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OSCULATING SURFACES ALONG A CURVE ON A SURFACE IN EUCLIDEAN 3-SPACE

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Abstract. We define an osculating surface to a surface along a curve on the surface in Euclidean 3-space \mathbb{E}^3 . Then, we analyze the necessary and sufficient condition for that surface to be ruled surface. Finally, we illustrate the convenience and efficiency of this approach by some representative examples.

Keywords: Darboux frame; ruled surfaces; marching-scale functions.

2010 AMS Subject Classification: 53A04, 53A05, 53A17.

1. INTRODUCTION

The problem of finding surfaces with a given common curve as a special curve play an important role in geometric design. The first paper related with this type of problem proposed by Wang et.al. [1]. They parameterized the surface by using the Serret–Frenet frame of the given curve and gave the necessary and sufficient condition to satisfy the geodesic requirement. The basic idea is to regard the wanted surface as an extension from the given characteristic curve, and represent it as a linear combination of the marching-scale functions $u(s, t)$, $v(s, t)$, $w(s, t)$ and the three vector functions $\mathbf{t}(s)$, $\mathbf{n}(s)$, $\mathbf{b}(s)$, which are the unit tangent, the principal

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normal and the binormal vector of the curve respectively. With the given geodesic curve and isoparametric constraints, they derived the necessary and sufficient conditions for the correct parametric representation of the surface pencil. The extension to ruled and developable surfaces is also outlined. Kasap et al. [2] generalized the marching-scale functions of Wang and gave a sufficient condition for a given curve to be a geodesic on a surface. With the inspiration of work of Wang, Li et al. [3] changed the characteristic curve from geodesic to line of curvature and defined the surface pencil with a common line of curvature. Bayram et al. [4] tackled the problem of constructing surfaces passing through a given asymptotic curve. Important contributions to surface passing through a given curve have been studied in [5, 6, 7, 8].

However, the relevant work on surfaces through characteristic curve on a surface depending on the Darboux frame is rare. So, this led us to offer an approach for designing a surface possessing a given curve on a surface. We call it an osculating surface along the curve. Then, we analyze the necessary and sufficient condition for that surface to be an osculating ruled surface. Moreover, we illustrate the convenience and efficiency of this approach by some representative examples.

2. PRELIMINARIES

In this section, we list some notions, formulas and conclusions for space curves, and ruled surfaces in Euclidean 3-space \mathbb{E}^3 which can be found in the textbooks on differential geometry (See for details [9, 10]).

Let $\alpha : I \subseteq \mathbb{R} \rightarrow \mathbb{E}^3$ be a unit speed curve; $\kappa(s)$ and $\tau(s)$ denote the natural curvature and torsion of $\alpha = \alpha(s)$, respectively. We assume $\alpha''(s) \neq 0$ for all $s \in [0, L]$, since this would give us a straight line. In this paper, $\alpha'(s)$ denote the derivative of α with respect to arc length parameter s . For each point of $\alpha(s)$, the set $\{\mathbf{t}(s), \mathbf{n}(s), \mathbf{b}(s)\}$ is called the Serret–Frenet frame along $\alpha(s)$, where $\mathbf{t}(s) = \alpha'(s)$ is the unit tangent, $\mathbf{n}(s) = \alpha''(s) / \|\alpha''(s)\|$ is the unit principal normal, and $\mathbf{b}(s) = \mathbf{t}(s) \times \mathbf{n}(s)$ is the unit binormal vector. The arc-length derivative of the Serret–Frenet frame is governed by the relations:

$$(1) \quad \begin{pmatrix} \mathbf{t}'(s) \\ \mathbf{n}'(s) \\ \mathbf{b}'(s) \end{pmatrix} = \begin{pmatrix} 0 & \kappa(s) & 0 \\ -\kappa(s) & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{pmatrix} \begin{pmatrix} \mathbf{t}(s) \\ \mathbf{n}(s) \\ \mathbf{b}(s) \end{pmatrix},$$

Let M be a regular surface, and $\alpha : I \subseteq \mathbb{R} \rightarrow M$ is a unit speed curve on M . If we denote the Darboux frame along the curve $\alpha = \alpha(s)$ by $\{\mathbf{e}_1(s), \mathbf{e}_2(s), \mathbf{e}_3(s)\}$; $\mathbf{t} = \mathbf{e}_1(s)$ be the unit tangent vector, $\mathbf{e}_3 = \mathbf{e}_3(s)$ is the surface unit normal restricted to α , and $\mathbf{e}_2 = \mathbf{e}_3 \times \mathbf{e}_1$ be the unit tangent to the surface M . Then, the rotation matrix between Serret–Frenet frame and Darboux frame is

$$(2) \quad \begin{pmatrix} \mathbf{t}(s) \\ \mathbf{n}(s) \\ \mathbf{b}(s) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \vartheta & \sin \vartheta \\ 0 & -\sin \vartheta & \cos \vartheta \end{pmatrix} \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}.$$

Hence, we have the derivative formulae of the Darboux frame as:

$$(3) \quad \begin{pmatrix} \mathbf{e}'_1 \\ \mathbf{e}'_2 \\ \mathbf{e}'_3 \end{pmatrix} = \begin{pmatrix} 0 & \kappa_g & \kappa_n \\ -\kappa_g & 0 & \tau_g \\ -\kappa_n & -\tau_g & 0 \end{pmatrix} \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix},$$

where

$$(4) \quad \left. \begin{aligned} \kappa_n &= \kappa \sin \vartheta = \langle \mathbf{e}'_1, \mathbf{e}_3 \rangle, \\ \kappa_g &= \kappa \cos \vartheta = \det(\alpha', \alpha'', \mathbf{e}_2), \\ \tau_g &= \tau + \vartheta' = \det(\alpha', \mathbf{e}_2, \mathbf{e}'_2). \end{aligned} \right\}$$

We call $\kappa_g = \kappa_g(s)$ a geodesic curvature, $\kappa_n = \kappa_n(s)$ a normal curvature, and $\tau_g = \tau - \vartheta'$ a geodesic torsion of $\alpha(s)$, respectively. In terms of these quantities, the geodesics, asymptotic lines, and line of curvatures on a smooth surface may be characterized, as loci along which $\kappa_g = 0$, $\kappa_n = 0$, and $\tau_g = 0$, respectively. Further, we have:

$$(5) \quad \left. \begin{aligned} \kappa(s) &= \sqrt{\kappa_g^2 + \kappa_n^2}, \\ \tau_g(s) &= \vartheta' + \tau. \end{aligned} \right\}$$

3. OSCULATING SURFACES

In this section, we consider an osculating surface along a regular curve $\alpha = \alpha(s)$ on the surface M such that the surface tangent plane is coincident with the subspace $Sp\{\mathbf{e}_1, \mathbf{e}_2\}$, that is expressing the surface in terms of the local Darboux frame along $\alpha(s)$ as:

$$(6) \quad M_O : \mathbf{P}(s, t) = \alpha(s) + u(s, t)\mathbf{e}_1(s) + v(s, t)\mathbf{e}_2(s); \quad 0 \leq t \leq T, \quad 0 \leq s \leq L,$$

where $u(s, t)$, and $v(s, t)$ are all C^1 functions. If the parameter t is seen as the time, the functions $u(s, t)$, and $v(s, t)$ can then be viewed as directed marching distances of a point unit in the time t in the direction \mathbf{e}_1 ; and \mathbf{e}_2 , respectively, and the position vector $\alpha(s)$ is seen as the initial location of this point on M . It is easily checked that the two tangent vectors of M_O are given by:

$$(7) \quad \left. \begin{aligned} \mathbf{P}_s(s, t) &= (1 + u_s - v\kappa_g)\mathbf{e}_1 + (u\kappa_g + u\kappa_n)\mathbf{e}_2 + (v\tau_g + u\kappa_n)\mathbf{e}_3, \\ \mathbf{P}_t(s, t) &= u_t\mathbf{e}_1 + v_t\mathbf{e}_2. \end{aligned} \right\}$$

The lower case subscript letters s , and t denote partial derivatives corresponding to the indicated variable, e.g., $\mathbf{P}_s = \frac{\partial \mathbf{P}}{\partial s}$, $\mathbf{P}_t = \frac{\partial \mathbf{P}}{\partial t}$. Thus, the normal vector of M_O is

$$(8) \quad \mathbf{N}(s, t) := \mathbf{P}_s \times \mathbf{P}_t = \eta_1(s, t)\mathbf{e}_1 + \eta_2(s, t)\mathbf{e}_2 + \eta_3(s, t)\mathbf{e}_3,$$

where

$$(9) \quad \left. \begin{aligned} \eta_1(s, t) &= -v_t(u\kappa_n + v\tau_g), \\ \eta_2(s, t) &= u_t(u\kappa_n + v\tau_g), \\ \eta_3(s, t) &= v_t(1 + u_s - v\kappa_g) - u_t(u_s + u\kappa_g). \end{aligned} \right\}$$

Our goal is to find the necessary and sufficient conditions for which the surface M_O is osculating to the surface M along $\alpha(s)$. First, since $\alpha(s)$ is an isoparametric curve on the surface M_O , there exists a parameter $t_0 \in [0, T]$ such that $\mathbf{P}(s, t_0) = \alpha(s)$, $0 \leq t_0 \leq T$, $0 \leq s \leq L$, that is,

$$(10) \quad \left. \begin{aligned} u(s, t_0) &= v(s, t_0) = 0, \\ u_s(s, t_0) &= v_s(s, t_0) = 0. \end{aligned} \right\}$$

Secondly, when $t = t_0$, i.e., along the curve $\alpha(s)$ of M , the surface normal is

$$(11) \quad \mathbf{N}(s, t_0) = v_t(s, t_0)\mathbf{e}_3.$$

This is the reason why we call M_O , the osculating surface of M along the curve $\alpha(s)$. Then we have the following theorem.

Theorem 1. The surface M_O is an osculating surface along the curve $\alpha(s)$ of M if and

only if

$$(12) \quad \left. \begin{aligned} u(s, t_0) = v(s, t_0) = 0, \\ u_s(s, t_0) = v_s(s, t_0) = 0, \\ v_t(s, t_0) \neq 0, \quad 0 \leq t_0 \leq T, \quad 0 \leq s \leq L. \end{aligned} \right\}$$

We will call the set of surfaces defined by Eqs. (6) and (12) isoparametric osculating surfaces, since the common curve is an isoparametric curve on these surfaces. Any osculating surface M_O defined by Eq. (6) and satisfying Eqs. (12) is a member of them. For the purposes of simplification and better analysis, next we study the case when the marching-scale functions $u(s, t)$, and $w(s, t)$ can be written as follows:

$$(13) \quad \begin{aligned} u(s, t) &= l(s)U(t), \\ v(s, t) &= n(s)V(t). \end{aligned}$$

Here $l(s)$, $n(s)$, $U(t)$ and $V(t)$ are C^1 functions, not identically zero. Thus, from Theorem 1, we can get the following corollary.

Corollary 1. The necessary and sufficient condition for M_O being an osculating surface along the curve $\alpha(s)$ of M is:

$$(14) \quad \left. \begin{aligned} U(t_0) = V(t_0) = 0, \quad l(s) = \text{const.} \neq 0, \quad n(s) = \text{const.} \neq 0, \\ \frac{dV(t_0)}{dt} = \text{const.} \neq 0, \quad 0 \leq t_0 \leq T, \quad 0 \leq s \leq L. \end{aligned} \right\}$$

Note that to obtain the osculating surfaces, we can first design the marching-scale functions in Eq. (14), and then apply them to Eq. (6) to derive the final parameterizations. For convenience in practice, the marching-scale functions can be further constrained to be in more restricted forms and still possess enough degrees of freedom to define a large class of osculating surfaces along the curve $\alpha(s)$ of M . Specifically, let us suppose that $u(s, t)$, and $v(s, t)$ can be chosen in two different forms:

(1) If we choose

$$(15) \quad \left\{ \begin{aligned} u(s, t) &= \sum_{k=1}^p a_{1k} l(s)^k U(t)^k, \\ v(s, t) &= \sum_{k=1}^p a_{2k} m(s)^k V(t)^k. \end{aligned} \right.$$

Thus, we can simply express the sufficient condition for which M_O being an osculating surface along the curve $\alpha(s)$ of M as:

$$(16) \quad \begin{cases} U(t_0) = V(t_0) = 0, \\ a_{21} \neq 0, m(s) \neq 0, \text{ and } \frac{dV(t_0)}{dt} \neq 0, \end{cases}$$

where $l(s), m(s), U(t)$, and $V(t)$ are C^1 functions, $a_{ij} \in \mathbb{R}$ ($i = 1, 2; j = 1, 2, \dots, p$) and $l(s)$, and $n(s)$ are not identically zero.

(2) If we choose

$$(17) \quad \begin{cases} u(s, t) = f\left(\sum_{k=1}^p a_{1k} l(s)^k U(t)^k\right), \\ v(s, t) = g\left(\sum_{k=1}^p a_{2k} n(s)^k V(t)^k\right), \end{cases}$$

then, we can rewrite the condition (14) as:

$$(18) \quad \begin{cases} U(t_0) = V(t_0) = v(t_0) = f(0) = g(0) = 0, \\ a_{21} \neq 0, \frac{dV(t_0)}{dt} = \text{const} \neq 0, n(s) \neq 0, g'(0) \neq 0, \end{cases}$$

where $l(s), n(s), U(t), V(t), f$, and g are C^1 functions.

Example 1. We consider a surface of revolution parameterized by

$$M : \mathbf{X}(s, t) = (s, e^s \sin t, e^s \cos t).$$

The curve

$$\alpha(s) = (s, e^s \sin s^2, e^s \cos s^2), \quad s \in \mathbb{R},$$

is a regular curve on the surface M . In this case, the Darboux frame is computed as follows:

$$\mathbf{e}_1(s) = \frac{\dot{\alpha}(s)}{\|\dot{\alpha}(s)\|} = \left(\frac{1}{\sqrt{1 + e^{2s}(1 + 4s^2)}}, \frac{e^s(2s \cos s^2 + \sin s^2)}{\sqrt{1 + e^{2s}(1 + 4s^2)}}, \frac{e^s(\cos s^2 - 2s \sin s^2)}{\sqrt{1 + e^{2s}(1 + 4s^2)}} \right),$$

and

$$\begin{aligned} \mathbf{e}_3(s) &= \frac{\mathbf{X}_s \times \mathbf{X}_t}{\|\mathbf{X}_s \times \mathbf{X}_t\|} \\ &= \left(-\frac{e^{2s}}{\sqrt{e^{2s} + e^{4s}}}, \frac{e^s \sin s^2}{\sqrt{e^{2s} + e^{4s}}}, \frac{e^s \cos s^2}{\sqrt{e^{2s} + e^{4s}}} \right), \end{aligned}$$

$$\mathbf{e}_2(s) = \begin{pmatrix} -\frac{2s e^{2s}}{\sqrt{e^{2s}+e^{4s}}\sqrt{1+e^{2s}(1+4s^2)}}, \\ \frac{e^s(\cos(s^2)+e^{2s}\cos(s^2)-2s e^{2s}\sin(s^2))}{\sqrt{e^{2s}+e^{4s}}\sqrt{1+e^{2s}(1+4s^2)}}, \\ -\frac{e^s(2s e^{2s}\cos(s^2)+\sin(s^2)+e^{2s}\sin(s^2))}{\sqrt{e^{2s}+e^{4s}}\sqrt{1+e^{2s}(1+4s^2)}} \end{pmatrix}.$$

Using Eq.(6), the osculating surface family can be represented as:

$$M_O : P(s,t) = \begin{pmatrix} s, \\ e^s \sin(s^2), \\ e^s \cos(s^2) \end{pmatrix} + u(s,t) \begin{pmatrix} \frac{1}{\sqrt{1+e^{2s}(1+4s^2)}}, \\ \frac{e^s(2s\cos(s^2)+\sin(s^2))}{\sqrt{1+e^{2s}(1+4s^2)}}, \\ \frac{e^s(\cos(s^2)-2s\sin(s^2))}{\sqrt{1+e^{2s}(1+4s^2)}} \end{pmatrix} + v(s,t) \begin{pmatrix} -\frac{2s e^{2s}}{\sqrt{e^{2s}+e^{4s}}\sqrt{1+e^{2s}(1+4s^2)}}, \\ \frac{e^s(\cos(s^2)+e^{2s}\cos(s^2)-2s e^{2s}\sin(s^2))}{\sqrt{e^{2s}+e^{4s}}\sqrt{1+e^{2s}(1+4s^2)}}, \\ -\frac{e^s(2s e^{2s}\cos(s^2)+\sin(s^2)+e^{2s}\sin(s^2))}{\sqrt{e^{2s}+e^{4s}}\sqrt{1+e^{2s}(1+4s^2)}} \end{pmatrix}.$$

It is very clear that the functions $u(s,t)$, and $v(s,t)$ can control the shape of the surface, and if these functions are given, then we immediately obtain an osculating surface in this family. So, we consider the following cases:

Case(1): We choose $u(s,t) = s \sin t$, and $v(s,t) = t \cos s$, and $t \in [0, T]$. Obviously, Eqs. (14) are satisfied, and the osculating surface is given by

$$M_O : \mathbf{P}(s,t) = \left\{ \begin{array}{l} s - \frac{2st e^{2s} \cos(s)}{\sqrt{e^{2s}+e^{4s}}\sqrt{1+e^{2s}(1+4s^2)}} + \frac{s \sin(s)}{\sqrt{1+e^{2s}(1+4s^2)}}, \\ e^s \sin(s^2) + \frac{e^s t \cos(s)((1+e^{2s})\cos(s^2)-2e^{2s}s\sin(s^2))}{\sqrt{e^{2s}+e^{4s}}\sqrt{1+e^{2s}(1+4s^2)}} + \frac{e^s s(2s \cos(s^2)+\sin(s^2)) \sin(s)}{\sqrt{1+e^{2s}(1+4s^2)}}, \\ e^s \cos(s^2) - \frac{e^s t \cos(s)(2e^{2s}s \cos(s^2)+(1+e^{2s})\sin(s^2))}{\sqrt{e^{2s}+e^{4s}}\sqrt{1+e^{2s}(1+4s^2)}} + \frac{e^s s(\cos(s^2)-2s \sin(s^2)) \sin(s)}{\sqrt{1+e^{2s}(1+4s^2)}} \end{array} \right\}.$$

The surfaces M , M_O , and $M \cup M_O$ along to the curve α are shown in Figs. 1(a,b), 2(a,b).

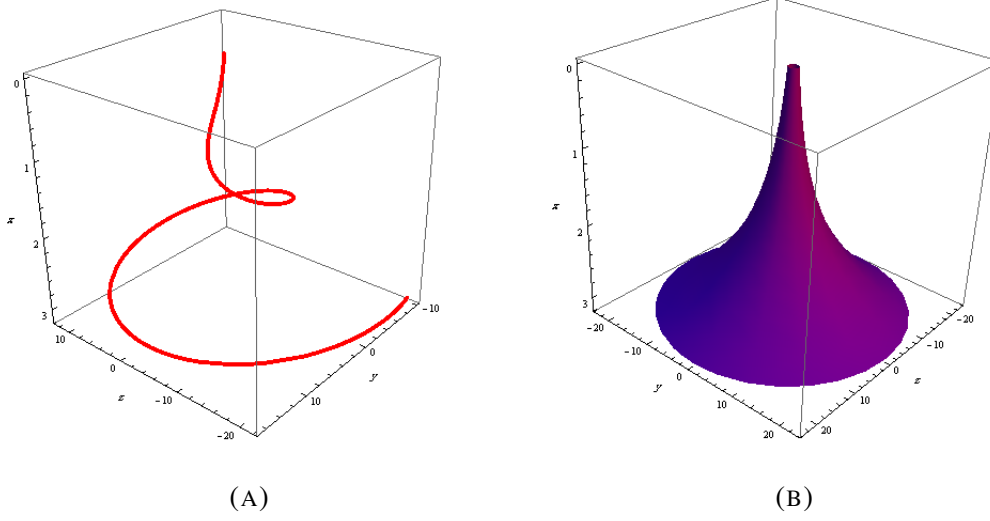


FIGURE 1. (a) The curve α (b) The surface M .

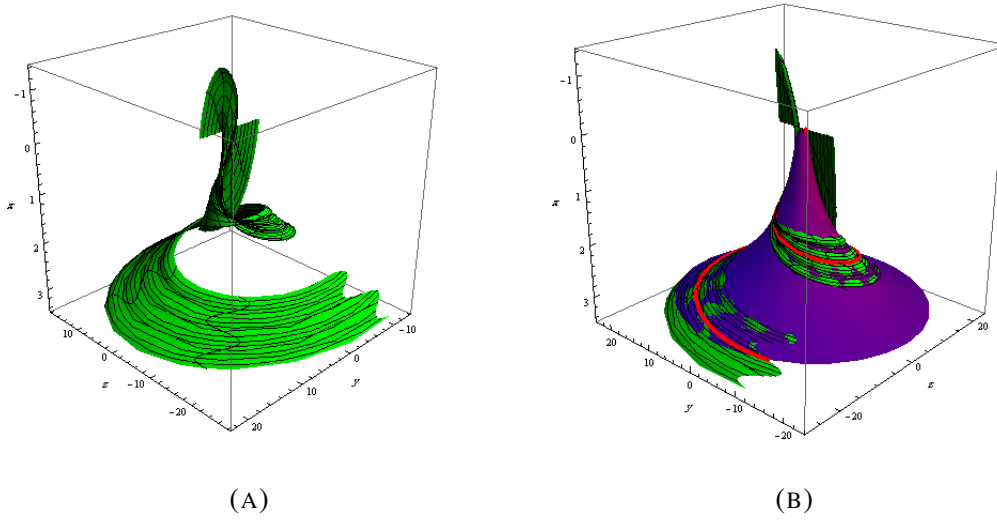


FIGURE 2. (a) The osculating surface M_O . (b) The surfaces M_O and M along the curve α .

Case(2): If we choose $u(s,t) = (1 + \sin(t)) + \sum_{k=2}^4 a_{1k}(1 + \sin(t))^k$,

$v(s,t) = \cos(t) + \sum_{k=2}^4 a_{2k} \cos^k(t)$, $t_0 = 0$, $t_1 = 3\pi/2$, $a_{1k}, a_{2k} \in \mathbb{R}$, and $t \in [0, 2\pi]$, then Eqs. (14) are satisfied.

Hence, the osculating surface can be represented as follows:

$$M_O : \mathbf{P}(s,t) = \{\mathbf{P}_1(s,t), \mathbf{P}_2(s,t), \mathbf{P}_3(s,t)\},$$

where

$$\mathbf{P}_1(s, t) = \left(\begin{array}{c} s - \frac{2e^{2s}s(1+\cos(t)+(1+\cos(t))^2+2(1+\cos(t))^3+3(1+\cos(t))^4)}{\sqrt{e^{2s}+e^{4s}}\sqrt{1+e^{2s}(1+4s^2)}} + \\ \frac{1+\sin(t)+(1+\sin(t))^2+2(1+\sin(t))^3+3(1+\sin(t))^4}{\sqrt{1+e^{2s}(1+4s^2)}} \end{array} \right),$$

$$\mathbf{P}_2(s, t) = \left(\begin{array}{c} \frac{e^s \sin(s^2) + e^s(7+21\cos(t)+25\cos^2(t)+14\cos^3(t)+3\cos^4(t))((1+e^{2s})\cos(s^2)-2e^{2s}s\sin(s^2))}{\sqrt{e^{2s}+e^{4s}}\sqrt{1+e^{2s}(1+4s^2)}} + \\ \frac{e^s(2s\cos(s^2)+\sin(s^2))(1+\sin(t)+(1+\sin(t))^2+2(1+\sin(t))^3+3(1+\sin(t))^4)}{\sqrt{1+e^{2s}(1+4s^2)}} \end{array} \right),$$

$$\mathbf{P}_3(s, t) = \left(\begin{array}{c} \frac{e^s \cos(s^2) - e^s(7+21\cos(t)+25\cos^2(t)+14\cos^3(t)+3\cos^4(t))(2e^{2s}s\cos(s^2)+(1+e^{2s})\sin(s^2))}{\sqrt{e^{2s}+e^{4s}}\sqrt{1+e^{2s}(1+4s^2)}} + \\ \frac{e^s(\cos(s^2)-2s\sin(s^2))(1+\sin(t)+(1+\sin(t))^2+2(1+\sin(t))^3+3(1+\sin(t))^4)}{\sqrt{1+e^{2s}(1+4s^2)}} \end{array} \right).$$

In this case, the surfaces M_O , and $M \cup M_O$ along the curve α are shown in Figs. 3(a,b).

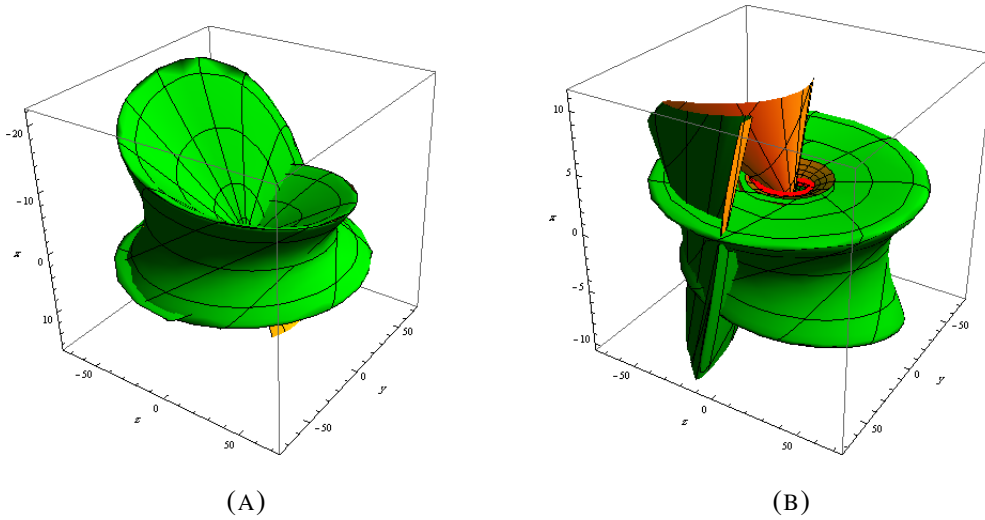


FIGURE 3. (a) The osculating surface M_O . (b) The surface M_O and M along the curve α .

Example 2. Let a surface M given by

$$M : \mathbf{X}(s, t) = \left(\cos(s) - \frac{t}{\sqrt{2}} \cos(s), \sin(s) - \frac{t}{\sqrt{2}} \sin(s), \frac{s}{\sqrt{2}} \right),$$

where the curve

$$\alpha(s) = \left(\cos(s) - \frac{s \cos(s)}{\sqrt{2}}, \sin(s) - \frac{s \sin(s)}{\sqrt{2}}, \frac{s}{\sqrt{2}} \right).$$

In this case, we get

$$\begin{aligned} \mathbf{e}_1(s) &= \begin{pmatrix} \frac{-\sqrt{2} \cos(s) + (-2 + \sqrt{2}s) \sin(s)}{\sqrt{2} \sqrt{4 - 2\sqrt{2}s + s^2}}, \\ -\frac{(-2 + \sqrt{2}s) \cos(s) + \sqrt{2} \sin(s)}{\sqrt{2} \sqrt{4 - 2\sqrt{2}s + s^2}}, \\ \frac{1}{\sqrt{4 - 2\sqrt{2}s + s^2}} \end{pmatrix}, \\ \mathbf{e}_3(s) &= \left(\frac{\sin(s)}{\sqrt{3 - 2\sqrt{2}v + v^2}}, -\frac{\cos(s)}{\sqrt{3 - 2\sqrt{2}v + v^2}}, \frac{\sqrt{2} - v}{\sqrt{3 - 2\sqrt{2}v + v^2}} \right), \\ \mathbf{e}_2(s) &= \begin{pmatrix} \frac{(-3 + \sqrt{2}s + (\sqrt{2} - s)s) \cos(s) + (\sqrt{2} - s) \sin(s)}{\sqrt{3 - 2\sqrt{2}s + s^2} \sqrt{4 - 2\sqrt{2}s + s^2}}, \\ \frac{(-\sqrt{2} + s) \cos(s) + (-3 + \sqrt{2}s + (\sqrt{2} - s)s) \sin(s)}{\sqrt{3 - 2\sqrt{2}s + s^2} \sqrt{4 - 2\sqrt{2}s + s^2}}, \\ -\frac{1}{\sqrt{3 - 2\sqrt{2}s + s^2} \sqrt{4 - 2\sqrt{2}s + s^2}} \end{pmatrix}. \end{aligned}$$

Also, using Eq. (6), the osculating surface family can be represented as:

$$\begin{aligned} M_O : P(s, t) &= \left(\cos(s) - \frac{s \cos(s)}{\sqrt{2}}, \sin(s) - \frac{s \sin(s)}{\sqrt{2}}, \frac{s}{\sqrt{2}} \right) \\ &+ u(s, t) \begin{pmatrix} \frac{-\sqrt{2} \cos(s) + (-2 + \sqrt{2}s) \sin(s)}{\sqrt{2} \sqrt{4 - 2\sqrt{2}s + s^2}}, \\ -\frac{(-2 + \sqrt{2}s) \cos(s) + \sqrt{2} \sin(s)}{\sqrt{2} \sqrt{4 - 2\sqrt{2}s + s^2}}, \\ \frac{1}{\sqrt{4 - 2\sqrt{2}s + s^2}} \end{pmatrix} \\ &+ v(s, t) \begin{pmatrix} \frac{(-3 + \sqrt{2}s + (\sqrt{2} - s)s) \cos(s) + (\sqrt{2} - s) \sin(s)}{\sqrt{3 - 2\sqrt{2}s + s^2} \sqrt{4 - 2\sqrt{2}s + s^2}}, \\ \frac{(-\sqrt{2} + s) \cos(s) + (-3 + \sqrt{2}s + (\sqrt{2} - s)s) \sin(s)}{\sqrt{3 - 2\sqrt{2}s + s^2} \sqrt{4 - 2\sqrt{2}s + s^2}}, \\ -\frac{1}{\sqrt{3 - 2\sqrt{2}s + s^2} \sqrt{4 - 2\sqrt{2}s + s^2}} \end{pmatrix}. \end{aligned}$$

It is very clear that the functions $u(s, t)$, and $v(s, t)$ can control the shape of the surface, and if these functions are given, then we immediately obtain an osculating surface in this family. So, we consider the following cases:

Case(1): We choose $u(s, t) = e^s \sin t$, and $v(s, t) = \sin t \cos s$, and $t \in [0, T]$. Obviously, Eqs. (14) are satisfied, and the osculating surface is given by

$$M_O : \mathbf{P}(s,t) = \left\{ \begin{array}{l} \cos(s) - \frac{s \cos(s)}{\sqrt{2}} + \frac{\cos(s)((-3+2\sqrt{2}s-s^2)\cos(s) + (\sqrt{2}-s)\sin(s))\sin(t)}{\sqrt{3-2\sqrt{2}s+s^2}\sqrt{4-2\sqrt{2}s+s^2}} + \\ \frac{e^s(-\sqrt{2}\cos(s) + (-2+\sqrt{2}s)\sin(s))\sin(t)}{\sqrt{2}\sqrt{4-2\sqrt{2}s+s^2}}, \\ \sin(s) - \frac{s \sin(s)}{\sqrt{2}} - \frac{e^s((-2+\sqrt{2}s)\cos(s) + \sqrt{2}\sin(s))\sin(t)}{\sqrt{2}\sqrt{4-2\sqrt{2}s+s^2}} + \\ \frac{\cos(s)((-\sqrt{2}+s)\cos(s) + (-3+2\sqrt{2}s-s^2)\sin(s))\sin(t)}{\sqrt{3-2\sqrt{2}s+s^2}\sqrt{4-2\sqrt{2}s+s^2}}, \\ \frac{s}{\sqrt{2}} + \frac{(e^s\sqrt{3-2\sqrt{2}s+s^2} - \cos(s))\sin(t)}{\sqrt{3-2\sqrt{2}s+s^2}\sqrt{4-2\sqrt{2}s+s^2}} \end{array} \right\}.$$

The surfaces M , M_O , and $M \cup M_O$ along the curve α are shown in Figs. 4(a,b), 5(a,b).

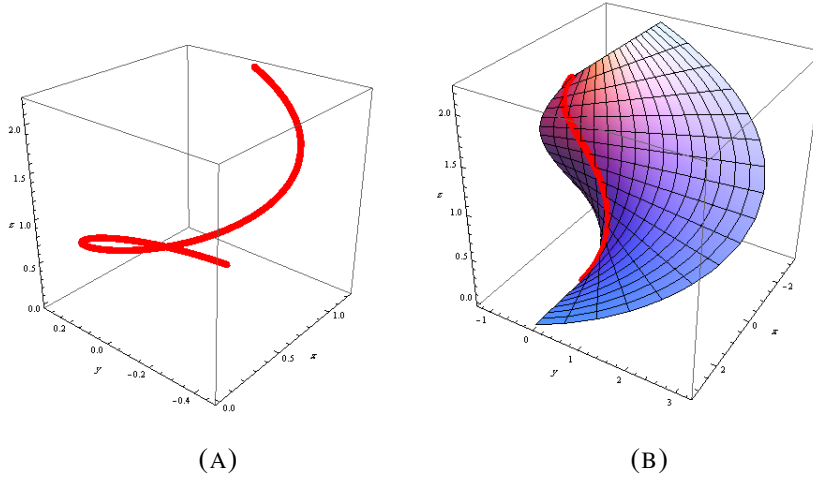


FIGURE 4. (a) The base curve α . (b) The surface M .

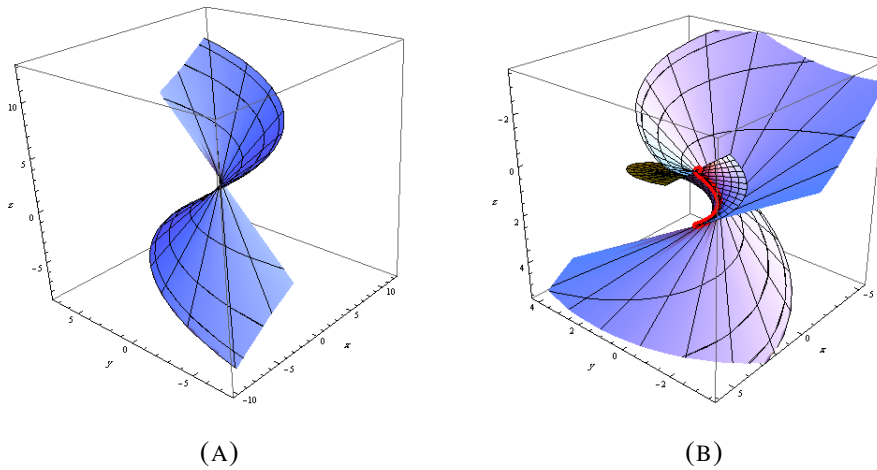


FIGURE 5. (a) The osculating surface M_O . (b) The surface M_O and M along to the curve α .

Case(2): If we choose $u(s, t) = (1 + \sin(t)) + \sum_{k=2}^4 a_{1k}(1 + \sin(t))^k$,

$v(s, t) = \cos(t) + \sum_{k=2}^4 a_{2k} \cos^k(t)$, $t_0 = 0$, $t_1 = 3\pi/2$, $a_{1k}, a_{2k} \in \mathbb{R}$, and $t \in [0, 2\pi]$, then Eqs.

(14) are satisfied. Hence, the osculating surface can be represented as follows:

$$M_O : \mathbf{P}(s, t) = \{\mathbf{P}_1(s, t), \mathbf{P}_2(s, t), \mathbf{P}_3(s, t)\},$$

where

$$\mathbf{P}_1(s, t) = \left(\begin{array}{c} \cos(s) - \frac{s \cos(s)}{\sqrt{2}} + \\ \frac{(7+21 \cos(t)+25 \cos^2(t)+14 \cos^3(t)+3 \cos^4(t)) \left(\frac{\sqrt{2}}{-3+2\sqrt{2}s-s^2} \cos(s) + (\sqrt{2}-s) \sin(s) \right) +}{\sqrt{3-2\sqrt{2}s+s^2} \sqrt{4-2\sqrt{2}s+s^2}} \\ \frac{(-\sqrt{2} \cos(s) + (-2+\sqrt{2}s) \sin(s)) (1+\sin(t) + (1+\sin(t))^2 + 2(1+\sin(t))^3 + 3(1+\sin(t))^4)}{\sqrt{2} \sqrt{4-2\sqrt{2}s+s^2}} \end{array} \right),$$

$$\mathbf{P}_2(s, t) = \left(\begin{array}{c} \sin(s) - \frac{s \sin(s)}{\sqrt{2}} - \\ \frac{(7+21 \cos(t)+25 \cos^2(t)+14 \cos^3(t)+3 \cos^4(t)) \left(\frac{\sqrt{2}}{(\sqrt{2}-s) \cos(s) + (3-2\sqrt{2}s+s^2) \sin(s)} \right) -}{\sqrt{3-2\sqrt{2}s+s^2} \sqrt{4-2\sqrt{2}s+s^2}} \\ \frac{((-2+\sqrt{2}s) \cos(s) + \sqrt{2} \sin(s)) (1+\sin(t) + (1+\sin(t))^2 + 2(1+\sin(t))^3 + 3(1+\sin(t))^4)}{\sqrt{2} \sqrt{4-2\sqrt{2}s+s^2}} \end{array} \right),$$

$$\mathbf{P}_3(s, t) = \left(\begin{array}{c} \frac{s}{\sqrt{2}} - \frac{1+\cos(t) + (1+\cos(t))^2 + 2(1+\cos(t))^3 + 3(1+\cos(t))^4}{\sqrt{3-2\sqrt{2}s+s^2} \sqrt{4-2\sqrt{2}s+s^2}} + \\ \frac{1+\sin(t) + (1+\sin(t))^2 + 2(1+\sin(t))^3 + 3(1+\sin(t))^4}{\sqrt{4-2\sqrt{2}s+s^2}} \end{array} \right).$$

In this case, the surfaces M_O , and $M \cup M_O$ along to the curve α are shown in Figs. 6(a,b).

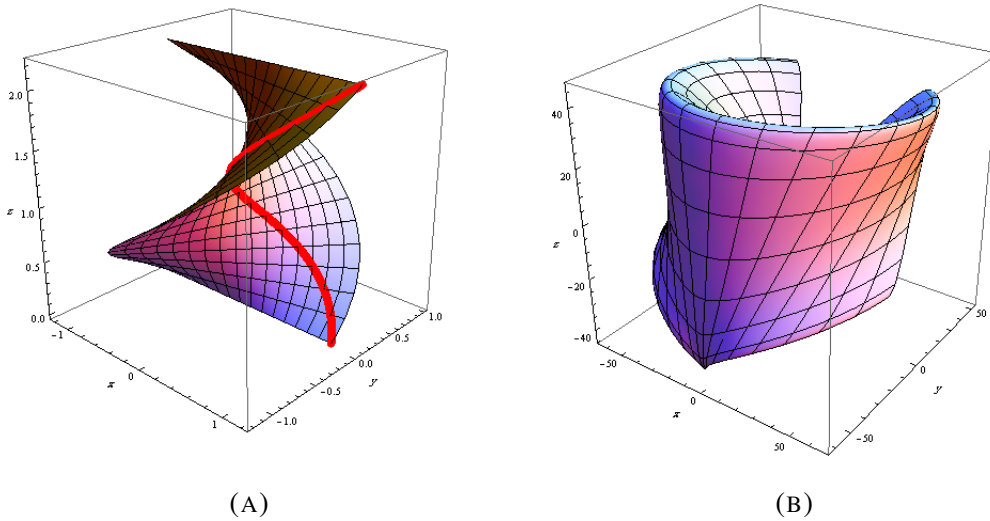


FIGURE 6. (a) The osculating surface M_O . (b) The surface M_O and M along to the curve α .

4. OSCULATING RULED SURFACES

A ruled surface is a surface generated by a straight line L moving along a curve. The various positions of the generating lines are called the rulings or generators of the surface. If its tangent planes in points of L are winding about L , the ruling L is called regular. If the tangent planes of points of L do not wind, i.e. there is only one tangent plane shared by all points of L , the ruling is called torsal. Ruled surfaces having only torsal rulings are called torsal ruled surfaces or developable ruled surfaces.

In this subsection, we will discuss the construction of osculating ruled surfaces. Suppose that M_O is a ruled surface along the curve $\alpha(s)$ of M , then there exists t_0 such that $\mathbf{P}(s, t_0) = \alpha(s)$. This follows that the ruled surface can be expressed as

$$M_O : \mathbf{P}(s, t) = \mathbf{P}(s, t_0) + (t - t_0)\mathbf{e}(s), \quad 0 \leq s \leq L, \quad \text{with } t, t_0 \in [0, T],$$

where $\mathbf{e}(s)$ denotes the direction of the rulings. According to the Eq. (6), we have

$$(19) \quad (t - t_0)\mathbf{e}(s) = u(s, t)\mathbf{e}_1(s) + v(s, t)\mathbf{e}_2(s), \quad 0 \leq s \leq L, \quad \text{with } t, t_0 \in [0, T],$$

which is a system of two equations with two unknown functions $u(s, t)$, and $v(s, t)$. For simplicity, we omit variable s . The solutions of the above system can be deduced as

$$(20) \quad \begin{aligned} u(s, t) &= (t - t_0) \langle \mathbf{e}, \mathbf{e}_1 \rangle = \det(\mathbf{e}, \mathbf{e}_2, \mathbf{e}_3), \\ v(s, t) &= (t - t_0) \langle \mathbf{e}, \mathbf{e}_2 \rangle = \det(\mathbf{e}, \mathbf{e}_3, \mathbf{e}_1). \end{aligned}$$

The above equations are just the necessary and sufficient conditions for which M_O is a ruled surface with a directrix $\alpha(s)$ on M .

Now, we need to check if M_O is an osculating ruled surface with a directrix $\alpha(s)$ of M by using the conditions given in Theorem 1. It is evident that in this case, these conditions become

$$(21) \quad \begin{aligned} \det(\mathbf{e}, \mathbf{e}_2, \mathbf{e}_3) &= \langle \mathbf{e}, \mathbf{e}_1 \rangle = 0, \\ \det(\mathbf{e}, \mathbf{e}_3, \mathbf{e}_1) &= \langle \mathbf{e}, \mathbf{e}_2 \rangle \neq 0. \end{aligned}$$

It follows that at any point on the curve $\alpha(s)$; the ruling direction $\mathbf{e}(s)$ must be in the plane $Sp\{\mathbf{e}_2, \mathbf{e}_3\}$. On the other hand, the ruling direction $\mathbf{e}(s)$ and the vector $\mathbf{e}_3(s)$ must not be parallel.

This leads to

$$(22) \quad \mathbf{e}(s) = \beta\mathbf{e}_2 + \gamma\mathbf{e}_3, \quad \beta \neq 0, \quad 0 \leq s \leq L,$$

for some real functions $\beta(s)$, and $\gamma(s)$. Substituting it into the expressions in Eq. (20), we get:

$$(23) \quad u(s,t) = \beta(s)t, \quad v(s,t) = \gamma(s)t; \quad \beta(s) \neq 0, \quad 0 \leq s \leq L.$$

Hence, the isoparametric ruled surfaces with a directrix $\alpha(s)$ on M can be expressed as:

$$(24) \quad \left. \begin{aligned} M_O : \mathbf{P}(s,t) &= \alpha(s) + t\mathbf{e}(s), \\ \mathbf{e}(s) &= \beta\mathbf{e}_2 + \gamma\mathbf{e}_3, \quad 0 \leq s \leq L, \quad 0 \leq t \leq T, \end{aligned} \right\}$$

where the functions $\gamma(s)$, and $\beta(s) \neq 0$ can control the shape of the ruled surfaces. However, the normal vector to the ruled surface M_O is

$$(25) \quad \mathbf{N}(s,t) = [1 - t(\beta\kappa_g + \gamma\kappa_n)]\mathbf{e}_1 + t(\beta' - \gamma\tau_g)\mathbf{e}_2 + t(\gamma' + \beta\tau_g)\mathbf{e}_3,$$

and thus when $t = 0$, i.e., along the curve $\alpha(s)$, the surface normal is

$$(26) \quad \mathbf{N}(s,0) = \mathbf{e}_3.$$

So, the normal vector of M_O at $\mathbf{P}(s,t_0) = \alpha(s)$ is coincident with the normal vector of M at $\alpha(s)$. This is the reason why we call M_O the osculating ruled surface of M along $\alpha(s)$.

Theorem 2. The necessary and sufficient condition for M_O being an osculating ruled surface along $\alpha(s)$ of M is that there exist a parameter $t_0 \in [0, T]$, and the functions $\gamma(s)$, and $\beta(s) \neq 0$, so that M_O can be represented by Eq. (24).

By Theorem 2, we not only prove the existence of the osculating surface, but also give the concrete expression of the surface. Every member of the isoparametric osculating ruled surfaces along $\alpha(s)$ of M is decided by two family parameters $\gamma(s)$, and $\beta(s) \neq 0$, i.e., by the direction vector function $\mathbf{e}(s)$.

Example 3. We consider a surface parameterized by

$$M : \mathbf{X}(s,t) = \left(1 + \cos s - \frac{\sqrt{2}t \sin s}{\sqrt{3 + \cos s}}, \frac{\sqrt{2}t \cos s}{\sqrt{3 + \cos s}} + \sin s, \frac{\sqrt{2}t \cos s}{\sqrt{3 + \cos s}} + 2 \sin \frac{s}{2} \right).$$

This surface is a ruled surface such that the base curve is

$$\alpha(s) = \left(1 + \cos s, \sin s, 2 \sin \left(\frac{s}{2} \right) \right), \quad s \in \mathbb{R}.$$

Thus, $\alpha(s)$ is a regular curve on the surface M . Then, we obtain

$$\mathbf{e}_1(s) = \frac{\dot{\alpha}(s)}{\|\dot{\alpha}(s)\|} = \left(-\frac{\sqrt{2} \sin s}{\sqrt{3 + \cos s}}, \frac{\sqrt{2} \cos s}{\sqrt{3 + \cos s}}, \frac{\sqrt{2} \cos\left(\frac{s}{2}\right)}{\sqrt{3 + \cos s}} \right),$$

and

$$\begin{aligned} \mathbf{e}_3(s) &= \frac{\mathbf{X}_s \times \mathbf{X}_t}{\|\mathbf{X}_s \times \mathbf{X}_t\|} \\ &= \frac{1}{\sqrt{13 + 3 \cos s}} \left(\frac{-3 \sin\left(\frac{s}{2}\right) - \sin\left(\frac{3s}{2}\right)}{\sqrt{2}}, 2\sqrt{2} \cos^3\left(\frac{s}{2}\right), -2\sqrt{2} \right), \end{aligned}$$

$$\mathbf{e}_2(s) = \frac{1}{\sqrt{(3 + \cos(s))(13 + 3 \cos(s))}} \begin{pmatrix} 4 \left(\cos^4\left(\frac{s}{2}\right) + \cos(s) \right), \\ (6 + \cos(s)) \sin(s), \\ 2 \sin\left(\frac{s}{2}\right) \end{pmatrix}.$$

Thus, using Eq. (23), the osculating surface family can be represented as:

$$\begin{aligned} M_O : \mathbf{P}(s, t) &= \left\{ 1 + \cos s, \sin s, 2 \sin \frac{s}{2} \right\} + \\ &t \left(\beta(s) \begin{pmatrix} -\frac{\sqrt{2} \sin s}{\sqrt{3 + \cos s}}, \\ \frac{\sqrt{2} \cos s}{\sqrt{3 + \cos s}}, \\ \frac{\sqrt{2} \cos\left(\frac{s}{2}\right)}{\sqrt{3 + \cos s}} \end{pmatrix} + \gamma(s) \begin{pmatrix} \frac{4 \left(\cos^4\left(\frac{s}{2}\right) + \cos(s) \right)}{\sqrt{(3 + \cos(s))(13 + 3 \cos(s))}}, \\ \frac{(6 + \cos(s)) \sin(s)}{\sqrt{(3 + \cos(s))(13 + 3 \cos(s))}}, \\ \frac{2 \sin\left(\frac{s}{2}\right)}{\sqrt{(3 + \cos(s))(13 + 3 \cos(s))}} \end{pmatrix} \right). \end{aligned}$$

The functions $\beta(s)$ and $\gamma(s)$ can control the shape of the surface and it is very clear that if these functions are given, then we immediately obtain an osculating surface in the family. In the following, we consider two cases:

Case(1): We choose $\beta(s) = \sin s$, and $\gamma(s) = s$. Obviously, Eqs. (19)-(23) are satisfied, and the osculating surface in this family is given by:

$$M_O : \mathbf{P}(s, t) = \left\{ \begin{aligned} &1 + \cos(s) + \frac{t \left(4s \cos^4\left(\frac{s}{2}\right) + 4s \cos(s) - \sqrt{26 + 6 \cos(s)} \sin(s)^2 \right)}{\sqrt{3 + \cos(s)} \sqrt{13 + 3 \cos(s)}}, \\ &\sin(s) + \frac{t \left((6s + \cos(s)) \left(s + \sqrt{26 + 6 \cos(s)} \right) \right) \sin(s)}{\sqrt{3 + \cos(s)} \sqrt{13 + 3 \cos(s)}}, \\ &\left(2 + \frac{2st}{\sqrt{3 + \cos(s)} \sqrt{13 + 3 \cos(s)}} \right) \sin\left(\frac{s}{2}\right) + \frac{\sqrt{2}t \cos\left(\frac{s}{2}\right) \sin(s)}{\sqrt{3 + \cos(s)}} \end{aligned} \right\}.$$

The surfaces M , M_O , and $M \cup M_O$ along the curve α , are shown in Figs. 7(a,b), 8(a,b).

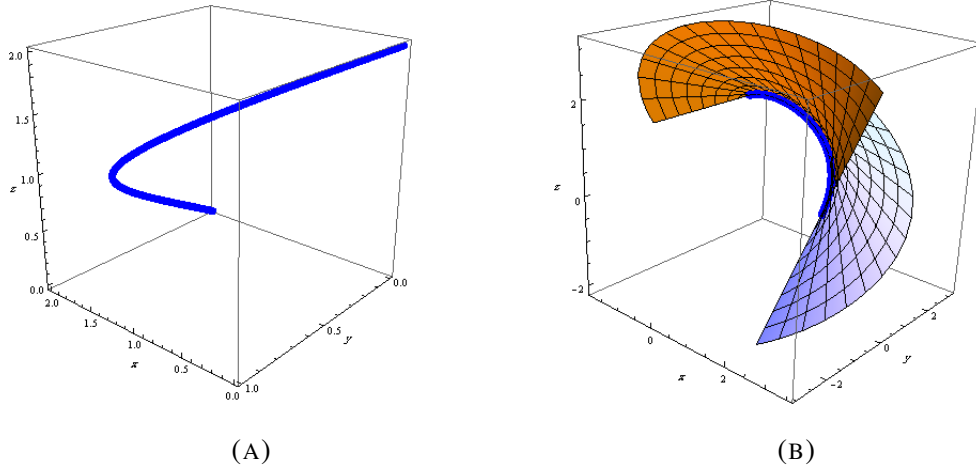


FIGURE 7. (a) The base curve α (b) The ruled surface M .

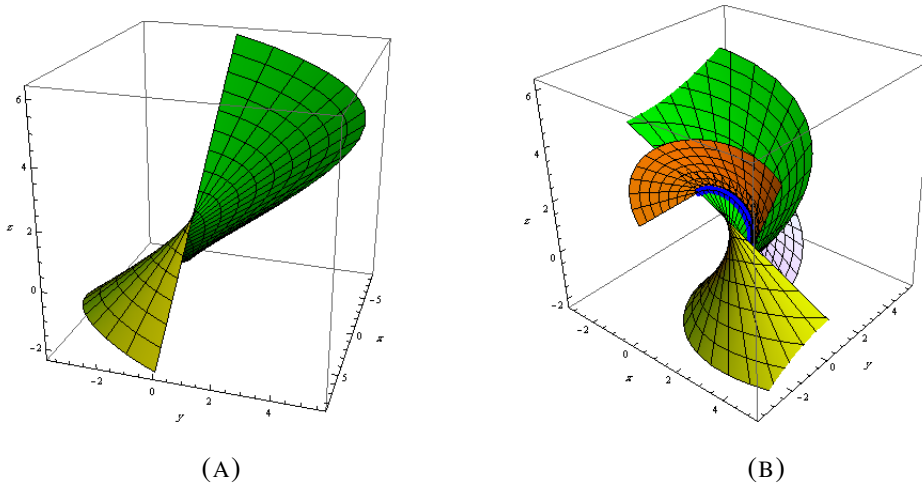


FIGURE 8. (a) The osculating surface M_O . (b) The surfaces M_O and M along the curve α .

Case(2): If we choose $\beta(s) = s^2$ and $\gamma(s) = 2s$, then the osculating surface in this family is given by:

$$M_O : \mathbf{P}(s, t) = \left\{ \begin{array}{l} 1 + \cos(s) + \frac{st(3 + 12\cos(s) + \cos(2s) - s\sqrt{26 + 6\cos(s)}\sin(s))}{\sqrt{3 + \cos(s)}\sqrt{13 + 3\cos(s)}}, \\ \sin(s) + \frac{st(s\cos(s)\sqrt{26 + 6\cos(s)} + 12\sin(s) + \sin(2s))}{\sqrt{3 + \cos(s)}\sqrt{13 + 3\cos(s)}}, \\ \frac{\sqrt{2}s^2t\cos(\frac{s}{2})}{\sqrt{3 + \cos(s)}} + 2\left(1 + \frac{2st}{\sqrt{3 + \cos(s)}\sqrt{13 + 3\cos(s)}}\right)\sin(\frac{s}{2}) \end{array} \right\}.$$

The surfaces M , M_O , and $M \cup M_O$ along the curve α , are shown in Figs. 9(a,b).

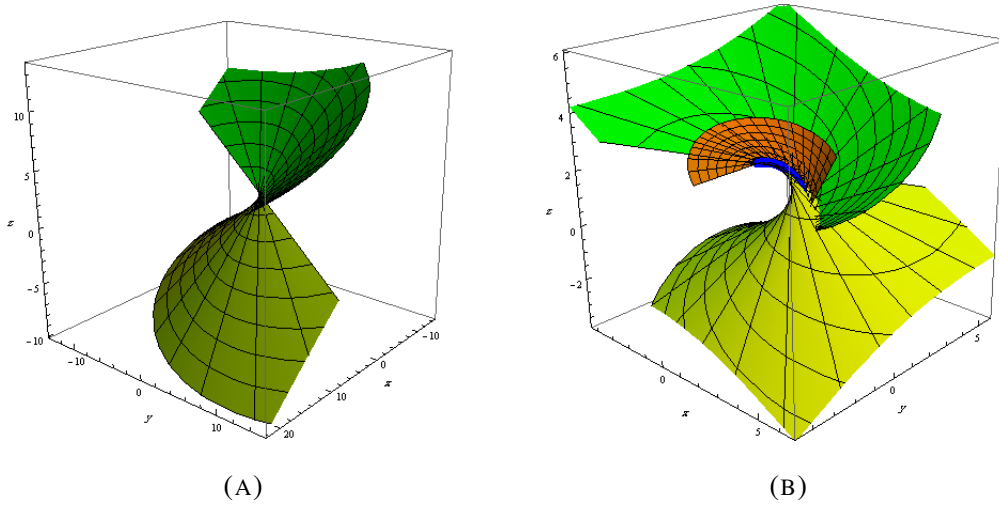


FIGURE 9. (a) The osculating surface M_O . (b) The surfaces M_O and M along the curve α .

5. CONCLUSION

In the three-dimensional Euclidean space E^3 , an osculating surface to a surface along a curve on this surface has been defined. Then, the necessary and sufficient conditions for that surface to be a ruled surface have been investigated. Meanwhile, we have illustrated the convenience and efficiency of this approach by some representative examples.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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