

Definition 2.9. Let A be a subset of a non-empty set X. Then cubic set characteristic function $\chi_A = \langle \overline{\mu}_{\chi_A}, \omega_{\chi_A} \rangle$ of is defined as

$$\overline{\mu}_{\chi_A}(x) = \begin{cases} [1,1], & \text{if} \quad x \in A, \\ [0,0], & \text{if} \quad x \notin A. \end{cases} \quad \text{and} \quad \omega_{\chi_A}(x) = \begin{cases} 0, & \text{if} \quad x \in A, \\ 1, & \text{if} \quad x \notin A. \end{cases}$$

Definition 2.10. For two cubic set $\mathscr{A} = \langle \overline{\mu}_1, \omega_1 \rangle$ and $\mathscr{B} = \langle \overline{\mu}_2, \omega_2 \rangle$ in an Γ-semiring S, we define

$$\mathscr{A} \sqsubseteq \mathscr{B} \Leftrightarrow \overline{\mu}_1 \preceq \overline{\mu}_2 \quad \text{and} \quad \omega_1 \geq \omega_2$$

Definition 2.11. Let $\mathscr{A} = \langle \overline{\mu}_1, \omega_1 \rangle$ and $\mathscr{B} = \langle \overline{\mu}_2, \omega_2 \rangle$ be two cubic set in an Γ-semiring S. Then

$$\mathscr{A} \circledcirc \mathscr{B} = \{ \langle x, (\overline{\mu}_1 \tilde{\circ} \overline{\mu}_2)(x), (\omega_1 \cdot \omega_2)(x) \rangle : x \in S \}$$

which is briefly denoted by $\mathscr{A} \circledcirc \mathscr{B} = \langle (\overline{\mu}_1 \tilde{\circ} \overline{\mu}_2), (\omega_1 \cdot \omega_2) \rangle$ where $\overline{\mu}_1 \tilde{\circ} \overline{\mu}_2$ and $\omega_1 \cdot \omega_2$ are defined as follows, respectively:

$$(\overline{\mu}_1 \tilde{\circ} \overline{\mu}_2)(x) = \begin{cases} r\sup_{x=y\beta z} \{ r\min\{ \overline{\mu}_1(y), \overline{\mu}_2(z) \} \} & \text{if} \quad x = y\beta z, \\ [0,0], & \text{otherwise} \end{cases}$$

and

$$(\omega_1 \cdot \omega_2)(x) = \begin{cases} \inf\limits_{x=y\beta z} \max\{ \omega_1(y), \omega_2(z) \} & \text{if} \quad x = y\beta z, \\ 1, & \text{otherwise} \end{cases}$$

for all $x, y, z \in S$ and $\beta \in \Gamma$. And

$$\mathscr{A} \circledast \mathscr{B} = \{ \langle x, (\overline{\mu}_1 \tilde{*} \overline{\mu}_2)(x), (\omega_1 * \omega_2)(x) \rangle : x \in S \}$$

which is briefly denoted by $\mathscr{A} \circledcirc \mathscr{B} = \langle (\overline{\mu}_1 \tilde{*} \overline{\mu}_2), (\omega_1 * \omega_2) \rangle$ where $\overline{\mu}_1 \tilde{*} \overline{\mu}_2$ and $\omega_1 * \omega_2$ are defined as follows, respectively:

$$(\overline{\mu}_1 \tilde{*} \overline{\mu}_2)(x) = \begin{cases} r\sup_{x=y\beta z} \{ r\min\{ \overline{\mu}_1(y), \overline{\mu}_2(z) \} \} & \text{if} \quad x = y\beta z, \\ [0,0], & \text{otherwise} \end{cases}$$

and

$$(\omega_1 * \omega_2)(x) = \begin{cases} \inf\limits_{x=y\beta z} \max\{ \omega_1(y), \omega_2(z) \} & \text{if} \quad x = y\beta z, \\ 1, & \text{otherwise} \end{cases}$$

for all $x,y,z \in S$ and $\beta \in \Gamma$.

Definition 2.12. Let $\mathscr{A} = \langle \overline{\mu}_A, f_A \rangle$ and $\mathscr{B} = \langle \overline{\mu}_B, f_B \rangle$ be two cubic set in a semiring S. Then the intersection of $\mathscr{A}$ and $\mathscr{B}$ denoted by $\mathscr{A} \overline{\sqcap} \mathscr{B}$ is the cubic set

$$\mathscr{A} \overline{\sqcap} \mathscr{B} = \langle \overline{\mu}_A \sqcap \overline{\mu}_B, f_A \vee f_B \rangle$$

where $(\overline{\mu}_A \sqcap \overline{\mu}_B)(x) = r\min\{\overline{\mu}_A(x), \overline{\mu}_B(x)\}$ and $(f_A \vee f_B)(x) = \max\{f_A(x), f_B(x)\}$ for all $x \in S$.

Definition 2.13. A cubic set $\mathscr{A} = \langle \overline{\mu}, \omega \rangle$ in an Γ-semiring S is called a *cubic* Γ*-subsemigroup* of S if it satisfies:

(1) $\overline{\mu}(x+y) \succeq r\min\{\mu(x), \mu(y)\}$ and $\omega(x+y) \leq \max\{\omega(x), \omega(y)\}$
(2) $\overline{\mu}(x\alpha y) \succeq r\min\{\overline{\mu}(x), \overline{\mu}(y)\}$ and $\omega(x\alpha y) \leq \max\{\omega(x), \omega(y)\}$,
for all $x,y \in S$ and $\alpha \in \Gamma$.

Definition 2.14. A cubic set $\mathscr{A} = \langle \overline{\mu}, \omega \rangle$ in an Γ-semiring S is called a *cubic left* (right) ideal of S if it satisfies:

(1) $\overline{\mu}(x+y) \succeq r\min\{\mu(x), \mu(y)\}$ and $\omega(x+y) \leq \max\{\omega(x), \omega(y)\}$
(2) $\overline{\mu}(x\alpha y) \succeq \overline{\mu}(y)$ $(\overline{\mu}(x\alpha y) \succeq \overline{\mu}(x))$ and $\omega(x\alpha y) \leq \omega(y)$, $(\omega(x\alpha y) \leq \omega(x))$
for all $x,y \in S$ and $\alpha \in \Gamma$.

A non-empty cubic set $\mathscr{A} = \langle \overline{\mu}, \omega \rangle$ of S is called *cubic ideal* of S if it is a cubic left ideal and a cubic right ideal of S.

3. Cubic Interior Ideal in Γ-Semiring.

Definition 3.1. A cubic Γ-subsemiring $\mathscr{A} = \langle \overline{\mu}, \omega \rangle$ in a Γ-semiring S is called a *cubic interior ideal* of S if $\overline{\mu}(x\alpha y\beta z) \succeq \overline{\mu}(y)$ and $\omega(x\alpha y\beta z) \leq \omega(y)$, for all $x,y,z \in S$ and $\alpha \in \Gamma$.

It is clarly every cubic ideal of an Γ-semiring S is cubic interior ideal of S. The following theorems we will study basic properties of a cubic interior ideal.

Theorem 3.1. Let $\mathscr{A} = \langle \overline{\mu}, \omega \rangle$ be cubic interior ideal and $\mathscr{B} = \langle \overline{\mu}_1, \omega_1 \rangle$ be cubic interior ideal of Γ-semiring S. Then $\mathscr{A} \sqcap \mathscr{B}$ is a cubic interior ideal of Γ-semiring S.

Proof. By assumptoin, we have $\mathscr{A} \sqcap \mathscr{B}$ is a cubic Γ-subsemiring of S. Let $a,x,y \in S$, $\gamma \in \Gamma$. Then

$$(\overline{\mu} \sqcap \overline{\mu}_1)(x\alpha a\beta y) = r\min\{\overline{\mu}(x\alpha a\beta y), \overline{\mu}_1(x\alpha a\beta y)\} \succeq r\min\{\overline{\mu}(a), \overline{\mu}_1(a)\} = (\overline{\mu} \sqcap \overline{\mu}_1)(a).$$

and

$$(\omega \cup \omega_1)(x\alpha a\beta y) = \max\{\omega(x\alpha a\beta y), \omega_1(x\alpha a\beta y)\} \leq \max\{\omega(y), \omega_1(a)\} = (\omega \cup \omega_1)(a).$$

Thus, $\mathscr{A} \sqcap \mathscr{B}$ is a cubic interior ideal of S. □

Corollary 3.2. The intersection of any family of cubic interior ideals of Γ-semiring S is a cubic interior ideal of Γ-semiring S.

Theorem 3.3. Let S be a Γ-semiring and let A be non-empty subset of S. Then A is an interior ideal of S if and only if the characteristic cubic set $\chi_A = \langle \overline{\mu}_{\chi_A}, \omega_{\chi_A} \rangle$ is a cubic interior ideal of S.

Proof. Suppose that A is a interior ideal of S and let $x, y \in S$ and $\alpha \in \Gamma$. Since A is an Γ-subsemiring of S we have $x + y \in A$ and $x\alpha y \in A$. Thus $\overline{\mu}_{\chi_A}(x) = \overline{\mu}_{\chi_A}(y) = \overline{\mu}_{\chi_A}(x+y) = [1,1]$ and $\omega_{\chi_A}(x) = \omega_{\chi_A}(y) = \omega_{\chi_A}(x+y) = 0$. So $\overline{\mu}_{\chi_A}(x+y) \succeq r\min\{\overline{\mu}_{\chi_A}(x), \overline{\mu}_{\chi_A}(y)\}$ and $\omega_{\chi_A}(x+y) \leq \max\{\omega_{\chi_A}(x), \omega_{\chi_A}(y)\}$. Similarly we have $\overline{\mu}_{\chi_A}(x) = \overline{\mu}_{\chi_A}(y) = \overline{\mu}_{\chi_A}(x\alpha y) = [1,1]$ and $\omega_{\chi_A}(x) = \omega_{\chi_A}(y) = \omega_{\chi_A}(x\alpha y) = 0$. It implies that

$$\overline{\mu}_{\chi_A}(x\alpha y) \succeq r\min\{\overline{\mu}_{\chi_A}(x), \overline{\mu}_{\chi_A}(y)\}$$

and

$$\omega_{\chi_A}(x\alpha y) \leq \max\{\omega_{\chi_A}(x), \omega_{\chi_A}(y)\}.$$

If $x \notin A$ or $y \notin A$, then $x\alpha y \in A$. Thus, $\overline{\mu}_{\chi_A}(x)$ or $\overline{\mu}_{\chi_A}(y) = [0,0]$, $\overline{\mu}_{\chi_A}(x\alpha y) = [1,1]$ and $\omega_{\chi_A}(x)$ or $\omega_{\chi_A}(y) = 1$, $\omega_{\chi_A}(x\alpha y) = 0$. It implies that

$$\overline{\mu}_{\chi_A}(x\alpha y) \succeq r\min\{\overline{\mu}_{\chi_A}(x), \overline{\mu}_{\chi_A}(y)\}$$

and

$$\omega_{\chi_A}(x\alpha y) \leq \max\{\omega_{\chi_A}(x), \omega_{\chi_A}(y)\}.$$

Thus $\chi_A = \langle \overline{\mu}_{\chi_A}, \omega_{\chi_A} \rangle$ is a cubic Γ-subsemigroup of S.

Let $x,y,z \in S$ and $\alpha,\beta \in \Gamma$. Then the following cases:

If $y \in A$, then $x\alpha y\beta z \in A$. Thus, $\overline{\mu}_{\chi_A}(y) = \overline{\mu}_{\chi_A}(x\alpha y\beta z) = [1,1]$ and $\omega_{\chi_A}(y) = \omega_{\chi_A}(x\alpha y\beta z) = 0$. It implies that

$$\overline{\mu}_{\chi_A}(x\alpha y\beta z) \succeq \overline{\mu}_{\chi_A}(y)$$

and

$$\omega_{\chi_A}(x\alpha y\beta z) \leq \omega_{\chi_A}(y).$$

If $y \notin A$, then $x\alpha y\beta z \in A$. Thus, $\overline{\mu}_{\chi_A}(y) = [0,0]$, $\overline{\mu}_{\chi_A}(x\alpha y\beta z) = [1,1]$ and $\omega_{\chi_A}(y) = 1$, $\omega_{\chi_A}(x\alpha y\beta z) = 0$. So $\overline{\mu}_{\chi_A}(x\alpha y\beta z) \succeq \overline{\mu}_{\chi_A}(y)$ and $\omega_{\chi_A}(x\alpha y\beta z) \leq \omega_{\chi_A}(y)$. We conclude that $\overline{\mu}_{\chi_A}(x\alpha y\beta z) \succeq \overline{\mu}_{\chi_A}(y)$ and $\omega_{\chi_A}(x\alpha y\beta z) \leq \omega_{\chi_A}(y)$.
This implies that $\chi_A = \langle \overline{\mu}_{\chi_A}, \omega_{\chi_A} \rangle$ is a cubic interior ideal of S.

Conversely, suppose that $\chi_A = \langle \overline{\mu}_{\chi_A}, \omega_{\chi_A} \rangle$ is a cubic interior ideal of S.
Let $x,y \in A$ and $\alpha \in \Gamma$ Then, $\overline{\mu}_{\chi_A}(x) = \overline{\mu}_{\chi_A}(y) = [1,1]$. Since $\chi_A = \langle \overline{\mu}_{\chi_A}, \omega_{\chi_A} \rangle$ is a cubic subsemiring of S

$$(1) \qquad \begin{cases} \overline{\mu}_{\chi_A}(x+y) \succeq r\min\{\overline{\mu}_{\chi_A}(x), \overline{\mu}_{\chi_A}(y)\} \\ \text{and } \omega_{\chi_A}(x+y) \leq \max\{\omega_{\chi_A}(x), \omega_{\chi_A}(y)\}. \end{cases}$$

$$(2) \qquad \begin{cases} \overline{\mu}_{\chi_A}(x\alpha y) \succeq r\min\{\overline{\mu}_{\chi_A}(x), \overline{\mu}_{\chi_A}(y)\} \\ \text{and } \omega_{\chi_A}(x\alpha y) \leq \max\{\omega_{\chi_A}(x), \omega_{\chi_A}(y)\}. \end{cases}$$

If $x+y \notin A$ and $x\alpha y \notin A$, then by (1) and (2), $\overline{\mu}_{\chi_A}(x+y) \prec r\min\{\overline{\mu}_{\chi_A}(x), \overline{\mu}_{\chi_A}(y)\}$ and $\omega_{\chi_A}(x+y) > \max\{\omega_{\chi_A}(x), \omega_{\chi_A}(y)\}$. and $\overline{\mu}_{\chi_A}(x\alpha y) \prec r\min\{\overline{\mu}_{\chi_A}(x), \overline{\mu}_{\chi_A}(y)\}$ and $\omega_{\chi_A}(x\alpha y) > \max\{\omega_{\chi_A}(x), \omega_{\chi_A}(y)\}$. It is a contradiction. Hence $x+y \in A$ and $x\alpha y \in A$. Therefore A is an Γ-subsemiring of S. Let $x,y,z \in S$ and $\alpha,\beta \in \Gamma$ with $y \in A$. Then $\overline{\mu}_{\chi_A}(y) = [1,1]$ and $\omega_{\chi_A}(y) = 0$. Since $\chi_A = \langle \overline{\mu}_{\chi_A}, \omega_{\chi_A} \rangle$ is a cubic interior ideal of S we have $\overline{\mu}_{\chi_A}(x\alpha y\beta z) \succeq \overline{\mu}_{\chi_A}(y)$ and $\omega_{\chi_A}(x\alpha y\beta z) \leq \omega_{\chi_A}(y)$. Thus $\overline{\mu}_{\chi_A}(x\alpha y\beta z) = [1,1]$ and $\omega_{\chi_A}(x\alpha y\beta z) = 0$. Hence $x\alpha y\beta z \in A$. Therefore A is an interior ideal of S. □

Definition 3.2. Let $\mathscr{A} = \langle \overline{\mu}, \omega \rangle$ is cubic set in X. For any $k \in [0,1]$ and $[s,t] \in D[0,1]$, we define $U(A,[s,t],k)$ as follows:

$$U(\mathscr{A},[s,t],k) = \{x \in X \mid \overline{\mu}(x) \succeq [s,t], \omega(x) \leq k\},$$

and we say it is a cubic level set of $\mathscr{A} = \langle \overline{\mu}, \omega \rangle$.

Theorem 3.4. Let S be an Γ-semiring. Then $\mathscr{A} = \langle \overline{\mu}, \omega \rangle$ is a cubic interior ideal of S if and only if the level set $U(\mathscr{A},[s,t],k)$ is an interior ideal of S.

Proof. Suppose that $\mathscr{A} = \langle \overline{\mu}, \omega \rangle$ is a cubic interior ideal of S.
Then $\mathscr{A} = \langle \overline{\mu}, \omega \rangle$ is a cubic Γ-subsemiring of S. Thus $\overline{\mu}(x+y) \succeq r\min\{\overline{\mu}(x), \overline{\mu}(y)\}$, $\omega(x+y) \leq \max\{\omega(x), \omega(y)\}$ and $\overline{\mu}(x\alpha y) \succeq r\min\{\overline{\mu}(x), \overline{\mu}(y)\}$, $\omega(x\alpha y) \leq \max\{\omega(x), \omega(y)\}$. Let $x, y \in U(\mathscr{A},[s,t],k)$ and $\alpha \in \Gamma$. Then $\overline{\mu}(x) \succeq [s,t]$, $\overline{\mu}(y) \succeq [s,t]$ and $\omega(x) \leq k$, $\omega(y) \leq k$. Thus

$$\overline{\mu}(x+y) \succeq r\min\{\overline{\mu}(x), \overline{\mu}(y)\} = r\min\{[s,t],[s,t]\} = [s,t],$$

$$\omega(x+y) \leq \max\{\omega(x), \omega(y)\} = \max\{k,k\} = k$$

and

$$\overline{\mu}(x\alpha y) \succeq r\min\{\overline{\mu}(x), \overline{\mu}(y)\} = r\min\{[s,t],[s,t]\} = [s,t],$$

$$\omega(x\alpha y) \leq \max\{\omega(x), \omega(y)\} = \max\{k,k\} = k.$$

So $x+y$ and $x\alpha y$ are elements of $U(\mathscr{A},[s,t],k)$. Hence $U(\mathscr{A},[s,t],k)$ is a Γ-subsemiring of S. Let $x,y,z \in S$ and $\alpha, \beta \in \Gamma$. with $y \in U(\mathscr{A},[s,t],k)$. Then $\overline{\mu}(y) \succeq [s,t]$ and $\omega(y) \leq k$. Thus

$$\overline{\mu}(x\alpha y\beta z) \succeq \overline{\mu}(y) \succeq [s,t]$$

and

$$\omega(x\alpha y\beta z) \leq \omega(y) \leq k.$$

It $x\alpha y\beta z \notin U(\mathscr{A},[s,t],k)$, then $\overline{\mu}(x\alpha y\beta z) \prec [s,t]$ and $\omega(x\alpha y\beta z) > k$. It is a contradiction. So $x\alpha y\beta z \in U(\mathscr{A},[s,t],k)$. Hence $U(\mathscr{A},[s,t],k)$ is an interor ideal of S. Therefore $U(\mathscr{A},[s,t],k)$ is a cubic level set is an interor ideal of S.

For the coverse, Syppose that $U(\mathscr{A},[s,t],k)$ is a cubic level set is an interor ideal of S. Let $x, y \in S$ and $\alpha \in \Gamma$. By assumption, $x+y$ and $x\alpha y$ are elements of $U(\mathscr{A},[s,t],k)$. Then $\overline{\mu}(x+y) \succeq [s,t]$, $\omega(x+y) \leq k$ and $\overline{\mu}(x\alpha y) \succeq [s,t]$, $\omega(x\alpha y) \leq k$. Since $U(\mathscr{A},[s,t],k)$ is a cubic level set is a Γ-subsemiring of S we have $x,y \in U(\mathscr{A},[s,t],k)$. Thus $\overline{\mu}(x) \succeq [s,t]$, $\overline{\mu}(y) \succeq [s,t]$ and $\omega(x) \leq k$, $\omega(y) \leq k$. So, $\overline{\mu}(x+y) \succeq r\min\{\overline{\mu}(x)\overline{\mu}(y)\}$, $\omega(x+y) \leq \max\{\omega(x)\omega(y)\}$ and $\overline{\mu}(x\alpha y) \succeq r\min\{\overline{\mu}(x)\overline{\mu}(y)\}$, $\omega(x\alpha y) \leq \max\{\omega(x)\omega(y)\}$. Hence $\mathscr{A} = \langle\overline{\mu},\omega\rangle$ is a cubic Γ-subsemiring of S.

Assume that $x,y,z \in S$ and $\alpha,\beta \in \Gamma$. By assumption, $x\alpha y\beta z$ and y are elements of $U(\mathscr{A},[s,t],k)$. Thus, $\overline{\mu}(x\alpha y\beta z) \succeq [s,t], \overline{\mu}(y) \succeq [s,t]$ and $\omega(x\alpha a\beta y) \leq k, \omega(y) \leq k$. So, $\overline{\mu}(x\alpha y\beta z) \succeq \overline{\mu}(y)$ and $\omega(x\alpha y\beta z) \leq \omega(y)$. Hence $\mathscr{A} = \langle\overline{\mu},\omega\rangle$ is is a cubic interor ideal of S. □

4. Homomorphic Inverse Image Operation to Get Cubic Set

In this section, we study some properties of homomorphic and inverse image of cubic set.

Definition 4.1. [4] Let $\mathscr{C}(X)$ be the family of cubic set in a set X.

Let X and Y be given classical sets. A mapping $h : X \to Y$ induces two mapping $\mathscr{C}_h : \mathscr{C}(X) \to \mathscr{C}(Y)$, $\mathscr{A} \mapsto \mathscr{C}_h(\mathscr{A})$ and $\mathscr{C}_h^{-1} : \mathscr{C}(Y) \to \mathscr{C}(X)$, $\mathscr{B} \mapsto \mathscr{C}_h^{-1}(\mathscr{B})$ where $\mathscr{C}_h(\mathscr{A})$ is given by

$$\mathscr{C}_h(\overline{\mu})(y) = \begin{cases} r\sup_{y=h(x)} \overline{\mu}(x), & \text{if} \quad h^{-1}(y) \neq 0, \\ [0,0], & \text{otherwise} \end{cases}$$

$$\mathscr{C}_h(\omega)(y) = \begin{cases} \inf_{y=h(x)} \omega(x), & \text{if} \quad h^{-1}(y) \neq 0, \\ 1, & \text{otherwise} \end{cases}$$

for all $y \in Y$. The *inverse image* $\mathscr{C}_h^{-1}(\mathscr{B})$ is defined by $\mathscr{C}_h^{-1}(\overline{\mu})(x) = \overline{\mu}(h(x))$ and $\mathscr{C}_h^{-1}(\omega)(x) = \omega(h(x))$ for all $x \in X$. Then the mapping $\mathscr{C}_h$ ($\mathscr{C}_h^{-1}$) is called a cubic transformation (inverse cubic transformation) induced by h. A cubic set $\mathscr{A} = \langle\overline{\mu},\omega\rangle$ in X has the *cubic property* if for any subset T of X there exists $x_0 \in T$ such that $\overline{\mu}(x_0) = r\sup_{x\in T} \overline{\mu}(x)$ and $\omega(x_0) = \inf_{x\in T} \omega(x)$.

Theorem 4.1. For a homomorphism $h : X \to Y$ of semirings,
let $\mathscr{C}_h : \mathscr{C}(X) \to \mathscr{C}(Y)$ be the cubic transformation induced by h. If $\mathscr{A} = \langle\overline{\mu},\omega\rangle \in \mathscr{C}(X)$ is a cubic interior ideal of X which has the cubic property, then $\mathscr{C}_h(A)$ is a cubic interior ideal of Y.

Proof. Let $\mathscr{A} = \langle\overline{\mu},\omega\rangle \in \mathscr{C}(X)$ be a cubic interior ideal of X and $\alpha \in \Gamma$.
Let $x_0 \in h^{-1}(h(x))$, $y_0 \in h^{-1}(h(y))$ be such that $\overline{\mu}(x_0) = r\sup_{a\in h^{-1}(h(x))} \overline{\mu}(a)$, $\omega(x_0) =$

$\inf_{a\in h^{-1}(h(x))}\omega(a)$ and $\overline{\mu}(y_0)=r\sup_{b\in h^{-1}(h(y))}\overline{\mu}(b),\ \omega(y_0)=\inf_{b\in h^{-1}(h(y))}\omega(b)$ respectively. Then

$$\begin{aligned}\mathscr{C}_h(\overline{\mu})(h(x)+h(y)) &= r\sup_{z\in h^{-1}(h(x)+h(y))}\overline{\mu}(z)\succeq\overline{\mu}(x_0+y_0)\\ &\succeq r\min\{\overline{\mu}(x_0),\overline{\mu}(y_0)\}\\ &= r\min\{r\sup_{a\in h^{-1}(h(x))}\overline{\mu}(a),r\sup_{b\in h^{-1}(h(y))}\overline{\mu}(b)\}\\ &= r\min\{\mathscr{C}_h(\overline{\mu}(a))(h(x)),\mathscr{C}_h(\overline{\mu}(b))(h(y))\},\end{aligned}$$

$$\begin{aligned}\mathscr{C}_h(\omega)(h(x)+h(y)) &= \inf_{z\in h^{-1}(h(x)+h(y))}\omega(z)\le\omega(x_0+y_0)\\ &\le \max\{\omega(x_0),\omega(y_0)\}\\ &= \max\{\inf_{a\in h^{-1}(h(x))}\omega(a),\inf_{b\in h^{-1}(h(y))}\omega(b)\}\\ &= \max\{\mathscr{C}_h(h(x)),\mathscr{C}_h(h(y))\}.\end{aligned}$$

And

$$\begin{aligned}\mathscr{C}_h(\overline{\mu})(h(x)\alpha h(y)) &= r\sup_{z\in h^{-1}((h(x)\alpha h(y))}\overline{\mu}(z)\succeq\overline{\mu}(x_0\alpha y_0)\\ &\succeq r\min\{\overline{\mu}(x_0),\overline{\mu}(y_0)\}\\ &= r\min\{r\sup_{a\in h^{-1}(h(x))}\overline{\mu}(a),r\sup_{b\in h^{-1}(h(y))}\overline{\mu}(b)\}\\ &= r\min\{\mathscr{C}_h(\overline{\mu}(a))(h(x)),\mathscr{C}_h(\overline{\mu}(a))(h(y))\},\end{aligned}$$

$$\begin{aligned}\mathscr{C}_h(\omega)(h(x)\alpha h(y)) &= \inf_{z\in h^{-1}((h(x)\alpha h(y))}\omega(z)\le\omega(x_0y_0)\\ &\le \max\{\omega(x_0),\omega(y_0)\}\\ &= \max\{\inf_{a\in h^{-1}(h(x))}\omega(a),\inf_{b\in h^{-1}(h(y))}\omega(b)\}\\ &= \max\{\mathscr{C}_h(\omega(a))(h(x)),\mathscr{C}_h(\omega(b))(h(y))\}.\end{aligned}$$

Thus $\mathscr{C}_h(\overline{\mu})(h(x)+h(y))\succeq r\min\{\mathscr{C}_h(\overline{\mu}(a))(h(x)),\mathscr{C}_h(\overline{\mu}(b))(h(y))\}$,
$\mathscr{C}_h(\omega)(h(x)+h(y))\le\max\{\mathscr{C}_h(h(x)),\mathscr{C}_h(h(y))\}$ and
$\mathscr{C}_h(\overline{\mu})(h(x)\alpha h(y))\succeq r\min\{\mathscr{C}_h(\overline{\mu}(a))(h(x)),\mathscr{C}_h(\overline{\mu}(a))(h(y))\}$,
$\mathscr{C}_h(\omega)(h(x)\alpha h(y))\le\max\{\mathscr{C}_h(\omega(a))(h(x)),\mathscr{C}_h(\omega(b))(h(y))\}$.
Hence $\mathscr{C}_h(\mathscr{A})$ is a cubic Γ-subsemiring of Y.

Similarly, let $h(a),h(x),h(y)\in h(X)$ and let $a_0\in h^{-1}(h(a))$, $x_0\in h^{-1}(h(x))$, $y_0\in h^{-1}(h(y))$ be such that $\overline{\mu}(a_0)=r\sup_{a\in h^{-1}(h(a))}\overline{\mu}(a)$, $\omega(a_0)=\inf_{a\in h^{-1}(h(a))}\omega(a)$, $\overline{\mu}(x_0)=r\sup_{b\in h^{-1}(h(x))}\overline{\mu}(b),\ \omega(x_0)=\inf_{b\in h^{-1}(h(x))}\omega(b)$ and $\overline{\mu}(y_0)=r\sup_{c\in h^{-1}(h(y))}\overline{\mu}(c),\ \omega(y_0)=$

$\inf_{c\in h^{-1}(h(y))}\omega(c)$ respectively and $\alpha,\beta\in\Gamma$. Then

$$\begin{aligned}\mathscr{C}_h(\overline{\mu})(h(a)\alpha h(x)\beta h(y)) &= r\sup_{k\in h^{-1}(h(a)\alpha h(x)\beta(h(y))}\overline{\mu}(k)\succeq\overline{\mu}(a_0\alpha x_0\beta y_0)\\ &\succeq \overline{\mu}(x_0)=r\sup_{b\in h^{-1}(h(x))}\overline{\mu}(b)=\mathscr{C}_h(\overline{\mu}(b))(h(x)).\end{aligned}$$

And

$$\begin{aligned}\mathscr{C}_h(\omega)(h(a)\alpha h(x)\beta h(y)) &= \inf_{k\in h^{-1}(h(a)\alpha h(x)\beta(h(y))}\omega(k)\leq\omega(a_0\alpha x_0\beta y_0)\\ &\leq \omega(x_0)=\inf_{b\in h^{-1}(h(x))}\omega(b)=\mathscr{C}_h(\omega(b))(h(x)).\end{aligned}$$

Thus $\mathscr{C}_h(\overline{\mu})(h(a)\alpha h(x)\beta h(y))\succeq\mathscr{C}_h(\overline{\mu}(b))(h(x))$ and

$\mathscr{C}_h(\omega)(h(a)\alpha h(x)\beta h(y))\leq\mathscr{C}_h(\omega(b))(h(x))$.

Hence $\mathscr{C}_h(\mathscr{A})$ is a cubic interior ideal of Y. □

Theorem 4.2. For a homomorphism $h:X\to Y$ of semigroups,

let $\mathscr{C}_h^{-1}:\mathscr{C}(Y)\to\mathscr{C}(X)$ be the inverse cubic transformation, induced by h.

If $\mathscr{A}=\langle\overline{\mu},\omega\rangle\in\mathscr{C}(Y)$ is a cubic interior ideal of Y then $\mathscr{C}_h^{-1}(B)$ is a cubic interior ideal of X.

Proof. Suppose that $\mathscr{A}=\langle\overline{\mu},\omega\rangle\in\mathscr{C}(Y)$ is a cubic interior ideal of Y,

let $x,y\in X$ and $\alpha\in\Gamma$. Then

$$\begin{aligned}\mathscr{C}_h^{-1}(\overline{\mu}(x+y)) &= \overline{\mu}(h(x+y))=\overline{\mu}(h(x)+h(y))\\ &\succeq r\min\{\overline{\mu}(h(x)),\overline{\mu}(h(y))\}\\ &= r\min\{\mathscr{C}_h^{-1}(\overline{\mu}(x)),\mathscr{C}_h^{-1}(\overline{\mu}(y))\},\end{aligned}$$

And

$$\begin{aligned}\mathscr{C}_h^{-1}(\omega(x+y)) &= \omega(h(x+y))=\omega(h(x)+h(y))\\ &\leq \max\{\omega(h(x)),\omega(h(y))\}\\ &= \max\{\mathscr{C}_h^{-1}(\omega(x)),\mathscr{C}_h^{-1}(\omega(y))\}.\end{aligned}$$

And

$$\begin{aligned}\mathscr{C}_h^{-1}(\overline{\mu}(x\alpha y)) &= \overline{\mu}(h(x\alpha y))=\overline{\mu}(h(x)\alpha h(y))\\ &\succeq r\min\{\overline{\mu}(h(x)),\overline{\mu}(h(y))\}\\ &= r\min\{\mathscr{C}_h^{-1}(\overline{\mu}(x)),\mathscr{C}_h^{-1}(\overline{\mu}(y))\},\end{aligned}$$

$$\begin{aligned}\mathscr{C}_h^{-1}(\omega(x\alpha y)) &= \omega(h(x\alpha y))=\omega(h(x)\alpha h(y))\\ &\leq \max\{\omega(h(x)),\omega(h(y))\}\\ &= \max\{\mathscr{C}_h^{-1}(\omega(x)),\mathscr{C}_h^{-1}(\omega(y))\}.\end{aligned}$$

Thus $\mathscr{C}_h^{-1}(\overline{\mu}(x+y)) \succeq r\min\{\mathscr{C}_h^{-1}(\overline{\mu}(x)), \mathscr{C}_h^{-1}(\overline{\mu}(y))\}$
and $\mathscr{C}_h^{-1}(\omega(x+y)) \leq r\min\{\mathscr{C}_h^{-1}(\omega(x)), \mathscr{C}_h^{-1}(\omega(y))\}$.
$\mathscr{C}_h^{-1}(\overline{\mu}(x\alpha y)) \succeq r\min\{\mathscr{C}_h^{-1}(\overline{\mu}(x)), \mathscr{C}_h^{-1}(\overline{\mu}(y))\}$,
$\mathscr{C}_h^{-1}(\omega(x\alpha y)) \leq r\min\{\mathscr{C}_h^{-1}(\omega(x)), \mathscr{C}_h^{-1}(\omega(y))\}$.
Hence $\mathscr{C}_h^{-1}(\mathscr{A})$ is a cubic Γ-subsemiring of S.

Let $a,x,y \in X$ and $\alpha, \beta \in \Gamma$. Then

$$\begin{aligned} \mathscr{C}_h^{-1}(\overline{\mu}(x\alpha a\beta y) &= \overline{\mu}(h(x\alpha a\beta y)) = \overline{\mu}(h(x)\alpha h(a)\beta h(y)) \\ &\succeq \overline{\mu}(h(a)) = \mathscr{C}_h^{-1}(\overline{\mu}(a)). \end{aligned}$$

And

$$\begin{aligned} \mathscr{C}_h^{-1}(\omega(x\alpha a\beta y)) &= \omega(h(x\alpha a\beta y)) = \omega(h(x)\alpha h(a)\beta h(y)) \\ &\leq \omega(h(a)) = \mathscr{C}_h^{-1}(\omega(a)). \end{aligned}$$

Thus $\mathscr{C}_h^{-1}(\overline{\mu}(x\alpha a\beta y)) \succeq \mathscr{C}_h^{-1}(\overline{\mu}(a))$ and $\mathscr{C}_h^{-1}(\omega(x\alpha a\beta y)) \leq \mathscr{C}_h^{-1}(\omega(a))$.
Therefore $\mathscr{C}_h^{-1}(B)$ is a cubic interior ideal of X. □

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CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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