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SOME PROPERTIES OF k-GENERALIZED MITTAG LEFFLER FUNCTION RELATED TO FRACTIONAL CALCULUS

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Abstract. This paper deals with the k-new generalized Mittag Leffler function. Some of its properties related to fractional calculus are presented viz. k-Weyl fractional integral and k-extended Euler beta integral transform, Whittaker integral transform. Some important special cases of the main results are also have been discussed.

Keywords: k-generalized Mittag-Leffler function; k-gamma function; k-beta function; k-pochhammer symbol.
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1. INTRODUCTION

The k-Mittag-Leffler functions are important for obtaining the solutions of the integral and differential equations of the fractional order. The most applicable ares of this functions are dynamical system, quantum mechanics, statistical distribution and the references cited therein. Due to great importance of the k-Mittag-Leffler functions, many researchers have extended the reserch work in the theory of special functions and fractional calculus. During the last decade the interest in k-generalized Mittag-Leffler functions is increased among mathematical researchers due to their vast potential of applications in several applied problems. For $k \in \mathbb{R}^+$; $z \in \mathbb{C}$, $(A_i, B_j \neq 0; i = 1,...,p; j = 1,...,q)$ and $(\alpha_i + A_i n), (\beta_j + B_j n) \in \mathbb{C} \setminus k\mathbb{Z}^-$, Gehlot & Prajapati [6] introduced the

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following k-extension of the Wright function, which known as generalized k-Wright function.

$$ {}_p\Psi_q^k(z) = {}_p\Psi_q^k\left[\begin{array}{c}(\alpha_i, A_i)_{1,p} \\ (\beta_j, B_j)_{1,q}\end{array}; z\right] = \sum_{n=0}^{\infty}\frac{\Gamma_k(\alpha_1+A_1n),..\Gamma_k(\alpha_p+A_pn)}{\Gamma_k(\beta_1+B_1n),...\Gamma_k(\beta_q+B_qn)}\frac{z^n}{n!} \tag{1}$$

Doorego and Cerutti [4] defined the k-Mittag Leffler function for $k \in \mathbb{R}$ and $\alpha, \beta, \gamma \in \mathbb{C}$, as follows:

$$E_{k,\alpha,\beta}^{\gamma}(z) = \sum_{n=0}^{\infty}\frac{(\gamma)_{n,k}}{\Gamma_k(\alpha n+\beta)}\frac{z^n}{n!} \tag{2}$$

where $Re(\alpha) > 0$, $Re(\beta) > 0$, $Re(\gamma) > 0$. and $(\gamma)_{n,k} = \gamma(\gamma+k)(\gamma+2k)...(\gamma+\overline{n-1}k)$.

Saxena et.al. [15] extended the generalized k-Mittag Leffler function for $k \in \mathbb{R}$ and $\alpha, \beta, \gamma, \tau \in \mathbb{C}$, such as

$$E_{k,\alpha,\beta}^{\gamma,\tau}(z) = \sum_{n=0}^{\infty}\frac{(\gamma)_{n\tau,k}}{\Gamma_k(\alpha n+\beta)}\frac{z^n}{n!} \tag{3}$$

Our purpose in this paper is to establised some fractional calculus results and properties of the k-new generalized Mittag-Leffler function, introduced by Gupta & Parihar [1] for $k \in \mathbb{R}$; $\alpha, \beta, \gamma, \delta \in \mathbb{C}$, such that

$$E_{k,\alpha,\beta,r}^{\gamma,\delta,s}(z) = \sum_{n=0}^{\infty}\frac{(\gamma)_{sn,k}}{\Gamma_k(\alpha n+\beta)}\frac{z^n}{(\delta)_{rn,k}} \tag{4}$$

where $Re(\alpha) > 0, Re(\beta) > 0, Re(\gamma) > 0, Re(\delta) > 0$ and $r, s > 0$ with $s \leq Re(\alpha) + r$ and $(\gamma)_{sn,k} = \frac{\Gamma_k(\gamma+snk)}{\Gamma_k(\gamma)}$ denotes the generalized Pochhammer symbol.

For particular $k = 1$, above equation reduces in generalized Mittag-Leffler functions defined by Salim and Faraj [18], which is absolutely convergent for all values of z provided that $s < r + Re(\alpha)$ and if $s = r + Re(\alpha)$ then $E_{\alpha,\beta,r}^{\gamma,\delta,s}(z)$ converges for $|z| < 1$.

$$E_{\alpha,\beta,r}^{\gamma,\delta,s}(z) = \sum_{n=0}^{\infty}\frac{(\gamma)_{sn}}{\Gamma(\alpha n+\beta)}\frac{z^n}{(\delta)_{rn}} \tag{5}$$

The functional relation between k-new generalized Mittag-Leffler function and Mittag-Leffler function (5) is given bellow [18];

$$E_{k,\alpha,\beta,r}^{\gamma,\delta,s}(z) = k^{1-\frac{\beta}{k}} \times E_{\frac{\alpha}{k},\frac{\beta}{k},r}^{\frac{\gamma}{k},\frac{\delta}{k},s}\left(k^{-\frac{\alpha}{k}}\frac{r^r}{s^s}z\right) \tag{6}$$

Proposition 1.1. Let $k, s \in \mathbb{R}$ and $\lambda \in \mathbb{C}$ then the following identity holds

$$\Gamma_s(\lambda) = \left(\frac{s}{k}\right)^{\frac{\lambda}{s}-1}\Gamma_k\left(\frac{k\lambda}{s}\right) \tag{7}$$

and in particular case

$$\Gamma_k(\lambda) = (k)^{\frac{\lambda}{k}-1}\Gamma\left(\frac{\lambda}{k}\right) \tag{8}$$

Proposition 1.2. Let $k, s \in \mathbb{R}$, $n \in \mathbb{N}$ and $\lambda \in \mathbb{C}$ then the following identity holds

$$(\lambda)_{nq,s} = \left(\frac{s}{k}\right)^{nq}\left(\frac{k\lambda}{s}\right)_{nq} \tag{9}$$

and in particular case

$$(\lambda)_{nq,k} = (k)^{nq}\left(\frac{\lambda}{k}\right)_{nq} \tag{10}$$

2. Preliminaries and Definitions

Definition 2.1. The k-Pochhammer symbol $(\lambda)_{n,k}$ was introduced by Diaz and Pariguan [14] and defined as

$$(\lambda)_{n,k} = \lambda(\lambda+k)(\lambda+2k)...(\lambda+\overline{n-1}k), \quad \text{where } \lambda \in \mathbb{C}, k \in \mathbb{R}, n \in \mathbb{N} \tag{11}$$

Definition 2.2. k-Gamma function $\Gamma_k(\lambda)$ was defined by Diaz and Pariguan [14] as

$$\Gamma_k(\lambda) = \int_0^\infty e^{-\frac{\xi^k}{k}}\xi^{\lambda-1}d\xi, \quad \text{where } \lambda \in \mathbb{C}, k \in \mathbb{R}, Re(\lambda) > 0 \tag{12}$$

$$\text{and} \qquad \Gamma_k(\lambda+k) = \lambda\,\Gamma_k(\lambda) \tag{13}$$

Definition 2.3. k-Beta function $\Gamma_k(\lambda)$ was defined by Diaz and Pariguan [14] as

$$B_k(x,y) = \frac{1}{k}\int_0^1 \xi^{\frac{x}{k}-1}(1-\xi)^{\frac{y}{k}-1}d\xi, k > 0, Re(x) > 0, Re(y) > 0 \tag{14}$$

$$\text{and} \qquad B_k(x,y) = \frac{\Gamma_k(x)\Gamma_k(y)}{\Gamma_k(x+y)} \tag{15}$$

Definition 2.4. Let λ be a real number, $0 < \lambda < 1$ and $k > 0$, then k-Weyl fractional integral [7] is defined as

$$W_k^\lambda f(x) = \frac{1}{k\Gamma_k(\lambda)}\int_x^\infty (\xi - x)^{\frac{\lambda}{k}-1}f(\xi)d\xi \tag{16}$$

Definition 2.5. Let $\lambda > 0$ be a real number, then k- Riemann Liouville fractional integral [1] is defined as

$$I_{k,a}^\lambda f(x) = \frac{1}{k\Gamma_k(\lambda)}\int_a^x (x-\xi)^{\frac{\lambda}{k}-1}f(\xi)d\xi; \quad k > 0 \tag{17}$$

Definition 2.6. Let $k > 0, Re(A) > 0, Re(x) > 0, Re(y) > 0$ then k-analogue of the extended Euler beta function [17]

$$B_k(x,y,A) = \frac{1}{k}\int_0^1 \xi^{\frac{x}{k}-1}(1-\xi)^{\frac{y}{k}-1}exp\left(-\frac{A^k}{k\xi(1-\xi)}\right)d\xi \tag{18}$$

where original beta k-function is given by $B_k(x,y) = \int_0^\infty B_k(x-1,y-1;A)dA$.

Definition 2.7. For $Re(\mu \pm m) > -\frac{1}{2}$ the following integral formula introduced by Whittaker and Watson [2]

$$\int_0^\infty e^{-\frac{1}{2}}\xi^{m-1}W_{\lambda,\mu}d\xi = \frac{\Gamma\left(\frac{1}{2}+\mu+m\right)\Gamma\left(\frac{1}{2}-\mu+m\right)}{\Gamma(1-\lambda+m)} \tag{19}$$

where $W_{\lambda,\mu}$ is the Whittaker confluent hypergeometric function.

3. Main Results

Theorem 3.1. If $k \in \mathbb{R}$; $\alpha, \beta, \gamma, \delta \in \mathbb{C}$, $Re(\alpha) > 0, Re(\beta) > 0, Re(\gamma) > 0, Re(\delta) > 0$; $r, s > 0$ and $s < Re(\alpha) + r$, then the following result hold true

$$W_k^{\lambda}\left[z^{-\frac{\beta+\lambda}{k}} E_{k,\alpha,\beta,r}^{\gamma,\delta,s}(\omega z^{-\frac{\alpha}{k}})\right] = z^{-\frac{\beta}{k}} E_{k,\alpha,\beta+\lambda,r}^{\gamma,\delta,s}(\omega z^{-\frac{\alpha}{k}}) \tag{20}$$

Proof. By applying (4) in (16), we have

$$W_k^{\lambda}\left[z^{-\frac{\beta+\lambda}{k}} E_{k,\alpha,\beta,r}^{\gamma,\delta,s}(\omega z^{-\frac{\alpha}{k}})\right] = \frac{1}{k\Gamma_k(\lambda)} \int_z^{\infty} (\xi - z)^{\frac{\lambda}{k}-1} \xi^{-\frac{\beta+\lambda}{k}} E_{k,\alpha,\beta,r}^{\gamma,\delta,s}(\omega \xi^{-\frac{\alpha}{k}}) d\xi$$

$$= \frac{1}{k\Gamma_k(\lambda)} \int_z^{\infty} (\xi - z)^{\frac{\lambda}{k}-1} \xi^{-\frac{\beta+\lambda}{k}} \sum_{n=0}^{\infty} \frac{(\gamma)_{sn,k}}{\Gamma_k(\alpha n + \beta)} \frac{\omega^n \xi^{-\frac{\alpha n}{k}}}{(\delta)_{rn,k}} d\xi$$

interchange the order of summation and integration to get

$$= \frac{1}{k\Gamma_k(\lambda)} \sum_{n=0}^{\infty} \frac{(\gamma)_{sn,k}}{\Gamma_k(\alpha n + \beta)} \frac{\omega^n}{(\delta)_{rn,k}} \int_z^{\infty} (\xi - z)^{\frac{\lambda}{k}-1} \xi^{-\frac{\alpha n+\beta+\lambda}{k}} d\xi$$

Let $\theta = \frac{\xi - z}{\xi}$, then

$$= \frac{z^{-\frac{\beta}{k}}}{k\Gamma_k(\lambda)} \sum_{n=0}^{\infty} \frac{(\gamma)_{sn,k}}{\Gamma_k(\alpha n + \beta)} \frac{\omega^n z^{-\frac{\alpha n}{k}}}{(\delta)_{rn,k}} \int_0^1 \theta^{\frac{\lambda}{k}-1} (1-\theta)^{\frac{\alpha n+\beta}{k}-1} d\theta$$

$$= z^{-\frac{\beta}{k}} E_{k,\alpha,\beta+\lambda,r}^{\gamma,\delta,s}\left(\omega z^{-\frac{\alpha}{k}}\right)$$

□

Theorem 3.2. If $k \in \mathbb{R}$; $\alpha, \beta, \gamma, \delta \in \mathbb{C}$, $Re(\alpha) > 0, Re(\beta) > 0, Re(\gamma) > 0, Re(\delta) > 0$; $r, s > 0$ and $s < Re(\alpha) + r$, then the following result hold true

$$I_{k,a}^{\lambda}\left[(z-a)^{\frac{b}{k}-1} E_{k,\alpha,\beta,r}^{\gamma,\delta,s}\left(\omega (z-a)^{\frac{\alpha}{k}}\right)\right] = (z-a)^{\frac{\lambda+b}{k}-1} E_{k,\alpha,\beta,r}^{\gamma,\delta,s}\left(\omega (z-a)^{\frac{\alpha}{k}}\right) \frac{\Gamma_k(\alpha n + b)}{\Gamma_k(\alpha n + b + \lambda)} \tag{21}$$

Proof. By applying (4) in (17), we have

$$I_{k,a}^{\lambda}\left[(z-a)^{\frac{b}{k}-1} E_{k,\alpha,\beta,r}^{\gamma,\delta,s}\left(\omega (z-a)^{\frac{\alpha}{k}}\right)\right] = \frac{1}{k\Gamma_k(\lambda)} \int_a^z (z-\xi)^{\frac{\lambda}{k}-1} (\xi - a)^{\frac{b}{k}-1} E_{k,\alpha,\beta,r}^{\gamma,\delta,s}\left(\omega (\xi - a)^{\frac{\alpha}{k}}\right) d\xi$$

$$= \frac{1}{k\Gamma_k(\lambda)} \int_a^z (z-\xi)^{\frac{\lambda}{k}-1} (\xi - a)^{\frac{b}{k}-1} \sum_{n=0}^{\infty} \frac{(\gamma)_{sn,k}}{\Gamma_k(\alpha n + \beta)} \frac{\omega^n (\xi - a)^{\frac{\alpha n}{k}}}{(\delta)_{rn,k}} d\xi$$

interchange the order of summation and integration to get

$$= \frac{1}{k\Gamma_k(\lambda)} \sum_{n=0}^{\infty} \frac{(\gamma)_{sn,k}}{\Gamma_k(\alpha n + \beta)} \frac{\omega^n}{(\delta)_{rn,k}} \int_a^z (z-\xi)^{\frac{\lambda}{k}-1} (\xi - a)^{\frac{\alpha n+b}{k}-1} d\xi$$

Let $\theta = \frac{\xi - a}{z - a}$, then

$$= \frac{1}{k\Gamma_k(\lambda)} (z-a)^{\frac{\lambda+b}{k}-1} \sum_{n=0}^{\infty} \frac{(\gamma)_{sn,k}}{\Gamma_k(\alpha n + \beta)} \frac{\omega^n (z-a)^{\frac{\alpha n}{k}}}{(\delta)_{rn,k}} \int_0^1 \theta^{\frac{\alpha n+b}{k}-1} (1-\theta)^{\frac{\lambda}{k}-1} d\theta$$

$$= (z-a)^{\frac{\lambda+b}{k}-1} E_{k,\alpha,\beta,r}^{\gamma,\delta,s}\left(\omega (z-a)^{\frac{\alpha}{k}}\right) \frac{\Gamma_k(\alpha n + b)}{\Gamma_k(\alpha n + b + \lambda)}$$

Special Case 1. For $a=0$, $b=\beta$ and $\omega=1$ in the Theorem 3.2, we deduce the following results, due to Gupta and Parihar [1]

$$I_k^{\lambda}\left[z^{\frac{\beta}{k}-1}E_{k,\alpha,\beta,r}^{\gamma,\delta,s}\left(z^{\frac{\alpha}{k}}\right)\right]=z^{\frac{\beta+\lambda}{k}-1}E_{k,\alpha,\beta+\lambda,r}^{\gamma,\delta,s}\left(z^{\frac{\alpha}{k}}\right) \tag{22}$$

□

Theorem 3.3. If $k\in\mathbb{R}$; $\alpha,\beta,\gamma,\delta\in\mathbb{C}$, $Re(\alpha)>0, Re(\beta)>0, Re(\gamma)>0, Re(\delta)>0; r,s>0$ and $s<Re(\alpha)+r$, then the following result hold true

$$\int_0^1 \xi^{\frac{a}{k}-1}(1-\xi)^{\frac{b}{k}-1}exp\left(-\frac{A^k}{k\xi(1-\xi)}\right)E_{k,\alpha,\beta,r}^{\gamma,\delta,s}(z\xi^{\frac{\alpha}{k}})d\xi=kE_{k,\alpha,\beta,r}^{\gamma,\delta,s}(z)B_k(\alpha n+a,b;A) \tag{23}$$

where $B_k(\alpha n+a,b;A)$, is the k-analogue of the extended Eulers beta function.

Proof. First we denote L.H.S. of (23) by integration symbol I_1 and then expanding $E_{k,\alpha,\beta,r}^{\gamma,\delta,s}(z\xi^{\frac{\alpha}{k}})$ by using (4)

$$I_1\equiv\int_0^1 \xi^{\frac{a}{k}-1}(1-\xi)^{\frac{b}{k}-1}exp\left(-\frac{A^k}{k\xi(1-\xi)}\right)\sum_{n=0}^{\infty}\frac{(\gamma)_{sn,k}}{\Gamma_k(\alpha n+\beta)}\frac{z^n\xi^{\frac{\alpha n}{k}}}{(\delta)_{rn,k}}d\xi$$

interchange the order of summation and integration to get

$$I_1\equiv\sum_{n=0}^{\infty}\frac{(\gamma)_{sn,k}}{\Gamma_k(\alpha n+\beta)}\frac{z^n}{(\delta)_{rn,k}}\int_0^1 \xi^{\frac{\alpha n+a}{k}-1}(1-\xi)^{\frac{b}{k}-1}exp\left(-\frac{A^k}{k\xi(1-\xi)}\right)d\xi$$

now using result of (18), we have

$$I_1\equiv\sum_{n=0}^{\infty}\frac{(\gamma)_{sn,k}}{\Gamma_k(\alpha n+\beta)}\frac{z^n}{(\delta)_{rn,k}}kB_k(\alpha n+a,b;A)$$

$$I_1\equiv kE_{k,\alpha,\beta,r}^{\gamma,\delta,s}(z)B_k(\alpha n+a,b;A)$$

□

Special Case 2. For $a=\beta$, $b=\delta$ and $A=0$ in the Theorem 3.3, we deduce the following results

$$\int_0^1 \xi^{\frac{\beta}{k}-1}(1-\xi)^{\frac{\delta}{k}-1}E_{k,\alpha,\beta,r}^{\gamma,\delta,s}(z\xi^{\frac{\alpha}{k}})d\xi=kE_{k,\alpha,\beta,r}^{\gamma,\delta,s}(z)B_k(\alpha n+\beta,\delta;0)$$

Now using the result $B_k(x,y;0)=B_k(x,y)$, due to Mubeen et.al [17]

$$\frac{1}{\Gamma_k(\delta)}\int_0^1 \xi^{\frac{\beta}{k}-1}(1-\xi)^{\frac{\delta}{k}-1}E_{k,\alpha,\beta,r}^{\gamma,\delta,s}(z\xi^{\frac{\alpha}{k}})d\xi=kE_{k,\alpha,\beta+\delta,r}^{\gamma,\delta,s}(z) \tag{24}$$

which is the Beta transform of k-new generalized Mitta-Leffler function, obtained by Gupta and Parihar [1]

Special Case 3. For $k=1$ and $A=0$ in the Theorem 3.3, we deduce the following results

$$\int_0^1 \xi^{a-1}(1-\xi)^{b-1}E_{\alpha,\beta,r}^{\gamma,\delta,s}(z\xi^{\alpha})d\xi=E_{\alpha,\beta,r}^{\gamma,\delta,s}(z)B(\alpha n+a,b;0)$$

Now using the (4) and result $B(x,y;0)=B(x,y)$, due to Mubeen et.al [17]

$$\int_0^1 \xi^{a-1}(1-\xi)^{b-1}E^{\gamma,\delta,s}_{\alpha,\beta,r}(z\xi^{\alpha})d\xi = \frac{\Gamma(b)\Gamma(\delta)}{\Gamma(\gamma)}{}_3\Psi_3\left[\begin{array}{c}(\gamma,s),(a,\alpha),(1,1);\\(\beta,\alpha),(\delta,r),(a+b,\alpha);\end{array} z\right] \tag{25}$$

which is the Beta transform of the generalized Mitta-Leffler function, obtained by salim and Faraj [18]

Theorem 3.4. If $k\in\mathbb{R}$; $\alpha,\beta,\gamma,\delta\in\mathbb{C}$, $Re(\alpha)>0, Re(\beta)>0, Re(\gamma)>0, Re(\delta)>0$; $r,s>0$ and $s<Re(\alpha)+r$, then the following result hold true

$$\frac{1}{\Gamma_k(a)}\int_t^x (x-z)^{\frac{a}{k}-1}(z-t)^{\frac{b}{k}-1}E^{\gamma,\delta,s}_{k,\alpha,\beta,r}\left(\lambda(z-t)^{\frac{\alpha}{k}}\right)dz$$

$$= (x-t)^{\frac{a+b}{k}-1}E^{\gamma,\delta,s}_{k,\alpha,\beta,r}\left(\lambda(x-t)^{\frac{\alpha}{k}}\right)\frac{\Gamma_k(\alpha n+b)}{\Gamma_k(\alpha n+a+b)} \tag{26}$$

Proof. First we denote L.H.S. of (26) by integration symbol I_2 and then by changing the variable z to $\xi=\frac{z-t}{x-t}$,

$$I_2 \equiv \frac{1}{\Gamma_k(a)}\int_0^1 (x-t)^{\frac{a+b}{k}-1}\xi^{\frac{b}{k}-1}(1-\xi)^{\frac{a}{k}-1}E^{\gamma,\delta,s}_{k,\alpha,\beta,r}\left(\lambda\xi^{\frac{\alpha}{k}}(x-t)^{\frac{\alpha}{k}}\right)d\xi$$

$$I_2 \equiv \frac{1}{\Gamma_k(a)}\int_0^1 (x-t)^{\frac{a+b}{k}-1}\xi^{\frac{b}{k}-1}(1-\xi)^{\frac{a}{k}-1}\sum_{n=0}^{\infty}\frac{(\gamma)_{sn,k}}{\Gamma_k(\alpha n+\beta)}\frac{\lambda^n\xi^{\frac{\alpha n}{k}}(x-t)^{\frac{\alpha n}{k}}}{(\delta)_{rn,k}}d\xi$$

interchange the order of summation and integration to get

$$I_2 \equiv \frac{(x-t)^{\frac{a+b}{k}-1}}{\Gamma_k(a)}\sum_{n=0}^{\infty}\frac{(\gamma)_{sn,k}}{\Gamma_k(\alpha n+\beta)}\frac{\lambda^n(x-t)^{\frac{\alpha n}{k}}}{(\delta)_{rn,k}}\int_0^1 \xi^{\frac{\alpha n+b}{k}-1}(1-\xi)^{\frac{a}{k}-1}d\xi$$

$$I_2 \equiv \frac{(x-t)^{\frac{a+b}{k}-1}}{\Gamma_k(a)}\sum_{n=0}^{\infty}\frac{(\gamma)_{sn,k}}{\Gamma_k(\alpha n+\beta)}\frac{\lambda^n(x-t)^{\frac{\alpha n}{k}}}{(\delta)_{rn,k}}\frac{\Gamma_k(\alpha n+b)\Gamma_k(a)}{\Gamma_k(\alpha n+a+b)}$$

$$I_2 \equiv (x-t)^{\frac{a+b}{k}-1}E^{\gamma,\delta,s}_{k,\alpha,\beta,r}\left(\lambda(x-t)^{\frac{\alpha}{k}}\right)\frac{\Gamma_k(\alpha n+b)}{\Gamma_k(\alpha n+a+b)}$$

□

Special Case 4. For $a=\delta$, $b=\beta$ and $k=1$ in the Theorem 3.4, we deduce the following results

$$\frac{1}{\Gamma(\delta)}\int_t^x (x-z)^{\delta-1}(z-t)^{\beta-1}E^{\gamma,\delta,s}_{\alpha,\beta,r}(\lambda(z-t)^{\alpha})dz = (x-t)^{\delta+\beta-1}E^{\gamma,\delta,s}_{\alpha,\beta+\delta,r}(\lambda(x-t)^{\alpha}) \tag{27}$$

which is the result obtained by Salim and Faraj [18], page 6

Theorem 3.5. If $k\in\mathbb{R}$; $\alpha,\beta,\gamma,\delta\in\mathbb{C}$, $Re(\alpha)>0, Re(\beta)>0, Re(\gamma)>0, Re(\delta)>0$; $r,s>0$ and $s<Re(\alpha)+r$,then the following result hold true

$$\int_0^{\infty} e^{-\frac{1}{2}\phi t}t^{\zeta-1}W_{\lambda,\mu}(\phi t)E^{\gamma,\delta,s}_{k,\alpha,\beta,r}(\omega t^{\sigma})dt$$

$$= k^{1-\lambda-\zeta}\times\phi^{-\zeta}\frac{\Gamma_k(\delta)}{\Gamma_k(\gamma)}\,{}_4\Psi^k_3\left[\begin{array}{c}(\gamma,sk),(\frac{k}{2}\pm\mu k+\zeta k,\sigma k),(k,k)\\(\beta,\alpha),(\delta,rk),(k-\lambda k+\zeta k,\sigma k)\end{array};\frac{\omega\phi^{-\sigma}}{k^{1+\sigma}}\right] \tag{28}$$

Proof. By apply (4) in the L.H.S. of (28), we have

$$\int_0^\infty e^{-\frac{1}{2}\phi t} t^{\zeta-1} W_{\lambda,\mu}(\phi t) E_{k,\alpha,\beta,r}^{\gamma,\delta,s}(\omega t^\sigma)dt$$

Let $\phi t = \theta$

$$= \int_0^\infty e^{-\frac{1}{2}\theta} \left(\frac{\theta}{\phi}\right)^{\zeta-1} W_{\lambda,\mu}(\theta) \sum_{n=0}^\infty \frac{(\gamma)_{sn,k}}{\Gamma_k(\alpha n+\beta)} \frac{\omega^n}{(\delta)_{rn,k}} \left(\frac{\theta}{\phi}\right)^{\sigma n} \frac{1}{\phi} d\theta$$

interchange the order of summation and integration to get

$$= \phi^{-\zeta} \frac{\Gamma_k(\delta)}{\Gamma_k(\gamma)} \sum_{n=0}^\infty \frac{\Gamma_k(\gamma+snk)\omega^n \phi^{-\sigma n}}{\Gamma_k(\alpha n+\beta)\Gamma_k(\delta+rnk)} \int_0^\infty e^{-\frac{1}{2}\theta}\, \theta^{\zeta+\sigma n-1} W_{\lambda,\mu}(\theta)d\theta$$

$$= \phi^{-\zeta} \frac{\Gamma_k(\delta)}{\Gamma_k(\gamma)} \sum_{n=0}^\infty \frac{\Gamma_k(\gamma+snk)\Gamma\left(\frac{\frac{k}{2}+k\mu+k\zeta+k\sigma n}{k}\right)\Gamma\left(\frac{\frac{k}{2}-k\mu+k\zeta+k\sigma n}{k}\right)\Gamma\left(\frac{k+kn}{k}\right)}{\Gamma_k(\alpha n+\beta)\Gamma_k(\delta+rnk)\Gamma\left(\frac{k-k\lambda+k\zeta+k\sigma n}{k}\right)} \frac{(\omega\phi^{-\sigma})^n}{n!}$$

Now using the identity $\Gamma\left(\frac{\eta}{k}\right) = k^{1-\frac{\eta}{k}}\,\Gamma_k(\eta)$ and definition of k-Wright function [6] in above, we get

$$= \phi^{-\zeta} k^{1-\lambda-\zeta} \frac{\Gamma_k(\delta)}{\Gamma_k(\gamma)} {}_4\Psi_3^k \left[\begin{array}{c} (\gamma, sk), (\frac{k}{2} \pm \mu k + \zeta k, \sigma k), (k,k) \\ (\beta,\alpha), (\delta, rk), (k-\lambda k+\zeta k, \sigma k) \end{array} ; \frac{\omega\phi^{-\sigma}}{k^{1+\sigma}} \right]$$

Special Case 5. For $r = p$, $s = q$ and $k = 1$ in the Theorem 3.5, we deduce the following results

$$\int_0^\infty e^{-\frac{1}{2}\phi t} t^{\zeta-1} W_{\lambda,\mu}(\phi t) E_{\alpha,\beta,p}^{\gamma,\delta,q}(\omega t^\sigma)dt$$

(29)
$$= \phi^{-\zeta} \frac{\Gamma(\delta)}{\Gamma(\gamma)} {}_4\Psi_3 \left[\begin{array}{c} (\gamma, q), (\frac{1}{2} \pm \mu + \zeta, \sigma), (1,1) \\ (\beta,\alpha), (\delta, p), (1-\lambda+\zeta, \sigma) \end{array} ; \omega\phi^{-\sigma} \right]$$

which is the Whittaker transform of generalized Mittag-Leffler function given by Salim and Faraj [18], page 8 □

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CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

REFERENCES

[1] A. Gupta, C.L. Parihar, *k*-New generalized Mittag Leffler function, J. Frac. Cal. Appl. 5(1) (2014), 165-176.

[2] E.T. Whittaker, G.N. Watson, A Course of modern analysis, Cambridge Univ. Press, Cambridge, 1962.

[3] F. Qi and K. S. Nisar, Some integral transform of the generalized *k*-Mittag-Leffler function, Publ. Inst. Math. 106(120) (2019), 125-133.

[4] G.A. Dorrego and R.A. Cerutti, The *k*-Mittag Leffler function, Int. J. Contemp. Math. Sci. 7(15) (2012), 705-716.

[5] H.J. Haubold, A.M. Mathai and R.K. Saxena, Mitag Leffler functions and their applications, J. Appl. Math. 2011 (2011), Article ID: 298628.

[6] K.S. Gehlot and J.C. Prajapati, Fractional calculus of generalized *k*-wright function, J. Fract. Calc. Appl. 4(2) (2013), 283-289.

[7] L.G. Romero, R.A. Cerutti and G.A. Dorrego, *k*-Weyl fractional integral, Int. J. Math. Anal. 6(34) (2012), 1685-1691.

[8] L.N. Mishra and M. Sen, On the concept of existence and local attractivity of solutions for some quadratic Volterra integral equation of fractional order, Appl. Math. Comput. 285 (2016), 174-183.

[9] L.N. Mishra, R.P. Agarwal and M. Sen, Solvability and asymptotic behavior for some nonlinear quadratic integral equation involving Erdélyi-Kober fractional integrals on the unbounded interval, Progr. Fract. Differ. Appl. 2(3) (2016), 153-168.

[10] L.N. Mishra, H.M. Srivastava and M. Sen, On existence results for some nonlinear functional-integral equations in Banach algebra with applications, Int. J. Anal. Appl. 11(1) (2016), 1-10.

[11] L.N. Mishra, M. Sen and R.N. Mohapatra, On existence theorems for some generalized nonlinear functional-integral equations with applications, Filomat, 31(7) (2017), 2081-2091.

[12] L.N. Mishra and R.P. Agarwal, On existence theorems for some nonlinear functional-integral equations, Dyn. Syst. Appl. 25 (2016), 303-320.

[13] L.N. Mishra, On existence and behavior of solutions to some nonlinear integral equations with applications, Ph.D. Thesis, National Institute of Technology, Silchar 788 010, Assam, India, (2017).

[14] R. Diaz and E. Pariguan, On hypergeometric functions and *k*-pochhammer symbol, Divulg. Mat. 15(2) (2007), 179-192.

[15] R. K. Saxena, J. Daiya and A. Singh, Integral transforms of the *k*-generalized Mitag Leffler function, Le Mat. LXIX (2014), 7-16.

[16] R.K. Saxena, Certain properties of generalized Mitag Leffler function, Conference of the Society of Special functions and their applications., 3rd Annual Conference, Chennai, India, (2002), 78-81.

[17] S. Mubeen, S.D. Purohit, M. Arshad and G. Rahman, Extension of *k*-gamma, *k*-beta functions and *k*-beta distribution, J. Math. Anal. 7(5) (2016), 118-131.

[18] T.O. Salim and A.W. Faraj, A generalization of Mittag-Leffler function and integral operator associated with fractional calculus, J. Fract. Calc. Appl. 3(5) (2012), 1-13.