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NEW GENERALIZED RATIONAL α_*-CONTRACTION FOR MULTIVALUED MAPPINGS IN b-METRIC SPACE

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Abstract. In this paper, we introduce the concept of generalized rational α_*-contraction for multivalued mappings in the setting of b-metric space. Further, we prove some common fixed point theorems for such rational contraction of multivalued mappings.

Keywords: fixed point; generalized rational α_*-contraction; multivalued mappings; b-metric space; α-admissible.

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1. Introduction

By the introduction of α-ψ-contraction, an important generalization of Banach contraction had been made by Samet, Vetro and Vetro [3]. They consider Ψ as a family of nondecreasing functions $\psi : [0,+\infty) \to [0,+\infty)$ so that $\Sigma_{n=1}^{+\infty}\psi^n(t) < +\infty$ for $t > 0$, where ψ^n denotes the n^{th} iterate of ψ. They also defined a new type of mapping called α-admissible mapping for their study. The concept of α-admissible draws the attention of many researchers and hence generalized further as triangular α-admissible [2], α-orbital admissible and triangular α-orbital admissible [3]. A new concept known as generalized α_*-ψ-Geraghty

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contraction type for multivalued mappings was introduced in 2017 by Ameer et al. [8]. Moreover, Bakhtin [4], by generalizing metric space, introduced b-metric space. For more results on various types of contraction mappings and b-metric space, one can see in [5, 7, 9, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28].

Here, the concept of generalized rational α_*-contraction for multivalued mappings in the setting of b-metric spaces is introduced.

2. PRELIMINARIES

We start this section with some definitions.

Definition 1. [4] *Let $s \geq 1$ be a real number and $d : X^2 \to [0, +\infty)$ be a mapping where $X \neq \phi$ such that for all $\kappa, \tau, \omega \in X$*

(i): $d(\kappa, \tau) = 0$ *implies and is implied by* $\kappa = \tau$,

(ii): $d(\kappa, \tau) = d(\tau, \kappa)$

(iii): $d(\kappa, \tau) \leq s[d(\kappa, \omega) + d(\omega, \tau)]$

Then we say that d is a b-metric on X.

Definition 2. [3] *Let $P : X \to X$ and $\alpha : X \times X \to [0, +\infty)$ be two mappings with the condition that if $\alpha(\kappa, \tau) \geq 1$ implies $\alpha(P\kappa, P\tau) \geq 1$, then P is said to be α-admissible .*

Definition 3. [10] *Let $P : X \to X$ and $\alpha : X \times X \to [0, +\infty)$ be two mappings such that P is α-admissible and satisfying the property that if $\alpha(\kappa, \omega) \geq 1$ and $\alpha(\omega, \tau) \geq 1$ imply $\alpha(\kappa, \tau) \geq 1$, then P is said to be triangular α-admissible.*

Definition 4. [11] *Let $P : X \to X$ and $\alpha : X \times X \to [0, +\infty)$ be two mappings with the condition that if $\alpha(\kappa, P\kappa) \geq 1$ implies $\alpha(P\kappa, P^2\kappa) \geq 1$, then P is said to be α-orbital admissible.*

Definition 5. [11] *Let $P : X \to X$ and $\alpha : X \times X \to [0, +\infty)$ be two mappings such that P is α-orbital admissible and satisfying the property that if $\alpha(\kappa, \tau) \geq 1$ and $\alpha(\tau, P\tau) \geq 1$ imply $\alpha(\kappa, P\tau) \geq 1$, then P is said to be triangular α-orbital admissible.*

Let us consider a b-metric space (X,d) and let $CB(X)$ denotes the family of all closed and bounded subsets of X.For $\kappa \in X$ and $M,N \in CB(X)$, we define

$$D(\kappa,M) = \inf_{a\in M} d(\kappa,a) \text{ and } D(M,N) = \sup_{a\in M} D(a,N).$$

Let $H : CB(X)^2 \to [0,+\infty)$ be a mapping defined as

$$H(M,N) = \max\{\sup_{\kappa\in M} D(\kappa,N), \sup_{\tau\in N} D(\tau,M)\},$$

for every $M,N \in CB(X)$. Then, H is a b-metric and it is named as a Hausdorff b-metric induced by a b-metric space (X,d).

Lemma 1. [6] *Let us consider a b-metric space (X,d). Then, for any $\kappa,\tau \in X$ and any $M,N \in CB(X)$, we have the following:*

- **(i):** $D(\kappa,N) \le d(\kappa,b)$*, for any $b \in N$,*
- **(ii):** $D(\kappa,N) \le H(M,N)$*,*
- **(iii):** $D(\kappa,M) \le s[d(\kappa,\tau) + D(\tau,N)]$.

Lemma 2. [6] *Let us consider two nonempty closed and bounded subsets, M and N of a b-metric space (X,d) and $q < 1$. Then, for every $a \in M$, there exists some $b \in N$ such that $qd(a,b) \le H(M,N)$.*

Definition 6. [12] *Let $\alpha : X \times X \to [0,+\infty)$ be a mapping and $P : X \to CB(X)$ be a multivalued mapping satisfying the property that if $\alpha(\kappa,\tau) \ge 1$ implies that $\alpha_*(P\kappa,P\tau) \ge 1$, where $\alpha_*(M,N) = \inf\{\alpha(\kappa,\tau) : \kappa \in M, \tau \in N\}$, then P is said to be α_*-admissible.*

Definition 7. [13] *Consider a b-metric space, (X,d) and a mapping $\alpha : X \times X \to [0,+\infty)$. If every Cauchy sequence $\{\kappa_n\}$ in X with $\alpha(\kappa_n,\kappa_{n+1}) \ge 1$ for all $n \in \mathbb{N}$ converges in X, then X is said to be complete.*

Lemma 3. [1] *Let us consider a b-metric space, (X,d) with $s \ge 1$ and a sequence $\{\kappa_n\}$ in X. If there exists $\gamma \in [0,1)$ satisfying $d(\kappa_{n+1},\kappa_n) \le \gamma d(\kappa_n,\kappa_{n-1})$ for all $n \in \mathbb{N}$, then $\{\kappa_n\}$ is a b-Cauchy sequence.*

Definition 8. [13] *Let (X,d) be a b-metric space. Let $P : X \to X$ be a mapping and $\alpha,\eta : X \times X \to [0,+\infty)$ be two functions. we say that P is α-η-continuous mapping on (X,d) if for given $\kappa \in X$ and a sequence $\{\kappa_n\}$ in X with $\alpha(\kappa_n,\kappa_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ such that $\kappa_n \to \kappa$ as $n \to +\infty$, then $P\kappa_n \to P\kappa$ as $n \to +\infty$.*
If $\eta(\kappa_n,\kappa_{n+1}) = 1$, then P is called an α-continuous mapping.

Let Ψ denote the class of functions $\psi : [0,+\infty) \to [0,+\infty)$ which is nondecreasing, continuous and $\psi(t) = 0$ if and only if $t = 0$.

3. Main Results

Following definitions and properties will be needed for our results.

Definition 9. [8] *Let $P,Q : X \to CB(X)$ be two multi-valued mappings and $\alpha : X \times X \to [0,+\infty)$ be a function. Then the pair (P,Q) is said to be triangular α_*-admissible if the following conditions hold:*

- **(i):** *(P,Q) is α_*-admissible; that is, $\alpha(\kappa,\tau) \geq 1$ implies $\alpha_*(P\kappa,Q\tau) \geq 1$ and $\alpha_*(Q\kappa,P\tau) \geq 1$, where*

$$\alpha_*(M,N) = \inf\{\alpha(\kappa,\tau) : \kappa \in M, \tau \in N\},$$

- **(ii):** *$\alpha(\kappa,\omega) \geq 1$ and $\alpha(\omega,\tau) \geq 1$ imply $\alpha(\kappa,\tau) \geq 1$.*

Definition 10. [8] *Let $P,Q : X \to CB(X)$ be two multi-valued mappings and $\alpha : X \times X \to [0,+\infty)$ be a function. Then the pair (P,Q) is said to be α_*-orbital admissible if the following condition holds:*

- **(i):** *$\alpha_*(\kappa,P\kappa) \geq 1$ and $\alpha_*(\kappa,Q\kappa) \geq 1$ imply $\alpha_*(P\kappa,Q^2\kappa) \geq 1$ and $\alpha_*(Q\kappa,P^2\kappa) \geq 1$.*

Definition 11. [8] *Let $P,Q : X \to CB(X)$ be two multi-valued mappings and $\alpha : X \times X \to [0,+\infty)$ be a function. Then the pair (P,Q) is said to be triangular α_*-orbital admissible, if the following conditions hold:*

- **(i):** *(P,Q) is α_*-orbital admissible.*
- **(ii):** *$\alpha(\kappa,\tau) \geq 1$, $\alpha_*(\tau,P\tau) \geq 1$ and $\alpha_*(\tau,Q\tau) \geq 1$ imply $\alpha_*(\kappa,P\tau) \geq 1$ and $\alpha_*(\kappa,Q\tau) \geq 1$.*

Lemma 4. [8] *Let $P,Q : X \to CB(X)$ be two multi-valued mappings such that the pair (P,Q) is triangular α_*-orbital admissible. Assume that there exists $\kappa_0 \in X$ such that $\alpha_*(\kappa_0, P\kappa_0) \geq 1$. Define a sequence $\{\kappa_n\}$ in X by $\kappa_{2i+1} \in P\kappa_{2i}$ and $\kappa_{2i+2} \in Q\kappa_{2i+1}$, where $i = 0,1,2,\ldots$. Then for $n,m \in \mathbb{N} \cup \{0\}$ with $m > n$, we have $\alpha(\kappa_n, \kappa_m) \geq 1$.*

Definition 12. [8] *Let (X,d) be a b-metric space. Let $P : X \to CB(X)$ be a multi-valued mapping and $\alpha : X \times X \to [0,+\infty)$ be a function. Then it is said that P is an α-continuous multi-valued mapping on $(CB(X),H)$ if whenever $\{\kappa_n\}$ is a sequence in X with $\alpha(\kappa_n, \kappa_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $\kappa \in X$ such that $\lim_{n\to+\infty} d(\kappa_n, \kappa) = 0$, then $\lim_{n\to+\infty} H(P\kappa_n, P\kappa) = 0$.*

Now, we introduce the concept of a pair of generalized rational α_*-contraction type for multi-valued mappings and used it to obtain common fixed point.

Definition 13. *In a b-metric space (X,d), $\alpha : X \times X \to [0,+\infty)$ be a function and $\varepsilon > 1$. We say that two multivalued mappings $P,Q : X \to CB(X)$ is a pair of generalized rational α_*-contraction type for multivalued mappings if there exists $\kappa, \tau \in X$ with $\alpha(\kappa, \tau) \geq 1$ and satisfies*

$$H(P\kappa, Q\tau) \leq \frac{1}{s^{\varepsilon}} M(\kappa, \tau) \tag{1}$$

where

$$\begin{aligned} M(\kappa,\tau) &= \max\Big\{ d(\kappa,\tau), D(\kappa,P\kappa), D(\tau,Q\tau), \frac{D(\kappa,P\kappa)D(\kappa,Q\tau)+D(\tau,Q\tau)D(\tau,P\kappa)}{1+s[D(\kappa,P\kappa)+D(\tau,D\tau)]}, \\ &\quad \frac{D(\kappa,P\kappa)D(\kappa,Q\tau)+D(\tau,Q\tau)D(\tau,P\kappa)}{1+D(\kappa,Q\tau)+D(\tau,P\kappa)} \Big\} \end{aligned} \tag{2}$$

Theorem 1. *In a b-metric space (X,d) with $s \geq 1$ and $\alpha : X \times X \to [0,+\infty)$ be a function. Let $P,Q : X \to CB(X)$ be a pair of generalized rational α_*-contraction type for multivalued mappings.*

- **(i):** *(X,d) is an α-complete;*
- **(ii):** *(P,Q) is triangular α_*-orbital admissible;*
- **(iii):** *$\alpha_*(\kappa_0, P\kappa_0) \geq 1$ for $\kappa_0 \in X$;*
- **(iv):** *P and Q are α-continuous.*

κ^* is a common fixed point of P and Q in X.

Proof. First, let $s>1$ and $\kappa_0\in X$ be so that $\alpha_*(\kappa_0,P\kappa_0)\geq 1$. Let $\kappa_1\in P\kappa_0$ so that $\alpha(\kappa_0,\kappa_1)\geq 1$ and $\kappa_1\neq\kappa_0$. Due to inequality (1)

$$0<D(\kappa_1,Q\kappa_1)\leq H(P\kappa_0,Q\kappa_1)\leq\frac{1}{s^{\varepsilon}}M(\kappa_0,\kappa_1)$$

Using Lemma 2 for $q=\frac{1}{s}<1$, there exists $\kappa_2\in Qx_1$ such that

$$\frac{1}{s}d(\kappa_1,\kappa_2)\leq H(P\kappa_0,Q\kappa_1)\leq\frac{1}{s^{\varepsilon}}M(\kappa_0,\kappa_1) \tag{3}$$

where

$$\begin{aligned}
M(\kappa_0,\kappa_1) &= \max\Bigg\{d(\kappa_0,\kappa_1),D(\kappa_0,P\kappa_0),D(\kappa_1,Q\kappa_1),\\
&\quad \frac{D(\kappa_0,P\kappa_0)D(\kappa_0,Q\kappa_1)+D(\kappa_1,Q\kappa_1)D(\kappa_1,P\kappa_0)}{1+s[D(\kappa_0,P\kappa_0)+D(\kappa_1,Q\kappa_1)]},\\
&\quad \frac{D(\kappa_0,P\kappa_0)D(\kappa_0,Q\kappa_1)+D(\kappa_1,Q\kappa_1)D(\kappa_1,P\kappa_0)}{1+D(\kappa_0,Q\kappa_1)+D(\kappa_1,P\kappa_0)}\Bigg\}\\
&= \max\Bigg\{d(\kappa_0,\kappa_1),d(\kappa_0,\kappa_1),D(\kappa_1,Q\kappa_1),\\
&\quad \frac{d(\kappa_0,\kappa_1)D(\kappa_0,Q\kappa_1)+D(\kappa_1,Q\kappa_1)d(\kappa_1,\kappa_1)}{1+s[d(\kappa_0,\kappa_1)+D(\kappa_1,Q\kappa_1)]},\\
&\quad \frac{d(\kappa_0,\kappa_1)D(\kappa_0,Q\kappa_1)+D(\kappa_1,Q\kappa_1)d(\kappa_1,\kappa_1)}{1+D(\kappa_0,Q\kappa_1)+d(\kappa_1,\kappa_1)}\Bigg\}\\
&= \max\Bigg\{d(\kappa_0,\kappa_1),D(\kappa_1,Q\kappa_1),\frac{d(\kappa_0,\kappa_1)s[d(\kappa_0,\kappa_1)+D(\kappa_1,Q\kappa_1)]}{1+s[d(\kappa_0,\kappa_1)+D(\kappa_1,Q\kappa_1)]},\\
&\quad \frac{d(\kappa_0,\kappa_1)D(\kappa_0,Q\kappa_1)}{1+D(\kappa_0,Q\kappa_1)}\Bigg\}\\
&= \max\{d(\kappa_0,\kappa_1),D(\kappa_1,Q\kappa_1)\}
\end{aligned}$$

If $\max\{d(\kappa_0,\kappa_1),D(\kappa_1,Q\kappa_1)\}=D(\kappa_1,Q\kappa_1)$, then by (3)

$$0<D(\kappa_1,Q\kappa_1)\leq\frac{1}{s^{\varepsilon}}D(\kappa_1,Q\kappa_1)$$

a contradiction, hence

$$\max\{d(\kappa_0,\kappa_1),D(\kappa_1,Q\kappa_1)\}=d(\kappa_0,\kappa_1)$$

By (3), we have

$$d(\kappa_1,\kappa_2)\leq\frac{1}{s^{\varepsilon-1}}d(\kappa_0,\kappa_1)$$

Similarly, for $\kappa_2 \in Q\kappa_1$ and $\kappa_3 \in P\kappa_2$, we have

$$\frac{1}{s}d(\kappa_2,\kappa_3) \le H(Q\kappa_1,P\kappa_2) \le \frac{1}{s^{\varepsilon}}M(\kappa_1,\kappa_2) \tag{4}$$

where

$$\begin{aligned}M(\kappa_1,\kappa_2) &= \max\Big\{d(\kappa_1,\kappa_2), D(\kappa_1,P\kappa_1), D(\kappa_2,Q\kappa_2),\\ &\quad \frac{D(\kappa_1,P\kappa_1)D(\kappa_1,Q\kappa_2)+D(\kappa_2,Q\kappa_2)D(\kappa_2,P\kappa_1)}{1+s[D(\kappa_1,P\kappa_1)+D(\kappa_2,Q\kappa_2)]}\\ &\quad \frac{D(\kappa_1,P\kappa_1)D(\kappa_1,Q\kappa_2)+D(\kappa_2,Q\kappa_2)D(\kappa_2,P\kappa_1)}{1+D(\kappa_1,Q\kappa_2)+D(\kappa_2,P\kappa_1)}\Big\}\\ &= \max\{d(\kappa_1,\kappa_2), D(\kappa_2,P\kappa_2)\}\end{aligned}$$

Due to inequality (1)

$$0 < D(\kappa_2,P\kappa_2) \le H(Q\kappa_1,P\kappa_2) \le \frac{1}{s^{\varepsilon}}M(\kappa_1,\kappa_2) \tag{5}$$

If $M(\kappa_1,\kappa_2) = D(\kappa_2,P\kappa_2)$, then by (5)

$$0 < D(\kappa_2,P\kappa_2) < \frac{1}{s^{\varepsilon}}D(\kappa_2,P\kappa_2)$$

which is impossible. Thus

$$\max\Big\{d(\kappa_1,\kappa_2), D(\kappa_2,P\kappa_2)\Big\} = d(\kappa_1,\kappa_2)$$

and by (4)

$$d(\kappa_2,\kappa_3) \le \frac{1}{s^{\varepsilon-1}}d(\kappa_1,\kappa_2)$$

Now, let $\{\kappa_n\}$ be a sequence in X so that $\kappa_{2i+1} \in P\kappa_{2i}$ and $\kappa_{2i+2} \in Q\kappa_{2i+1}$, $i = 0,1,2,\ldots$. So $\alpha_*(\kappa_0,P\kappa_0) \ge 1$ and (P,Q) is triangular α_*-orbital admissible, by Lemma 4

$$\alpha(\kappa_n,\kappa_{n+1}) \ge 1, \forall\, n \in \mathbb{N}\cup\{0\}.$$

Then,

$$0 < D(\kappa_{2i+1},Q\kappa_{2i+1}) \le H(P\kappa_{2i},Q\kappa_{2i+1}) \le \frac{1}{s^{\varepsilon}}M(\kappa_{2i},\kappa_{2i+1}) \tag{6}$$

and

$$\frac{1}{s}d(\kappa_{2i+1},\kappa_{2i+2}) \le H(P\kappa_{2i},Q\kappa_{2i+1}) \le \frac{1}{s^{\varepsilon}}M(\kappa_{2i},\kappa_{2i+1}) \tag{7}$$

for $n \in \mathbb{N} \cup \{0\}$, where

$$\begin{aligned}
& M(\kappa_{2i}, \kappa_{2i+1}) \\
= \ & \max\Big\{ d(\kappa_{2i}, \kappa_{2i+1}), D(\kappa_{2i}, P\kappa_{2i}), D(\kappa_{2i+1}, Q\kappa_{2i+1}), \\
& \frac{D(\kappa_{2i}, P\kappa_{2i})D(\kappa_{2i}, Q\kappa_{2i+1}) + D(\kappa_{2i+1}, Q\kappa_{2i+1})D(\kappa_{2i+1}, P\kappa_{2i})}{1 + s[D(\kappa_{2i}, P\kappa_{2i}) + D(\kappa_{2i+1}, Q\kappa_{2i+1})]}, \\
& \frac{D(\kappa_{2i}, P\kappa_{2i})D(\kappa_{2i}, Q\kappa_{2i+1}) + D(\kappa_{2i+1}, Q\kappa_{2i+1})D(\kappa_{2i+1}, P\kappa_{2i})}{1 + D(\kappa_{2i}, Q\kappa_{2i+1}) + D(\kappa_{2i+1}, P\kappa_{2i})} \Big\} \\
= \ & \max\Big\{ d(\kappa_{2i}, \kappa_{2i+1}), d(\kappa_{2i}, \kappa_{2i+1}), d(\kappa_{2i+1}, \kappa_{2i+2}), \\
& \frac{d(\kappa_{2i}, \kappa_{2i+1}), D(\kappa_{2i}, Q\kappa_{2i+1}) + D(\kappa_{2i+1}, Q\kappa_{2i+1})d(\kappa_{2i+1}, \kappa_{2i+1})}{1 + s[d(\kappa_{2i}, \kappa_{2i+1}) + D(\kappa_{2i+1}, Q\kappa_{2i+1})]}, \\
& \frac{d(\kappa_{2i}, \kappa_{2i+1})D(\kappa_{2i}, Q\kappa_{2i+1}) + D(\kappa_{2i+1}, Q\kappa_{2i+1})d(\kappa_{2i+1}, \kappa_{2i+1})}{1 + D(\kappa_{2i}, Q\kappa_{2i+1}) + d(\kappa_{2i+1}, \kappa_{2i+1})} \Big\} \\
= \ & \max\Big\{ d(\kappa_{2i}, \kappa_{2i+1}), D(\kappa_{2i+1}, Q\kappa_{2i+1}), \\
& \frac{d(\kappa_{2i}, \kappa_{2i+1})s[d(\kappa_{2i}, \kappa_{2i+1}) + D(\kappa_{2i+1}, Q\kappa_{2i+1})]}{1 + s[d(\kappa_{2i}, \kappa_{2i+1}) + D(\kappa_{2i+1}, Q\kappa_{2i+1})]}, \\
& \frac{d(\kappa_{2i}, \kappa_{2i+1})D(\kappa_{2i}, Q\kappa_{2i+1})}{1 + D(\kappa_{2i}, Q\kappa_{2i+1})} \Big\} \\
= \ & \max\{d(\kappa_{2i}, \kappa_{2i+1}), D(\kappa_{2i+1}, Q\kappa_{2i+1})\}
\end{aligned}$$

we get,

$$M(\kappa_{2i}, \kappa_{2i+1}) \leq \max\{d(\kappa_{2i}, \kappa_{2i+1}), D(\kappa_{2i+1}, Q\kappa_{2i+1})\}.$$

If $\max\{d(\kappa_{2i}, \kappa_{2i+1}), D(\kappa_{2i+1}, Q\kappa_{2i+1})\} = D(\kappa_{2i+1}, Q\kappa_{2i+1})$, then from (6), we have

$$0 < D(\kappa_{2i+1}, Q\kappa_{2i+1}) \leq \frac{1}{s^{\varepsilon}} D(\kappa_{2i+1}, Q\kappa_{2i+1})$$

a contradiction and hence

$$\max\{d(\kappa_{2i}, \kappa_{2i+1}), D(\kappa_{2i+1}, Q\kappa_{2i+1})\} = d(\kappa_{2i}, \kappa_{2i+1}).$$

Further, by (7) we get

$$d(\kappa_{2i+2}, \kappa_{2i+1}) \leq \frac{1}{s^{\varepsilon-1}} d(\kappa_{2i+1}, \kappa_{2i}).$$

Thus $d(\kappa_{n+1}, \kappa_{n+2}) \leq \frac{1}{s^{\varepsilon-1}} d(\kappa_n, \kappa_{n+1})$ holds for all $n \in \mathbb{N} \cup \{0\}$ and hence $\{x_n\}$ is a Cauchy sequence.

Due to the α-completeness of (X,d) and $\alpha(\kappa_n,\kappa_{n+1}) \geq 1$ we have, $\kappa^* \in X$ for all $n \in \mathbb{N} \cup \{0\}$ such that

$$\lim_{n\to+\infty} d(\kappa_n,\kappa_*) = 0 \Rightarrow \lim_{i\to+\infty} d(\kappa_{2i+1},\kappa^*) = 0 \text{ and } \lim_{i\to+\infty} d(\kappa_{2i+2},\kappa^*) = 0.$$

Due to α-continuity of Q, $\lim_{n\to+\infty} H(Q\kappa_{2i+1}, Q\kappa^*) = 0$.

Thus,

$$\begin{aligned} D(\kappa^*, Q\kappa^*) &\leq s[d(\kappa^*,\kappa_{2i+1}) + D(\kappa_{2i+1}, Q\kappa^*)] \\ &\leq s[d(\kappa^*,\kappa_{2i+1}) + H(Q\kappa_{2i+1}, Q\kappa^*)] \\ &\to s[0+0] = 0 \end{aligned}$$

so, $\kappa^* \in Q\kappa^*$. Similarly, $\kappa^* \in P\kappa^*$.

Hence, $\kappa^* \in X$ is a common fixed point of P and Q. □

Theorem 2. *In a b-metric space (X,d) with $s \geq 1$ and $\alpha : X \times X \to [0,+\infty)$ be a function. Let $P,Q : X \to CB(X)$ be a pair of generalized rational α_*-contraction type for multivalued mappings.*

- **(i):** *(X,d) is an α-complete;*
- **(ii):** *(P,Q) is triangular α_*-orbital admissible;*
- **(iii):** *$\alpha_*(\kappa_0, P\kappa_0) \geq 1$ for $\kappa_0 \in X$;*
- **(iv):** *if $\{\kappa_n\}$ is a sequence in X such that $\alpha(\kappa_n,\kappa_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $\kappa_n \to \kappa^* \in X$ as $n \to +\infty$, then there exists a subsequence $\{\kappa_{n(k)}\}$ of $\{\kappa_n\}$ such that $\alpha(\kappa_{n(k)},\kappa^*) \geq 1$ for all $k \in \mathbb{N} \cup \{0\}$.*

κ^* is a common fixed point of P and Q in X.

Proof. Similar to the proof of Theorem 1, let $\{\kappa_n\}$ be a sequene in X as $\kappa_{2i+1} \in P\kappa_{2i}$ and $\kappa_{2i+2} \in Q\kappa_{2i+1}$ where $i = 0,1,2,\ldots$ with $\alpha(\kappa_n,\kappa_{n+1}) \geq 1$, for all $n \in \mathbb{N} \cup \{0\}$ such that $\{x_n\}$ converges to $\kappa^* \in X$. By condition (iv), there exists a subsequence $\{\kappa_{n(k)}\}$ of $\{\kappa_n\}$ such that

$\alpha(\kappa_{n(k)},\kappa^*)\geq 1$ for all k. Therefore,

$$\begin{aligned}\frac{1}{s}D(\kappa^*,Q\kappa^*) &\leq d(\kappa^*,\kappa_{2n(k)+1})+D(\kappa_{2n(k)+1},Q\kappa^*)\\ &\leq d(\kappa^*,\kappa_{2n(k)+1})+H(P\kappa_{2n(k)},Q\kappa^*)\\ &\leq d(\kappa^*,\kappa_{2n(k)+1})+\frac{1}{s^{\varepsilon}}M(\kappa_{2n(k)},\kappa^*) \qquad (8)\end{aligned}$$

where

$$\begin{aligned}&M(\kappa_{2n(k)},\kappa^*)\\ =\ &\max\Big\{d(\kappa_{2n(k)},\kappa^*),D(\kappa_{2n(k)},P\kappa_{2n(k)}),D(\kappa^*,Q\kappa^*),\\ &\frac{D(\kappa_{2n(k)},P\kappa_{2n(k)})D(\kappa_{2n(k)},Q\kappa^*)+D(\kappa^*,Q\kappa^*)D(\kappa^*,P\kappa_{2n(k)})}{1+s[D(\kappa_{2n(k)},P\kappa_{2n(k)})+D(\kappa^*,Q\kappa^*)]},\\ &\frac{D(\kappa_{2n(k)},P\kappa_{2n(k)})D(\kappa_{2n(k)},Q\kappa^*)+D(\kappa^*,Q\kappa^*)D(\kappa^*,P\kappa_{2n(k)})}{1+D(\kappa_{2n(k)},Q\kappa^*)+D(\kappa^*,P\kappa_{2n(k)})}\Big\}\\ =\ &\max\Big\{d(\kappa_{2n(k)},\kappa^*),D(\kappa_{2n(k)},P\kappa_{2n(k)}),D(\kappa^*,Q\kappa^*),\\ &\frac{D(\kappa_{2n(k)},P\kappa_{2n(k)})D(\kappa_{2n(k)},Q\kappa^*)+D(\kappa^*,Q\kappa^*)D(\kappa^*,P\kappa_{2n(k)})}{1+s[D(\kappa_{2n(k)},P\kappa_{2n(k)})+D(\kappa^*,Q\kappa^*)]},\\ &\frac{D(\kappa_{2n(k)},P\kappa_{2n(k)})D(\kappa_{2n(k)},Q\kappa^*)+D(\kappa^*,Q\kappa^*)D(\kappa^*,P\kappa_{2n(k)})}{1+D(\kappa_{2n(k)},Q\kappa^*)+D(\kappa^*,P\kappa_{2n(k)})}\Big\} \qquad (9)\end{aligned}$$

Applying $k\to+\infty$, we get $\lim_{k\to+\infty}M(\kappa_{2n(k)},\kappa^*)=D(\kappa^*,Q\kappa^*)$. Let $\kappa^*\notin Q\kappa^*$, then $D(\kappa^*,Q\kappa^*)>0$,

a contradiction. Applying $k\to+\infty$, we get

$$\frac{1}{s}D(x^*,Qx^*)\leq d(x^*,x_{2n(k)+1})+D(x_{2n(k)+1},Qx^*),$$

which contradicts $\varepsilon>1$, and hence $\kappa^*\in Q\kappa^*$ i.e. κ^* is the fixed point of Q. Similarly, we have $\kappa^*\in P\kappa^*$. Thus, $\kappa^*\in X$ is the common fixed point of P and Q. □

Corollary 1. *In a complete b-metric space (X,d) with $s\geq 1$ and $\alpha: X\times X\to[0,+\infty)$ be a function. Let $P: X\to CB(X)$ be a generalized rational α_*-contraction type for multi-valued mappings*

(i): *(X,d) is an α-complete;*

(ii): *P is triangular α_*-orbital admissible;*

(iii): $\alpha_*(\kappa_0, P\kappa_0) \geq 1$ *for* $\kappa_0 \in X$*;*

(iv): *P is an α-continuous multi-valued mapping or if $\{\kappa_n\}$ is a sequence in X such that $\alpha(\kappa_n, \kappa_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ such that $\kappa_n \to \kappa^* \in X$ as $n \to +\infty$, then there exists a subsequence $\{\kappa_{n(k)}\}$ of $\{\kappa_n\}$ such that $\alpha(\kappa_{n(k)}, \kappa^*) \geq 1$ for all $k \in \mathbb{N} \cup \{0\}$.*

κ^* is a fixed point of P in X.

Following corollary can be otained by putting $\psi(t) = t$ in Theorems 1 and 2.

Corollary 2. *In a complete b-metric space (X, d) with $s \geq 1$ and $\alpha : X \times X \to [0, +\infty)$ be a function. Let $P, Q : X \to CB(X)$ be two multivalued mappings*

(i): *(X, d) is an α-complete;*

(ii): *there exists $\mathfrak{g} \in \mathscr{G}$ such that for $\kappa, \tau \in X$ with $\alpha(\kappa, \tau) \geq 1$, the pair (P, Q) satisfies the following inequality:*

$$s^3 H(P\kappa, Q\tau) \leq \mathfrak{g}(M(\kappa, \tau)).(M(\kappa, \tau)),$$

where

$$\begin{aligned} M(\kappa, \tau) \;=\; \max\Bigg\{ & d(\kappa, \tau), D(\kappa, P\kappa), D(\tau, Q\tau), \\ & \frac{D(\kappa, P\kappa)D(\kappa, Q\tau) + D(\tau, Q\tau)D(\tau, P\kappa)}{1 + s[D(\kappa, P\kappa) + D(\tau, Q\tau)]}, \\ & \frac{D(\kappa, P\kappa)D(\kappa, Q\tau) + D(\tau, Q\tau)D(\tau, P\kappa)}{1 + D(\kappa, Q\tau) + D(\tau, P\kappa)} \Bigg\}; \end{aligned}$$

(iii): *(P, Q) is triangular α_*-orbital admissible;*

(iv): $\alpha_*(\kappa_0, P\kappa_0) \geq 1$ *for* $\kappa_0 \in X$*;*

(v): *P and Q are α-continuous or if $\{\kappa_n\}$ is a sequence in X such that $\alpha(\kappa_n, \kappa_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ such that $\kappa_n \to \kappa^* \in X$ as $n \to +\infty$, then there exists a subsequence $\{\kappa_{n(k)}\}$ of $\{\kappa_n\}$ such that $\alpha(\kappa_{n(k)}, \kappa^*) \geq 1$ for all $k \in \mathbb{N} \cup \{0\}$.*

κ^* is a common fixed point of P and Q.

4. Consequences

Definition 14. *Let (X,d) be a b-metric space. Let $\alpha : X \times X \to [0,+\infty)$ be a function and $P,Q : X \to X$ be two mappings. The pair (P,Q) is said to be a generalized rational α-ψ-Geraghty contraction type mapping, if there exists $\mathfrak{g} \in \mathscr{G}$ and $\psi \in \Psi$ such that for all $\kappa, \tau \in X$ with $\alpha(\kappa,\tau) \geq 1$, the pair (P,Q) satisfies the following inequality*

$$\psi(s^3 d(P\kappa, Q\tau)) \leq \mathfrak{g}(\psi(M(\kappa,\tau))).\psi(M(\kappa,\tau)).$$

where

$$\begin{aligned} M(\kappa,\tau) &= \max\Bigg\{ d(\kappa,\tau), D(\kappa,P\kappa), D(\tau,Q\tau), \\ &\quad \frac{d(\kappa,P\kappa)d(\kappa,Q\tau)+d(\tau,Q\tau)d(\tau,P\kappa)}{1+s[d(\kappa,P\kappa)+d(\tau,Q\tau)]}, \\ &\quad \frac{d(\kappa,P\kappa)d(\kappa,Q\tau)+d(\tau,Q\tau)d(\tau,P\kappa)}{1+d(\kappa,Q\tau)+d(\tau,P\kappa)} \Bigg\}. \end{aligned}$$

Theorem 3. *In a b-metric space (X,d) with $s \geq 1$ and $\alpha : X \times X \to [0,+\infty)$ be a function. Let $P,Q : X \to X$ be a pair of generalized rational α-ψ-Geraghty contraction type for multi-valued mappings*

- **(i):** (X,d) *is an α-complete;*
- **(ii):** (P,Q) *is triangular α-orbital admissible;*
- **(iii):** $\alpha(\kappa_0, P\kappa_0) \geq 1$ *for* $\kappa_0 \in X$*;*
- **(iv):** *P and Q are α-continuous.*

κ^* is a common fixed point of P and Q in X.

Theorem 4. *In a b-metric space (X,d) with $s \geq 1$ and $\alpha : X \times X \to [0,+\infty)$ be a function. Let $P,Q : X \to X$ be a pair of generalized rational α-ψ-Geraghty contraction type for multivalued mappings*

- **(i):** (X,d) *is an α-complete b-metric space;*
- **(ii):** (P,Q) *is triangular α-orbital admissible;*
- **(iii):** $\alpha(\kappa_0, P\kappa_0) \geq 1$ *for* $\kappa_0 \in X$*;*

(iv): *if* $\{\kappa_n\}$ *is a sequence in* X *such that* $\alpha(\kappa_n,\kappa_{n+1}) \geq 1$ *for all* $n \in \mathbb{N}\cup\{0\}$ *and* $\kappa_n \to \kappa^* \in X$ *as* $n \to +\infty$*, then there exists a subsequence* $\{\kappa_{n(k)}\}$ *of* $\{\kappa_n\}$ *such that* $\alpha(\kappa_{n(k)},\kappa^*) \geq 1$ *for all* $k \in \mathbb{N}\cup\{0\}$.

κ^* *is a common fixed point of* P *and* Q *in* X.

Corollary 3. *Let* $(X,\preceq)$ *be a partially ordered set. Let there exists a complete b-metric space* (X,d)*. Suppose* $P,Q : X \to X$ *are two mappings satisfying the following conditions:*

(i): *there exists* $\mathfrak{g} \in \mathscr{G}$ *and* $\psi \in \psi$ *such that*

$$\psi(s^3 d(P\kappa,Q\tau)) \leq \mathfrak{g}(\psi(M(\kappa,\tau))).\psi(M(\kappa,\tau)),$$

where

$$\begin{aligned} M(\kappa,\tau) \;\;=\;\; \max\Big\{ & d(\kappa,\tau), D(\kappa,P\kappa), D(\tau,Q\tau), \\ & \frac{d(\kappa,P\kappa)d(\kappa,Q\tau)+d(\tau,Q\tau)d(\tau,P\kappa)}{1+s[d(\kappa,P\kappa)+d(\tau,Q\tau)]}, \\ & \frac{d(\kappa,P\kappa)d(\kappa,Q\tau)+d(\tau,Q\tau)d(\tau,P\kappa)}{1+d(\kappa,Q\tau)+d(\tau,P\kappa)}\Big\}; \end{aligned}$$

for all $\kappa,\tau \in X$ *with* $\kappa \preceq \tau$*;*

(ii): *P and Q are nondecreasing;*

(iii): $\kappa_0 \preceq P\kappa_0$ *for* $\kappa_0 \in X$*;*

(iv): *either P and Q are continuous or if* $\{\kappa_n\}$ *is a nondecreasing sequence such that* $\kappa_n \to \kappa^* \in X$ *as* $n \to +\infty$*, then there exists a subsequence* $\{\kappa_{n(k)}\}$ *of* $\{\kappa_n\}$ *such that* $\kappa_{n(k)} \preceq \kappa^*$ *for all* $k \in \mathbb{N}\cup\{0\}$.

x^* *is a common fixed point of* P *and* Q *in* X.

CONCLUSION

In this paper, the concept of generalized rational α_*-ψ-Geraghty contraction for multivalued mappings is introduced. Further, the concept is used in the setting of b-metric space to prove three common fixed point theorems and some corollaries. Some consequences are also discussed. An application is also presented in differential equation.

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CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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