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COMMON COUPLED FIXED AND COINCIDENCE POINTS RESULTS FOR RATIONAL TYPE CONTRACTION MAPPINGS IN COMPLEX VALUED S_b-METRIC SPACES

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Abstract. In this paper, we introduce a new rational type contraction mapping in complex valued S_b-metric space and find some common coupled fixed and coincidence points. Some results are also given as corollaries.

Keywords: complex valued S_b-metric space; coupled common fixed point; coupled coincidence point; rational type contraction.

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1. INTRODUCTION AND PRELIMINARIES

Azam et al. [1] introduced the concept of complex valued metric space and proved some fixed point results for a pair of mappings for contraction condition satisfying a rational expression. Moreover, Shin Min Kang et al. [2] introduced the notion of complex valued G-metric space and proved contraction principle in this space. In 2014, Nabil M. Mlaiki [3] introduced the

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complex valued S-metric space and proved the existence and the uniqueness of a common fixed point of two self mappings in this space. Recently, Priyobarta et al. [4] introduced the concept of complex valued S_b-metric space and some topological properties. They also proved some fixed point theorems. Some more results on complex valued can be seen in [5-6].

The concept of rational type contraction is one of the interest for researchers and these can be found in [7-15]. On the other hand, there are various forms of generalization of metric space in the literature. Some of them can be found in [16-24]. The concept of coupled fixed point was introduced by Guo and Lakshmikantham [25]. The concept is further used by various authors in [26-27].

In this paper, we prove some common coupled and coincidence points theorems for rational type contractive mappings in complex valued S_b-metric space.

Let $\mathbb{C}$ be the set of complex numbers and $z_1, z_2 \in \mathbb{C}$. Define a partial order $\precsim$ on $\mathbb{C}$ as follows:

$$z_1 \precsim z_2 \text{ if and only if } Re(z_1) \leq Re(z_2) \text{ and } Im(z_1) \leq Im(z_2).$$

If follows that $z_1 \precsim z_2$ if one of the following conditions is satisfied :

(C_1)**:** $Re(z_1) = Re(z_2)$ and $Im(z_1) = Im(z_2)$,

(C_2)**:** $Re(z_1) < Re(z_2)$ and $Im(z_1) = Im(z_2)$,

(C_3)**:** $Re(z_1) = Re(z_2)$ and $Im(z_1) < Im(z_2)$,

(C_4)**:** $Re(z_1) < Re(z_2)$ and $Im(z_1) < Im(z_2)$.

Particularly, we write $z_1 \precnsim z_2$ if $z_1 \neq z_2$ and one (C_2), (C_3) and (C_4) is satisfied and we write $z_1 \prec z_2$ if only (C_4) is satisfied. The following statements hold:

(1) If $a, b \in \mathbb{R}$ with $a \leq b$, then $az \prec bz$ for all $z \in \mathbb{C}$.

(2) If $0 \lesssim z_1 \precnsim z_2$, then $|z_1| < |z_2|$.

(3) If $z_1 \lesssim z_2$ and $z_2 \prec z_3$, then $z_1 \prec z_3$.

Definition 1.1. [1] *Let X be a nonempty set whereas $\mathbb{C}$ be the set of complex numbers. Suppose that the mapping $d : X \times X \to \mathbb{C}$, satisfies the following conditions:*

(d_1)**:** $0 \precsim d(x,y)$, *for all $x, y \in X$ and $d(x,y) = 0$ if and only if $x = y$;*

(d_2)**:** $d(x,y) = d(y,x)$ *for all $x, y \in X$;*

(d_3)**:** $d(x,y) \precsim d(x,z) + d(z,y)$, *for all $x, y, z \in X$.*

Then d is called a complex valued metric on X, and (X,d) is called a complex valued metric space.

Example 1. [5] *Let $X=\mathbb{C}$ be a set of complex number. Define $d:\mathbb{C}\times\mathbb{C}\to\mathbb{C}$, by*

$$d(z_1,z_2)=|x_1-x_2|+i|y_1-y_2|$$

where $z_1=x_1+iy_1$ and $z_2=x_2+iy_2$. Then (X,d) is a complex valued metric space.

Example 2. [6] *Let $X=\mathbb{C}$ be a set of complex number. Define $d:\mathbb{C}\times\mathbb{C}\to\mathbb{C}$, by*

$$d(z_1,z_2)=e^{ik}|z_1-z_2|$$

where $0\leq k\leq\frac{\pi}{2}$, $z_1=x_1+iy_1$ and $z_2=x_2+iy_2$. Then (X,d) is a complex valued metric space.

Definition 1.2. [4] *Let X be a nonempty set and $s\geq 1$ be a given real number. Suppose that a mapping $S:X^3\to\mathbb{C}$ satisfies:*

- **(CS_b1):** $0\prec S(x,y,z)$ *for all $x,y,z\in X$ with $x\neq y\neq z\neq x$,*
- **(CS_b2):** $S(x,y,z)=0\Leftrightarrow x=y=z$,
- **(CS_b3):** $S(x,x,y)=S(y,y,x)$, *for all $x,y\in X$,*
- **(CS_b4):** $S(x,y,z)\precsim s(S(x,x,a)+S(y,y,a)+S(z,z,a))$ *for all $x,y,z,a\in X$.*

Then, S is called a complex valued S_b-metric and (X,S) is called a complex valued S_b-metric space.

Definition 1.3. [4] *A complex valued S_b-metric space (X,S) is said to be symmetric if*

$$S(x,x,y)=S(y,y,x).$$

Definition 1.4. [4] *Let (X,S) be a complex valued S_b-metric space, let $\{x_n\}$ be a sequence in X.*

- **(i):** *$\{x_n\}$ is a complex valued S_b-convergent to x if for every $a\in\mathbb{C}$ with $0<a$, there exists $k\in\mathbb{N}$ such that $S(x_n,x_n,x)\prec a$ or $S(x,x,x_n)\prec a$ for all $n\succeq k$ and denoted by $\lim_{n\to\infty}x_n=x$.*
- **(ii):** *A sequence $\{x_n\}$ is called complex valued S_b-Cauchy if for every $a\in\mathbb{C}$ with $0<a$,there exists $k\in\mathbb{N}$ such that $S(x_n,x_n,x_m)\prec a$ for each $m,n\geq k$.*

(iii): *If every complex valued S_b-Cauchy sequence is complex valued S_b-convergent in (X,S), then (X,S) is said to be complex valued S_b-complete.*

Proposition 1.1. [4] *Let (X,S) be a complex valued S_b-metric space and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is complex valued S_b-convergent to x if and only if $|S(x_n,x_n,x)| \to 0$ as $n \to \infty$ or $|S(x,x,x_n)| \to 0$ as $n \to \infty$.*

Theorem 1.2. [4] *Let (X,S) be a complex valued S_b-metric space, then for a sequence $\{x_n\}$ in X and point $x \in X$, the following are equivalent*

(1): *$\{x_n\}$ is a complex valued S_b-convergent to x.*

(2): *$|S(x_n,x_n,x)| \to 0$ as $n \to \infty$.*

Theorem 1.3. [4] *Let (X,S) be a complex valued S_b-metric space and $\{x_n\}$ be a sequence in X. Then, $\{x_n\}$ is complex valued S_b-Cauchy sequence if and only if $|S(x_n,x_m,x_l)| \to 0$ as $n,m,l \to \infty$.*

Definition 1.5. [25] *An element $(x,y) \in X \times X$ is called a*

(1) coupled fixed point of a mapping $A : X \times X \to X$ if $x = A(x,y)$ and $y = A(y,x)$;

(2) coupled common fixed point of two mappings $A,B : X \times X \to X$ if $x = A(x,y) = B(x,y)$ and $y = A(y,x) = B(y,x)$.

Definition 1.6. [25, 26] *Let X be a nonempty set. An element $(x,y) \in X \times X$ is called*
i) a coupled fixed point of the mapping $f : X \times X \to X$ if $x = f(x,y)$ and $y = f(y,x)$.
ii) a coupled coincidence point of mappings $f : X \times X \to X$ and $T : X \to X$ if $T(x) = f(x,y)$ and $T(y) = f(y,x)$.
iii) a common coupled fixed point of mappings $f : X \times X \to X$ and $T : X \to X$ if $x = T(x) = f(x,y)$ and $y = T(y) = f(y,x)$.

2. Main Results

Now we prove the following theorems

Theorem 2.1. *Let (X,S) be a complete complex valued symmetric S_b-metric space with parameter $s \geq 1$ and let the mappings $f,g : X^2 \to X$ satisfying*

$$\begin{aligned} S(f(x,y),f(x,y),g(u,v)) &\precsim a_1\frac{S(x,x,u)+S(y,y,v)}{2} \\ &+a_2\frac{S(f(x,y),f(x,y),g(u,v))S(x,x,u)}{1+S(x,x,u)+S(y,y,v)} \\ &+a_3\frac{S(f(x,y),f(x,y),g(u,v))S(y,y,v)}{1+S(x,x,u)+S(y,y,v)} \\ &+a_4\frac{S(x,x,f(x,y))S(x,x,u)}{1+S(x,x,u)+S(y,y,v)} \\ &+a_5\frac{S(x,x,f(x,y))S(y,y,v)}{1+S(x,x,u)+S(y,y,v)} \\ &+a_6\frac{S(u,u,g(u,v))S(x,x,u)}{1+S(x,x,u)+S(y,y,v)} \\ &+a_7\frac{S(u,u,g(u,v))S(y,y,v)}{1+S(x,x,u)+S(y,y,v)} \qquad (1) \end{aligned}$$

for all $x,y,u,v \in X$ and $a_i \geq 0$ with $\sum_{i=1}^{7} a_i < 1$, $i = 1,2,\ldots,7$ and $s < \frac{1-a_2-a_3-a_6-a_7}{a_1+a_4+a_5}$. Then f and g have a unique common coupled fixed point in X.

Proof. Let $x_0, y_0 \in X$ be arbitrary points.

Define

$$x_{2k+1} = f(x_{2k},y_{2k}) \quad , \quad y_{2k+1} = f(y_{2k},x_{2k})$$

$$x_{2k+2} = g(x_{2k+1},y_{2k+1}) \quad , \quad y_{2k+2} = g(y_{2k+1},x_{2k+1})$$

for $k = 0,1,2,\ldots$. Then

$$\begin{aligned} S(x_{2k+1},x_{2k+1},x_{2k+2}) &= S(f(x_{2k},y_{2k}),f(x_{2k},y_{2k}),g(x_{2k+1},y_{2k+1})) \\ &\precsim a_1\frac{S(x_{2k},x_{2k},x_{2k+1})+S(y_{2k},y_{2k},y_{2k+1})}{2} \\ &+a_2\frac{S(f(x_{2k},y_{2k}),f(x_{2k},y_{2k}),g(x_{2k+1},y_{2k+1}))S(x_{2k},x_{2k},x_{2k+1})}{1+S(x_{2k},x_{2k},x_{2k+1})+S(y_{2k},y_{2k},y_{2k+1})} \\ &+a_3\frac{S(f(x_{2k},y_{2k}),f(x_{2k},y_{2k}),g(x_{2k+1},y_{2k+1}))S(y_{2k},y_{2k},y_{2k+1})}{1+S(x_{2k},x_{2k},x_{2k+1})+S(y_{2k},y_{2k},y_{2k+1})} \end{aligned}$$

$$
\begin{aligned}
& +a_4\frac{S(x_{2k},x_{2k},f(x_{2k},y_{2k}))S(x_{2k},x_{2k},x_{2k+1})}{1+S(x_{2k},x_{2k},x_{2k+1})+S(y_{2k},y_{2k},y_{2k+1})} \\
& +a_5\frac{S(x_{2k},x_{2k},f(x_{2k},y_{2k}))S(y_{2k},y_{2k},y_{2k+1})}{1+S(x_{2k},x_{2k},x_{2k+1})+S(y_{2k},y_{2k},y_{2k+1})} \\
& +a_6\frac{S(x_{2k+1},x_{2k+1},g(x_{2k+1},y_{2k+1}))S(x_{2k},x_{2k},x_{2k+1})}{1+S(x_{2k},x_{2k},x_{2k+1})+S(y_{2k},y_{2k},y_{2k+1})} \\
& +a_7\frac{S(x_{2k+1},x_{2k+1},g(x_{2k+1},y_{2k+1}))S(y_{2k},y_{2k},y_{2k+1})}{1+S(x_{2k},x_{2k},x_{2k+1})+S(y_{2k},y_{2k},y_{2k+1})} \\
= \; & a_1\frac{S(x_{2k},x_{2k},x_{2k+1})+S(y_{2k},y_{2k},y_{2k+1})}{2} \\
& +a_2\frac{S(x_{2k+1},x_{2k+1},x_{2k+2})S(x_{2k},x_{2k},x_{2k+1})}{1+S(x_{2k},x_{2k},x_{2k+1})+S(y_{2k},y_{2k},y_{2k+1})} \\
& +a_3\frac{S(x_{2k+1},x_{2k+1},x_{2k+2})S(y_{2k},y_{2k},y_{2k+1})}{1+S(x_{2k},x_{2k},x_{2k+1})+S(y_{2k},y_{2k},y_{2k+1})} \\
& +a_4\frac{S(x_{2k},x_{2k},x_{2k+1})S(x_{2k},x_{2k},x_{2k+1})}{1+S(x_{2k},x_{2k},x_{2k+1})+S(y_{2k},y_{2k},y_{2k+1})} \\
& +a_5\frac{S(x_{2k},x_{2k},x_{2k+1})S(y_{2k},y_{2k},y_{2k+1})}{1+S(x_{2k},x_{2k},x_{2k+1})+S(y_{2k},y_{2k},y_{2k+1})} \\
& +a_6\frac{S(x_{2k+1},x_{2k+1},x_{2k+2})S(x_{2k},x_{2k},x_{2k+1})}{1+S(x_{2k},x_{2k},x_{2k+1})+S(y_{2k},y_{2k},y_{2k+1})} \\
& +a_7\frac{S(x_{2k+1},x_{2k+1},x_{2k+2})S(y_{2k},y_{2k},y_{2k+1})}{1+S(x_{2k},x_{2k},x_{2k+1})+S(y_{2k},y_{2k},y_{2k+1})} \\
\precsim \; & a_1\frac{S(x_{2k},x_{2k},x_{2k+1})+S(y_{2k},y_{2k},y_{2k+1})}{2} \\
& +(a_2+a_3)S(x_{2k+1},x_{2k+1},x_{2k+2})+(a_4+a_5)S(x_{2k},x_{2k},x_{2k+1}) \\
& +(a_6+a_7)S(x_{2k+1},x_{2k+1},x_{2k+2})
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow (1-a_2-a_3-a_6-a_7)S(x_{2k+1},x_{2k+1},x_{2k+2}) &\precsim (\frac{a_1}{2}+a_4+a_5)S(x_{2k},x_{2k},x_{2k+1}) \\
&\quad +\frac{a_1}{2}S(y_{2k},y_{2k},y_{2k+1}) \\
\Rightarrow S(x_{2k+1},x_{2k+1},x_{2k+2}) &\precsim \frac{\frac{a_1}{2}+a_4+a_5}{1-a_2-a_3-a_6-a_7}S(x_{2k},x_{2k},x_{2k+1}) \\
&\quad +\frac{\frac{a_1}{2}}{1-a_2-a_3-a_6-a_7}S(y_{2k},y_{2k},y_{2k+1})
\end{aligned}
$$

(2)

Proceeding similarly one can prove that

$$\Rightarrow S(y_{2k+1},y_{2k+1},y_{2k+2}) \precsim \frac{\frac{a_1}{2}+a_4+a_5}{1-a_2-a_3-a_6-a_7}S(y_{2k},y_{2k},y_{2k+1}) + \frac{\frac{a_1}{2}}{1-a_2-a_3-a_6-a_7}S(x_{2k},x_{2k},x_{2k+1}) \tag{3}$$

Adding (2) and (3) we have

$$\begin{aligned} S(x_{2k+1},x_{2k+1},x_{2k+2}) &+ S(y_{2k+1},y_{2k+1},y_{2k+2}) \\ &\precsim \frac{a_1+a_4+a_5}{1-a_2-a_3-a_6-a_7}[S(x_{2k},x_{2k},x_{2k+1})+S(y_{2k},y_{2k},y_{2k+1})] \end{aligned}$$

Therefore

$$\begin{aligned} S(x_{2k+1},x_{2k+1},x_{2k+2}) &+ S(y_{2k+1},y_{2k+1},y_{2k+2}) \\ &\precsim h[S(x_{2k},x_{2k},x_{2k+1})+S(y_{2k},y_{2k},y_{2k+1})] \end{aligned}$$

where $h=\frac{a_1+a_4+a_5}{1-a_2-a_3-a_6-a_7}<1$.

Also, we can show that

$$\begin{aligned} S(x_{2k+2},x_{2k+2},x_{2k+3}) &+ S(y_{2k+2},y_{2k+2},y_{2k+3}) \\ &\precsim h[S(x_{2k+1},x_{2k+1},x_{2k+2})+S(y_{2k+1},y_{2k+1},y_{2k+2})] \\ &\precsim h^2[S(x_{2k},x_{2k},x_{2k+1})+S(y_{2k},y_{2k},y_{2k+1})] \end{aligned}$$

Continuing this way, we have

$$\begin{aligned} S(x_n,x_n,x_{n+1}) &+ S(y_n,y_n,y_{n+1}) \\ &\precsim h[S(x_{n-1},x_{n-1},x_n)+S(y_{n-1},y_{n-1},y_n)] \\ &\precsim h^2[S(x_{n-2},x_{n-2},x_{n-1})+S(y_{n-2},y_{n-2},y_{n-1})] \\ &\precsim \cdots \precsim h^n[S(x_0,x_0,x_1)+S(y_0,y_0,y_1)] \end{aligned}$$

If $S(x_n,x_n,x_{n+1})+S(y_n,y_n,y_{n+1})=S_n$, then

$$S_n \precsim hS_{n-1} \precsim h^2S_{n-2} \precsim \cdots \precsim h^nS_0$$

So for $m > n$,

$$
\begin{aligned}
S(x_n,x_n,x_m)+S(y_n,y_n,y_m) &\precsim s[2S(x_n,x_n,x_{n+1})+S(x_{n+1},x_{n+1},x_m) \\
&\quad +2S(y_n,y_n,y_{n+1})+S(y_{n+1},y_{n+1},y_m)] \\
&= 2s[S(x_n,x_n,x_{n+1})+S(y_n,y_n,y_{n+1})] \\
&\quad +s[S(x_{n+1},x_{n+1},x_m)+S(y_{n+1},y_{n+1},y_m)] \\
&\precsim 2s[S(x_n,x_n,x_{n+1})+S(y_n,y_n,y_{n+1})] \\
&\quad +s^2[2S(x_{n+1},x_{n+1},x_{n+2})+S(x_{n+2},x_{n+2},x_m) \\
&\quad +2S(y_{n+1},y_{n+1},y_{n+2})+S(y_{n+2},y_{n+2},y_m) \\
&= 2s[S(x_n,x_n,x_{n+1})+S(y_n,y_n,y_{n+1})] \\
&\quad +2s^2[S(x_{n+1},x_{n+1},x_{n+2})+S(y_{n+1},y_{n+1},y_{n+2}) \\
&\quad +s^2[S(x_{n+2},x_{n+2},x_m)+S(y_{n+2},y_{n+2},y_m)] \\
&\precsim 2s[S(x_n,x_n,x_{n+1})+S(y_n,y_n,y_{n+1})] \\
&\quad +2s^2[S(x_{n+1},x_{n+1},x_{n+2})+S(y_{n+1},y_{n+1},y_{n+2}) \\
&\quad +\ldots+2s^{m-n-1}[S(x_{m-2},x_{m-2},x_{m-1})+S(y_{m-2},y_{m-2},y_{m-1})] \\
&\quad +s^{m-n}[S(x_{m-1},x_{m-1},x_m)+S(y_{m-1},y_{m-1},y_m)] \\
&\precsim 2s[S(x_n,x_n,x_{n+1})+S(y_n,y_n,y_{n+1})] \\
&\quad +2s^2[S(x_{n+1},x_{n+1},x_{n+2})+S(y_{n+1},y_{n+1},y_{n+2}) \\
&\quad +2s^3[S(x_{n+2},x_{n+2},x_{n+3})+S(y_{n+2},y_{n+2},y_{n+3})] \\
&\quad +\ldots+2s^{m-n-1}[S(x_{m-2},x_{m-2},x_{m-1})+S(y_{m-2},y_{m-2},y_{m-1})] \\
&\quad +2s^{m-n}[S(x_{m-1},x_{m-1},x_m)+S(y_{m-1},y_{m-1},y_m)] \\
&\precsim 2\{sh^n+s^2h^{n+1}+s^3h^{n+2}+\ldots+s^{m-n}h^{m-1}\}S_0 \\
&< 2sh^n[1+sh+(sh)^2+\ldots]S_0 \\
&= \frac{2sh^n}{1-sh}S_0 \to 0 \text{ as } n\to\infty
\end{aligned}
$$

which shows that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in X. As X is complete S_b-metric space, so there exists $x,y \in X$ such that $x_n \to x$ and $y_n \to y$ as $n \to \infty$.

Now we will prove that $x = f(x,y)$ and $y = f(y,x)$. On the contrary suppose that $x \neq f(x,y)$ and $y \neq f(y,x)$. Then $S(x,x,f(x,y)) = l_1 > 0$ and $S(y,y,f(y,x)) = l_2 > 0$.
Using inequality (1)

$$\begin{aligned}
l_1 &= S(x,x,f(x,y) \\
&\precsim s[2S(x,x,x_{n+1}) + S(x_{n+1},x_{n+1},f(x,y))] \\
&= s[2S(x,x,x_{n+1}) + S(f(x_n,y_n),f(x_n,y_n),f(x,y))] \\
&\precsim 2sS(x,x,x_{n+1}) + s\Bigg[a_1\frac{S(x_n,x_n,x)+S(y_n,y_n,y)}{2} \\
&\quad +a_2\frac{S(f(x_n,y_n),f(x_n,y_n),f(x,y))S(x_n,x_n,x)}{1+S(x_n,x_n,x)+S(y_n,y_n,y)} \\
&\quad +a_3\frac{S(f(x_n,y_n),f(x_n,y_n),f(x,y))S(y_n,y_n,y)}{1+S(x_n,x_n,x)+S(y_n,y_n,y)} \\
&\quad +a_4\frac{S(x_n,x_n,f(x_n,y_n))S(x_n,x_n,x)}{1+S(x_n,x_n,x)+S(y_n,y_n,y)} + a_5\frac{S(x_n,x_n,f(x_n,y_n))S(y_n,y_n,y)}{1+S(x_n,x_n,x)+S(y_n,y_n,y)} \\
&\quad +a_6\frac{S(x,x,f(x,y))S(x_n,x_n,x)}{1+S(x_n,x_n,x)+S(y_n,y_n,y)} + a_7\frac{S(x,x,f(x,y))S(y_n,y_n,y)}{1+S(x_n,x_n,x)+S(y_n,y_n,y)}\Bigg].
\end{aligned}$$

Since $\{x_n\}$ and $\{y_n\}$ are convergent to x and y, therefore by taking limit as $n \to \infty$ we get $l_1 \leq 0$, which is a contradiction, so $|S(x,x,f(x,y)| = 0$ which gives $x = f(x,y)$.

Similarly, we can prove that $y = f(y,x)$. Also, we can prove that $x = g(x,y)$ and $y = g(y,x)$. Hence (x,y) is a common coupled fixed point of f and g.

In order to prove the uniqueness of the coupled fixed point, if possible let (p,q) be the second common coupled fixed point of f and g.

Then by using inequality (1), we have

$$\begin{aligned}
S(x,x,p) &= S(f(x,y),f(x,y),g(p,q)) \\
&\precsim \frac{a_1}{2}\{S(x,x,p)+S(y,y,q)\} \\
&\quad +a_2\frac{S(f(x,y),f(x,y),g(p,q))S(x,x,p)}{1+S(x,x,p)+S(y,y,q)} + a_3\frac{S(f(x,y),f(x,y),g(p,q))S(y,y,q)}{1+S(x,x,p)+S(y,y,q)} \\
&\quad +a_4\frac{S(x,x,f(x,y))S(x,x,p)}{1+S(x,x,p)+S(y,y,q)} + a_5\frac{S(x,x,f(x,y))S(y,y,q)}{1+S(x,x,p)+S(y,y,q)}
\end{aligned}$$

$$
\begin{aligned}
&+a_6\frac{S(p,p,g(p,q))S(x,x,p)}{1+S(x,x,p)+S(y,y,q)}+a_7\frac{S(p,p,g(p,q)))S(y,y,q)}{1+S(x,x,p)+S(y,y,q)}\\
= \;& \frac{a_1}{2}\{S(x,x,p)+S(y,y,q)\}+a_2\frac{S(x,x,p)S(x,x,p)}{1+S(x,x,p)+S(y,y,q)}\\
&+a_3\frac{S(x,x,p)S(y,y,q)}{1+S(x,x,p)+S(y,y,q)}+a_4\frac{S(x,x,x)S(x,x,p)}{1+S(x,x,p)+S(y,y,q)}\\
&+a_5\frac{S(x,x,x)S(y,y,q)}{1+S(x,x,p)+S(y,y,q)}+a_6\frac{S(p,p,p)S(x,x,p)}{1+S(x,x,p)+S(y,y,q)}\\
&+a_7\frac{S(p,p,p)S(y,y,q)}{1+S(x,x,p)+S(y,y,q)}
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow S(x,x,p) &\precsim \frac{a_1}{2}\{S(x,x,p)+S(y,y,q)\}+a_2S(x,x,p)+a_3S(x,x,p)\\
\Rightarrow (1-a_2-a_3)S(x,x,p) &\precsim \frac{a_1}{2}S(x,x,p)+\frac{a_1}{2}S(y,y,q)\\
\Rightarrow (1-\frac{a_1}{2}-a_2-a_3)S(x,x,p) &\precsim \frac{a_1}{2}S(y,y,q)\\
\Rightarrow S(x,x,p) &\precsim \frac{a_1}{2-a_1-2a_2-2a_3}S(y,y,q) \qquad (4)
\end{aligned}
$$

Similarly,

$$
S(y,y,q) \precsim \frac{a_1}{2-a_1-2a_2-2a_3}S(x,x,p) \qquad (5)
$$

Adding (4) and (5) we have

$$
\begin{aligned}
S(x,x,p)+S(y,y,q) &\precsim \frac{a_1}{2-a_1-2a_2-2a_3}[S(x,x,p)+S(y,y,q)]\\
\Rightarrow [1-\frac{a_1}{2-a_1-2a_2-2a_3}][S(x,x,p)+S(y,y,q)] &\precsim 0\\
\Rightarrow \frac{2(1-a_1-a_2-a_3)}{2-a_1-2a_2-a_3}[S(x,x,p)+S(y,y,q)] &\precsim 0
\end{aligned}
$$

Since $a_1+a_2+a_3<1$, $\frac{2(1-a_1-a_2-a_3)}{2-a_1-2a_2-2a_3}>0$.

Hence $|S(x,x,p)+S(y,y,q)|=0$,

which implies that $x=p$ and $y=q \Rightarrow (x,y)=(p,q)$.

Thus f and g have unique coupled common fixed point. This completes the proof.

□

Corollary 2.1. *Let (X,S) be a complete complex valued symmetric S_b-metric space with parameter $s \geq 1$ and let the mapping $f : X^2 \to X$ satisfying*

$$\begin{aligned} S(f(x,y),f(x,y),f(u,v)) \precsim\ & a_1 \frac{S(x,x,u)+S(y,y,v)}{2} \\ & +a_2 \frac{S(f(x,y),f(x,y),f(u,v))S(x,x,u)}{1+S(x,x,u)+S(y,y,v)} \\ & +a_3 \frac{S(f(x,y),f(x,y),f(u,v))S(y,y,v)}{1+S(x,x,u)+S(y,y,v)} \\ & +a_4 \frac{S(x,x,f(x,y))S(x,x,u)}{1+S(x,x,u)+S(y,y,v)} + a_5 \frac{S(x,x,f(x,y))S(y,y,v)}{1+S(x,x,u)+S(y,y,v)} \\ & +a_6 \frac{S(u,u,f(u,v))S(x,x,u)}{1+S(x,x,u)+S(y,y,v)} + a_7 \frac{S(u,u,f(u,v))S(y,y,v)}{1+S(x,x,u)+S(y,y,v)} \end{aligned}$$

for all $x,y,u,v \in X$ and $a_i \geq 0$ with $\sum_{i=1}^{7} a_i < 1$, $i = 1,2,\ldots,7$. Then f has a unique coupled fixed point in X.

Theorem 2.2. *Let (X,S) be a complete complex valued symmetric S_b-metric space with parameter $s \geq 1$ and let the mappings $f,g : X^2 \to X$ satisfy*

$$\begin{aligned} S(f(x,y),f(x,y),g(u,v)) \precsim\ & \beta_1 \frac{S(x,x,u)+S(y,y,v)}{2} \\ & +\beta_2 \frac{S(x,x,f(x,y))S(u,u,g(u,v))}{1+s[S(x,x,g(x,y)+S(u,u,f(u,v))+S(x,x,u)+S(y,y,v))]} \end{aligned} \tag{6}$$

for all $x,y,u,v \in X$ and β_1, β_2 are non-negative real numbers with $\beta_1+\beta_2 < 1$ and $s < \frac{1-\beta_2}{\beta_1}$. Then f and g have unique common coupled fixed point.

Proof. Let $x_0.y_0$ be arbitrary points. Define

$$x_{2k+1} = f(x_{2k},x_{2k}) \quad , \quad y_{2k+1} = f(y_{2k},x_{2k})$$

$$x_{2k+2} = g(x_{2k+1},y_{2k+1}) \quad , \quad y_{2k+2} = g(y_{2k+1},x_{2k+1})$$

for $k = 0,1,2,\ldots$. Then

$$\begin{aligned} & S(x_{2k+1},x_{2k+1},x_{2k+2}) \\ & = S(f(x_{2k},y_{2k}),f(x_{2k},y_{2k}),g(x_{2k+1},y_{2k+1})) \end{aligned}$$

$$\precsim \beta_1 \frac{S(x_{2k},x_{2k},x_{2k+1})+S(y_{2k},y_{2k},y_{2k+1})}{2}$$
$$+\beta_2 \frac{S(x_{2k},x_{2k},f(x_{2k},y_{2k}))S(x_{2k+1},x_{2k+1},g(x_{2k+1},y_{2k+1}))}{1+s[S(x_{2k},x_{2k},g(x_{2k+1},y_{2k+1}))+S(x_{2k+1},x_{2k+1},f(x_{2k},y_{2k}))+S(x_{2k},x_{2k},x_{2k+1})+S(y_{2k},y_{2k},y_{2k+1}])}$$
$$= \beta_1 \frac{S(x_{2k},x_{2k},x_{2k+1})+S(y_{2k},y_{2k},y_{2k+1})}{2}$$
$$+\beta_2 \frac{S(x_{2k},x_{2k},x_{2k+1})S(x_{2k+1},x_{2k+1},x_{2k+2})}{1+s[S(x_{2k},x_{2k},x_{2k+2})+S(x_{2k+1},x_{2k+1},x_{2k+1})+S(x_{2k},x_{2k},x_{2k+1})+S(y_{2k},y_{2k},y_{2k+1})]}$$
$$\precsim \frac{\beta_1}{2}\{S(x_{2k},x_{2k},x_{2k+1})+S(y_{2k},y_{2k},y_{2k+1})\}+\beta_2 S(x_{2k+1},x_{2k+1},x_{2k+2})$$

$$\Rightarrow (1-\beta_2)S(x_{2k+1},x_{2k+1},x_{2k+2}) \precsim \frac{\beta_1}{2}[S(x_{2k},x_{2k},x_{2k+1})+S(y_{2k},y_{2k},y_{2k+1})]$$
$$\Rightarrow S(x_{2k+1},x_{2k+1},x_{2k+2}) \precsim \frac{\beta_1}{2(1-\beta_2)}[S(x_{2k},x_{2k},x_{2k+1})+S(y_{2k},y_{2k},y_{2k+1})] \tag{7}$$

Similarly we can show that

$$S(y_{2k+1},y_{2k+1},y_{2k+2}) \precsim \frac{\beta_1}{2(1-\beta_2)}[S(x_{2k},x_{2k},x_{2k+1})+S(y_{2k},y_{2k},y_{2k+1})] \tag{8}$$

Adding (7) and (8) we have

$$\begin{aligned} S(x_{2k+1},x_{2k+1},x_{2k+2}) &+ S(y_{2k+1},y_{2k+1},y_{2k+2}) \\ &\precsim \frac{\beta_1}{1-\beta_2}[S(x_{2k},x_{2k},x_{2k+1})+S(y_{2k},y_{2k},y_{2k+1})] \\ &= k[S(x_{2k},x_{2k},x_{2k+1})+S(y_{2k},y_{2k},y_{2k+1})] \end{aligned}$$

where $k=\frac{\beta_1}{1-\beta_2}$.

Similarly, we can show that

$$\begin{aligned} S(x_{2k+2},x_{2k+2},x_{2k+3}) &+ S(y_{2k+2},y_{2k+2},y_{2k+3}) \\ &\precsim k[S(x_{2k+1},x_{2k+1},x_{2k+2})+S(y_{2k+1},y_{2k+1},y_{2k+2})] \\ &\precsim k^2[S(x_{2k},x_{2k},x_{2k+1})+S(y_{2k},y_{2k},y_{2k+1})] \end{aligned}$$

Now, if $S(x_n,x_n,x_{n+1})+S(y_n,y_n,y_{n+1})=S_n$ then

$$S_n \precsim kS_{n-1} \precsim k^2 S_{n-2} \precsim \cdots \precsim k^n S_0$$

So, for $m>n$ we have

$$\begin{aligned}
S(x_n,x_n,x_m)+S(y_n,y_n,y_m) &\precsim s[2S(x_n,x_n,x_{n+1})+S(x_{n+1},x_{n+1},x_m) \\
&\quad +2S(y_n,y_n,y_{n+1})+S(y_{n+1},y_{n+1},y_m)] \\
&= s[2S(x_n,x_n,x_{n+1})+2S(y_n,y_n,y_{n+1})] \\
&\quad +s[S(x_{n+1},x_{n+1},x_m)+S(y_{n+1},y_{n+1},y_m)] \\
&\precsim 2s[S(x_n,x_n,x_{n+1})+S(y_n,y_n,y_{n+1})] \\
&\quad +s^2[2S(x_{n+1},x_{n+1},x_{n+2})+S(x_{n+2},x_{n+2},x_m) \\
&\quad +2S(y_{n+1},y_{n+1},y_{n+2})+S(y_{n+2},y_{n+2},y_m) \\
&= 2s[S(x_n,x_n,x_{n+1})+S(y_n,y_n,y_{n+1})] \\
&\quad +2s^2[S(x_{n+1},x_{n+1},x_{n+2})+S(y_{n+1},y_{n+1},y_{n+2}) \\
&\quad +s[S(x_{n+2},x_{n+2},x_m)+S(y_{n+2},y_{n+2},y_m)] \\
&\precsim 2s[S(x_n,x_n,x_{n+1})+S(y_n,y_n,y_{n+1})] \\
&\quad +2s^2[S(x_{n+1},x_{n+1},x_{n+2})+S(y_{n+1},y_{n+1},y_{n+2}) \\
&\quad +\cdots+2s^{m-n-1}[S(x_{m-2},x_{m-2},x_{m-1})+S(y_{m-2},y_{m-2},y_{m-1})] \\
&\quad +s^{m-n}[S(x_{m-1},x_{m-1},x_m)+S(y_{m-1},y_{m-1},y_m)] \\
&\precsim 2s[S(x_n,x_n,x_{n+1})+S(y_n,y_n,y_{n+1})] \\
&\quad +2s^2[S(x_{n+1},x_{n+1},x_{n+2})+S(y_{n+1},y_{n+1},y_{n+2}) \\
&\quad +\cdots+2s^{m-n-1}[S(x_{m-2},x_{m-2},x_{m-1})+S(y_{m-2},y_{m-2},y_{m-1})] \\
&\quad +2s^{m-n}[S(x_{m-1},x_{m-1},x_m)+S(y_{m-1},y_{m-1},y_m)] \\
&\precsim 2\{sk^n+s^2k^{n+1}+\ldots+s^{m-n}k^{m-1}\}S_0 \\
&< 2sk^n[1+sk+(sk)^2+\ldots]S_0 \\
&= \frac{2sk^n}{1-sk}S_0 \to 0 \text{ as } n\to\infty
\end{aligned}$$

Therefore $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in X. Since X is complete S_b-metric space, there exist $x,y \in X$ such that $x_n \to x$ and $y_n \to y$ as $n \to \infty$.

Now, we will show that $x = f(x,y)$ and $y = f(y,x)$. Suppose on contrary that $x \neq f(x,y)$ and $y \neq f(y,x)$, so that $S(x,x,f(x,y)) = l_1 > 0$ and $S(y,y,f(y,x)) = l_2 > 0$. Consider the following and using inequality (6), we get

$$\begin{aligned}
l_1 &= S(x,x,f(x,y) \\
&\precsim s[2S(x,x,x_{n+1}) + S(x_{n+1},x_{n+1},f(x,y))] \\
&= sS(x,x,x_{n+1}) + sS(f(x_n,y_n),f(x_n,y_n),f(x,y)) \\
&\precsim sS(x,x,x_{n+1}) + s\Bigg[\beta_1 \frac{S(x_n,x_n,x) + S(y_n,y_n,y)}{2} \\
&\quad + \beta_2 \frac{S(x_n,x_n,f(x_n,y_n))S(x,x,f(x,y))}{1 + s[S(x_n,x_n,f(x,y)) + S(x,x,f(x_n,y_n)) + S(x_n,x_n,x) + S(y_n,y_n,y)} \\
&= sS(x,x,x_{n+1}) + \frac{s\beta_1}{2}[S(x_n,x_n,x) + S(y_n,y_n,y)] \\
&\quad + \beta_2 \frac{S(x_n,x_n,x_{n+1})S(x,x,f(x,y))}{1 + s[S(x_n,x_n,f(x,y)) + S(x,x,x_{n+1}) + S(x_n,x_n,x) + S(y_n,y_n,y)}
\end{aligned}$$

Taking the limit as $n \to \infty$ we get

$$S(x,x,f(x,y)) \leq 0$$

Therefore

$$S(x,x,f(x,y)) = 0$$

which implies that $x = f(x,y)$. Similarly, we can prove that $y = f(y,x)$. Also, we can prove that $x = g(x,y)$ and $y = g(y,x)$. Hence, (x,y) is a common coupled fixed point of f and g.

In order to prove the uniqueness of the common coupled fixed point of f and g, if possible let (p,q) be the second common coupled fixed point of f and g.

Then by using inequality (6), we have

$$\begin{aligned}
S(x,x,p) &= S(f(x,y),f(x,y),g(p,q)) \\
&\precsim \frac{\beta_1}{2}\{S(x,x,p) + S(y,y,q)\} \\
&\quad + \beta_2 \frac{S(x,x,f(x,y))S(p,p,g(p,q))}{1 + s[S(x,x,g(p,q)) + S(p,p,f(x,y)) + S(x,x,p) + S(y,y,q)]}
\end{aligned}$$

$$
\begin{aligned}
\Rightarrow S(x,x,p) &\precsim \frac{\beta_1}{2}\{S(x,x,p)+S(y,y,q)\} \\
\Rightarrow (1-\frac{\beta_1}{2})S(x,x,p) &\precsim \frac{\beta_1}{2}S(y,y,q) \\
\Rightarrow S(x,x,p) &\precsim \frac{\beta_1}{2-\beta_1}S(y,y,q) \qquad (9)
\end{aligned}
$$

Similarly

$$S(y,y,q) \precsim \frac{\beta_1}{2-\beta_1}S(x,x,p) \qquad (10)$$

Adding (9) and (10) we have

$$
\begin{aligned}
S(x,x,p)+S(y,y,q) &\precsim \frac{\beta_1}{2-\beta_1}[S(x,x,p)+S(y,y,q)] \\
\Rightarrow (1-\frac{\beta_1}{2-\beta_1})|S(x,x,p)+S(y,y,q)| &\leq 0 \\
\Rightarrow \frac{2(1-\beta_1)}{2-\beta_1}|S(x,x,p)+S(y,y,q)| &\leq 0.
\end{aligned}
$$

But $\frac{2(1-\beta_1)}{2-\beta_1} > 0$. Therefore $|S(x,x,p)+S(y,y,q)| = 0$. Which implies that $x = p$ and $y = q$ $\Rightarrow (x,y) = (p,q)$. Thus f and g have a unique common coupled fixed point. □

Corollary 2.2. *Let (X,S) be a complete complex valued symmetric S_b-metric space with parameter $s \geq 1$ and let the mapping $f : X^2 \to X$ satisfying*

$$
\begin{aligned}
S(f(x,y),f(x,y),f(u,v)) \precsim\ & \beta_1\frac{S(x,x,u)+S(y,y,v)}{2} \\
&+\beta_2\frac{S(x,x,f(x,y))S(u,u,f(u,v))}{1+s[S(x,x,f(u,v)+S(u,u,f(x,y))+S(x,x,u)+S(y,y,v))]}
\end{aligned}
$$

for all $x,y,u,v \in X$ and β_1,β_2 are non-negative real numbers with $\beta_1+\beta_2 < 1$. Then f has a unique coupled fixed point.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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