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ON WEAKLY α-SHIFTING RING

MANJURI DUTTA[1,*], KHWAIRAKPAM HERACHANDRA SINGH[2], NAZEER ANSARI[3]

[1]Department of Mathematics, North Eastern Regional Institute of Science and Technology, Nirjuli-791109, Arunachal Pradesh, India.

[2]Department of Mathematics, Manipur University, Canchipur-795003, Manipur, India.

[3]Department of Mathematics, National Institute of Technology, Calicut, Kozhikode-673601, Kerala, India.

Abstract. For a ring endomorphism α, we introduce weakly α-shifting ring which is an extension of reduced as well as α-shifting ring. The notion of weakly α-shifting ring is a generalization of weak α-compatible ring. We investigate various properties of this ring including some kinds of examples in the process of development of this new concept.

Keywords: weak α-symmetric ring; weak α-reversible ring; weak α-compatible ring; weak α-rigid ring; semi-commutative ring.

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1. Introduction

Throughout this article, all rings are associative with identity 1 and $\alpha : R \longrightarrow R$ is an ring endomorphism of a ring R. An element x of a ring R is nilpotent whenever $x^m = 0$ for some positive integer m. We denote $Nil(R)$, the set of nilpotent elements of R. We recall that a ring is said to be reduced whenever it has no non zero nilpotent elements. Again a ring is defined as symmetric in [1] whenever $xyz = 0 \Rightarrow xzy = 0$ for any $x, y, z \in R$. In 1999, Cohn [2] defined that

*Corresponding author
E-mail address: mdutta499@gmail.com
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a ring is said to be reversible if $xy = 0$ implies $yx = 0$ for any $x, y \in R$. Again a ring is called semicommutative if for any $x, y \in R$, $xy = 0$ implies $xRy = 0$, this ring is also called ZI ring in [14]. If a ring is commutative, then it is always reversible, symmetric and semicommutative. It is mentioned that reduced rings are symmetric [3, Theorem I.3]. We can see that symmetric rings are reversible and reversible rings are always semicommutative by using their definitions. D.D. Anderson and V. Camillo provided the examples of non reduced symmetric ring [3, Example II.5]and a non symmetric reversible ring [3, Example I.5]. Moreover Example 1.5 of [15] is given to establish that a semicommutative ring may not be reversible.

In 1996, J. Krempa furnished a new concept rigid endomorphism of a ring in [5]. An endomorphism α of R is stated as rigid if $x\alpha(x) = 0$ implies $x = 0$ for any $x \in R$. A ring is said to be α-rigid if there exists a some rigid endomorphism α. Motivated by that new term, L. Ouyang defined weak α-rigid ring [6] in the context of $Nil(R)$ in 2008. A ring is weak α-rigid if $x\alpha(x) \in Nil(R) \Leftrightarrow x \in Nil(R)$ for any $x \in R$. Another term α-reversible ring [7] was introduced in 2009. A ring R is right (left) α-reversible whenever $xy = 0$ implies $y\alpha(x) = 0$ ($\alpha(y)x = 0$) for any $x, y \in R$. A ring is said to be α-reversible if it satisfies the both conditions of right and left α-reversible. In 2014, A. Bahlekeh introduced weak α-reversible ring [8] by extending α-reversible ring with the help of the set $Nil(R)$. Whenever $xy \in Nil(R)$ for any $x, y \in R$ implies $y\alpha(x) \in Nil(R)$, then R is said to be weak α-reversible. On the other hand, T.K. Kwak extended the concept of symmetric ring to α-symmetric [9] by using ring endomorphism α in 2007. A ring R is right(left) α-symmetric if $xyz = 0 \Rightarrow xz\alpha(y) = 0$ ($\alpha(y)xz = 0$) for any $x, y, z \in R$. Motivated by this above definition, L.Ouyang and H.Chen introduced weak α-symmetric ring [10] in 2010. A ring R is weak α-symmetric ring if $xyz \in Nil(R)$ implies $xz\alpha(y) \in Nil(R)$ for any $x, y, z \in R$. A ring R is α-compatible [11] if $xy = 0 \Leftrightarrow x\alpha(y) = 0$ for any $x, y \in R$. Again in 2011, weak α-compatible ring [12] was introduced by using the weak condition to α-compatible ring. A ring R is weak α-compatible if $xy \in Nil(R) \Leftrightarrow x\alpha(y) \in Nil(R)$ for any $x, y \in R$. In 2010, the concept of reversible ring extend to α-shifting ring [13] by using ring endomorphism α. They defined R is right(left) α-shifting whenever $x\alpha(y) = 0$ ($\alpha(x)y = 0$) implies $y\alpha(x) = 0$ ($\alpha(y)x = 0$) for any $x, y \in R$. The ring is α-shifting whenever it satisfies both the conditions of right and left α-shifting.

Motivated by all of the above definitions, we have introduced the concept of weakly α-shifting ring which is an extension of reduced as well as α-shifting ring. The notion of weakly α-shifting ring is a generalization of weak α-compatible ring.

2. Weakly α-Shifting Ring

In this section we introduce and study a class of rings, called weakly α-shifting ring which is an extension of α-shifting rings. We prove that weakly α-shifting ring is a generalization of weak α-compatible ring. We investigate the connections of weakly α-shifting ring to weak α-reversible ring, weak α-rigid ring and weak α-symmetric rings. Moreover some results of α-shifting rings can be extended to weakly α-shifting ring. We now start with the following definition:

Definition 2.1. *A ring R is called weakly α-shifting if $x\alpha(y) \in Nil(R) \Rightarrow y\alpha(x) \in Nil(R)$ for any $x, y \in R$.*

It is very easy to check that

Lemma 2.1. *If $xy \in Nil(R)$ for any $x, y \in R$ then $yx \in Nil(R)$.*

We get the following remark from the above Lemma and the definition of weakly α-shifting ring.

Remark 2.1. *All rings are always weakly Id-shifting rings where Id is the identity ring endomorphism.*

It is shown that the concept of reduced ring and α-shifting ring do not depend on each other by the Example 1.1(2) and Example 2.3 of [13]. Now the next proposition shows the connection between α-shifting and weakly α-shifting ring.

Proposition 2.1. *If R is reduced and α-shifting ring then R is weakly α-shifting ring.*

Proof. Let $x\alpha(y) \in Nil(R)$ for any $x, y \in R$. It implies $x\alpha(y) = 0$ as R is reduced ring. Since R is α-shifting, so $x\alpha(y) = 0$ implies $y\alpha(x) = 0$. As R is reduced, $y\alpha(x) \in Nil(R)$. Thus R is weakly α-shifting ring.

Let $T_n(R)$ denote $n\times n$ upper triangular matrix ring over R. Then the map $\bar{\alpha}: T_n(R)\longrightarrow T_n(R)$ defined by $\bar{\alpha}((a_{ij}))=(\alpha(a_{ij}))$ for all $(a_{ij})\in T_n(R)$ is a ring endomorphism of $T_n(R)$.

Proposition 2.2. *R is weakly α-shifting ring if and only if $T_n(R)$ is weakly $\bar{\alpha}$-shifting ring for any $n\in\mathbb{N}$.*

Proof. Let R be a weakly α-shifting ring. Let us consider $A\bar{\alpha}(B)\in Nil(T_n(R))$ for

$$A=\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n}\\ 0 & a_{22} & \dots & a_{2n}\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \dots & a_{nn}\end{pmatrix} \text{ and } B=\begin{pmatrix} b_{11} & b_{12} & \dots & b_{1n}\\ 0 & b_{22} & \dots & b_{2n}\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \dots & b_{nn}\end{pmatrix} \text{ in } T_n(R).$$

Therefore $(A\bar{\alpha}(B))^k=0$ for some positive integer k. It implies $(a_{ii}\alpha(b_{ii}))^k=0$ for $i=1,2,\dots,n$. Then $a_{ii}\alpha(b_{ii})\in Nil(R)$. Consequently $b_{ii}\alpha(a_{ii})\in Nil(R)$ as R is weakly α-shifting ring. So $(b_{ii}\alpha(a_{ii}))^{k_i}=0$ for some positive integer k_i. Now $(B\bar{\alpha}(A))^{\bar{k}}\in Nil(T_n(R))$ where $\bar{k}=max\{k_1,k_2,...,k_n\}$. Thus $B\bar{\alpha}(A)\in Nil(T_n(R))$ and so $T_n(R)$ is weakly $\bar{\alpha}$-shifting ring.
Conversely let $T_n(R)$ be weakly $\bar{\alpha}$-shifting ring. Now let us consider $x\alpha(y)\in Nil(R)$ for $x,y\in R$. It implies $(x\alpha(y))^m=0$ for some positive integer m. It leads to

$$\left(\begin{pmatrix} x & 0 & \dots & 0\\ 0 & 0 & \dots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \dots & a\end{pmatrix}\bar{\alpha}\left(\begin{pmatrix} y & 0 & \dots & 0\\ 0 & 0 & \dots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \dots & 0\end{pmatrix}\right)\right)^m=0.$$

Now by using the definition of weakly $\bar{\alpha}$-shifting of $T_n(R)$,

$$\begin{pmatrix} y & 0 & \dots & 0\\ 0 & 0 & \dots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \dots & 0\end{pmatrix}\begin{pmatrix} \alpha(x) & 0 & \dots & 0\\ 0 & 0 & \dots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \dots & 0\end{pmatrix}\in Nil(T_n(R)).$$

Now it is very easy to check that $y\alpha(x)\in Nil(R)$. Thus we have R is weakly α-shifting ring.

The next example shows that there exists a weakly α-shifting ring which is not α-shifting.

Example 2.1. *We prove that the ring $R \oplus R$ is weakly α-shifting ring as shown in Example[2.15]. Then $T_2(R \oplus R)$ is weakly $\bar{\alpha}$-shifting by immediate consequence of above Proposition 2.5. Let us consider* $A = \begin{pmatrix} (0,0) & (0,0) \\ (0,0) & (1,0) \end{pmatrix}$ *and* $B = \begin{pmatrix} (0,0) & (0,1) \\ (0,0) & (0,0) \end{pmatrix}$ *in $T_2(R \oplus R)$. Therefore we have $A\bar{\alpha}(B) = 0$ but* $B\bar{\alpha}(A) = \begin{pmatrix} (0,0) & (0,1) \\ (0,0) & (0,0) \end{pmatrix} \neq 0$. *Thus $T_2(R \oplus R)$ is not $\bar{\alpha}$-shifting.*

Lemma 2.2. *Let $\alpha : R \longrightarrow S$ be any ring endomorphism. If $x \in Nil(R)$ for any $x \in R$, then $\alpha(x) \in Nil(S)$.*

Remark 2.2. *The converse of the Lemma* **??** *holds whenever α is a monomorphism.*

Proposition 2.3. *Let R be a weak α-compatible ring. Then we have the following:*

(i) *R is weak α-reversible.*

(ii) *R is weakly α-shifting ring.*

Proof. (i) Let $xy \in Nil(R)$ for any $x, y \in R$. Then $yx \in Nil(R) \Rightarrow y\alpha(x) \in Nil(R)$ by using Lemma 2.2 and the condition that R is weak α-compatible ring. Thus R is a weak α-reversible ring.
(ii) Let $x\alpha(y) \in Nil(R)$ for any $x, y \in R$. It implies $xy \in Nil(R)$ as R is weak α-compatible ring. Again $xy \in Nil(R)$ implies $y\alpha(x) \in Nil(R)$ by using Proposition 2.9(i). Thus R is weakly α -shifting ring.

Proposition 2.4. *Let R be a weak α-reversible ring for a monomorphism α. Then we have the following:*

(i) *R is weak α-compatible.*

(ii) *R is weakly α-shifting.*

Proof. (i) Let us consider $xy \in Nil(R)$ for any $x, y \in R$. Now $xy \in Nil(R) \Rightarrow yx \in Nil(R) \Rightarrow x\alpha(y) \in Nil(R)$ by using Lemma 2.2 and R is weak α-reversible ring.
Conversely, let $x\alpha(y) \in Nil(R)$ for any $x, y \in R$. Then $\alpha(y)\alpha(x) \in Nil(R)$ by using the definition of weak α-reversible ring of R. It implies $\alpha(yx) \in Nil(R) \Rightarrow yx \in Nil(R)$ by using Remark 2.8. Now we have $xy \in Nil(R)$ by using Lemma 2.2. So for any $x, y \in R$, $xy \in Nil(R) \Leftrightarrow x\alpha(y) \in Nil(R)$. Thus R is weak α-compatible.

(ii) From Proposition 2.10(i), we have R is weak α-compatible. Now R is weakly α-shifting by using Proposition 2.9(ii).

Proposition 2.5. *Weak α-symmetric rings are always weak α-reversible.*

Proof. Let R be a weak α-symmetric ring. Let $xy \in Nil(R)$ for any $x, y \in R$. Since R is weak α-symmetric ring, so $xy = 1.x.y \in Nil(R)$ implies $y\alpha(x) = 1.y.\alpha(x) \in Nil(R)$. Thus R is weak α-reversible.

The next corollary is a direct deduction of Proposition 2.11 and Proposition 2.10(ii).

Corollary 2.1. *Weak α-symmetric rings are weakly α-shifting whenever α is monomorphism.*

The next example provides a weak α-symmetric which is not weak α-compatible.

Example 2.2. *Let us consider that F be any field and $R = F[x]$. Let $\alpha : R \longrightarrow R$ such that $\alpha(f(x)) = f(0)$ for all $f(x) \in F[x]$. Clearly α is a ring endomorphism of $F[x]$. We know that R is a domain. We can easily show that for any ring endomorphism α, domains are weak α-symmetric ring. Thus $F[x]$ is weak α-symmetric. Now let $f(x) = x \neq 0$ and $g(x) = a \neq 0$. So clearly $g(x)\alpha(f(x)) = 0 \in Nil(R)$. But $g(x)f(x) \neq 0 \notin Nil(R)$ where $Nil(R) = 0$ as R is domain. Thus we can see that R is not weak α-compatible.*

Remark 2.3. *Since the ring $R = F[x]$ given in Example 2.13 is also a weak α-reversible ring by Proposition 2.11. Therefore the above example also provides a weak α-reversible ring which is not weak α-compatible.*

In the next example, we give a weakly α-shifting ring which is not weak α-reversible.

Example 2.3. *Let R be a commutative ring. Let $\alpha : R \oplus R \longrightarrow R \oplus R$ such that $\alpha((a,b)) = (b,a)$ for all$(a,b) \in R \oplus R$. Then α is a ring endomorphism of $R \oplus R$. Now our first motive is to show that $R \oplus R$ is weakly α-shifting ring. Therefore let us consider $(a,b)\alpha((c,d)) \in Nil(R \oplus R)$ for any $(a,b),(c,d) \in R \oplus R$. It implies $(ad,bc) \in Nil(R \oplus R)$. So there exists a positive integer m such that $((ad,bc))^m = 0$. So we have $((ad))^m = ((bc))^m = 0$. Since R is commutative, so $((da))^m = ((cb))^m = 0$. Now $((c,d)\alpha(a,b))^m = ((cb,da))^m = 0 \Rightarrow (c,d)\alpha((a,b)) \in Nil(R \oplus R)$.*

Therefore $R\oplus R$ is weakly α-shifting ring.
Now we see that $(1,0)(0,1)=0\in Nil(R\oplus R)$. *But* $(0,1)\alpha(1,0)=(0,1)$ *is not nilpotent element of $R\oplus R$. So $R\oplus R$ is not weak α-reversible.*

Remark 2.4. *We can see that $R\oplus R$, the weakly α-shifting ring given in Example 2.15 is neither weak α-compatible nor weak α-symmetric by using Proposition 2.9(i) and Proposition 2.11 respectively.*

Proposition 2.6. *If R is weak α-rigid ring and Nil(R) forms an ideal, then R is weakly α-shifting.*

Proof. Let us consider R is weak α-rigid ring. Let $x\alpha(y)\in Nil(R)$ for any $x,y\in R$. It implies $yx\alpha(y)\alpha(x)=yx\alpha(yx)\in Nil(R)$ as $Nil(R)$ forms an ideal. Since R is weak α-rigid ring, we have $yx\in Nil(R)$. Now $yx\in Nil(R)\Rightarrow\alpha(yx)\in Nil(R)$ by using Lemma 2.7. Since $Nil(R)$ forms an ideal, then we have $\alpha^2(x)\alpha(y)\alpha(x)y\in Nil(R)$. It implies $\alpha(\alpha(x)y)\alpha(x)y\in Nil(R)$. Now by using the definition of weak α-rigid ring, we have $\alpha(x)y\in Nil(R)$. Now $\alpha(x)y\in Nil(R)\Rightarrow$ $y\alpha(x)\in Nil(R)$ by using Lemma 2.2. Thus R is weakly α-shifting ring.

Example 2.4. *From the example of weakly α-shifting ring given in Example 2.15, we can see that* $(1,0)\alpha(1,0)=(1,0)(0,1)=0\in Nil(R\oplus R)$ *but* $(1,0)$ *is not nilpotent element of $R\oplus R$. Thus $R\oplus R$ is not weak α-rigid ring.*

Lemma 2.3. [16] *R is semicommutative $\Rightarrow$ Nil(R) forms an ideal.*

We have the following corollary from the Proposition 2.17 and Lemma 2.19.

Corollary 2.2. *If R is weak α-rigid ring and semicommutative then R is weakly α-shifting.*

Proposition 2.7. *Let R be weakly α-shifting ring. Then we have the following:*

(i) *If $x\alpha^k(y)\in Nil(R)$, then $y\alpha^k(x)\in Nil(R)$ for any positive integer k.*
(ii) *If $xy\in Nil(R)$, then $x\alpha^k(y),y\alpha^k(x)\in Nil(R)$ for any positive integer $k=2m$.*

Proof. (i) For $k=1$, $x\alpha(y)\in Nil(R)$ implies $y\alpha(x)\in Nil(R)$ by the definition of weakly α-shifting ring. Let us consider $x\alpha^m(y)\in Nil(R)$ implies $y\alpha^m(x)\in Nil(R)$ for some $m>1$.

Now let $x\alpha^{m+1}(y) \in Nil(R)$. It implies $x\alpha(\alpha^m(y)) \in Nil(R) \Rightarrow \alpha^m(y)\alpha(x) \in Nil(R)$ as R is weakly α-shifting ring. By using the Lemma 2.2, we have $\alpha(x)\alpha^m(y) \in Nil(R)$. Again by using our assumption $y\alpha^{m+1}(x) = y\alpha^m(\alpha(x)) \in Nil(R)$. Thus $x\alpha^k(y) \in Nil(R)$ implies $y\alpha^k(x) \in Nil(R)$ for any positive integer k by using principle of mathematical induction.
(ii) Let $xy \in Nil(R)$. By using Lemma 2.7, we have $\alpha(xy) = \alpha(x)\alpha(y) \in Nil(R)$. Since R is weakly α-shifting, then $y\alpha^2(x) = y\alpha(\alpha(x)) \in Nil(R)$. Again by using Lemma 2.7, we have $\alpha(y\alpha^2(x)) \in Nil(R)$. It implies $\alpha(y)\alpha^3(x) \in Nil(R)$. Now by using Lemma 2.2, we have $\alpha^3(x)\alpha(y) \in Nil(R$. Since R is weakly α-shifting ring, therefore $y\alpha^4(x) = y\alpha(\alpha^3(x)) \in Nil(R)$. Continuing the same process, we get $y\alpha^k(x) \in Nil(R)$ for any positive integer $k = 2m$. On the other hand, if $xy \in Nil(R)$, then $yx \in Nil(R)$ by using Lemma 2.2. Using the above method for yx in lieu xy, we get $x\alpha^k(y) \in Nil(R)$ for any positive integer $k = 2m$.

Proposition 2.8. *Let R be weakly α-shifting ring for monomorphism α. Then the following are equivalent:*

(i) *$xy \in Nil(R)$ for any $x, y \in R$.*
(ii) *$x\alpha^k(y) \in Nil(R)$ for any positive integer $k = 2m$.*

Proof. (i)$\Rightarrow$ (ii) is obvious by Proposition 2.21 (ii).
(ii)$\Rightarrow$ (i). If $x\alpha^k(y) \in Nil(R)$ for any positive integer $k = 2m$, then $x\alpha(\alpha^{k-1}(y)) \in Nil(R)$. Since R is weakly α-shifting ring, we get $\alpha^{k-1}(y)\alpha(x) \in Nil(R)$. It implies $\alpha(\alpha^{k-2}(y)x) \in Nil(R)$. By using Remark 2.8, we have $\alpha^{k-2}(y)x \in Nil(R)$. Again by using Lemma 2.2, we get $x\alpha^{k-2}(y) \in Nil(R)$. It implies $x\alpha(\alpha^{k-3}(y)) \in Nil(R)$. Since R is weakly α-shifting ring, we get $\alpha^{k-3}(y)\alpha(x) \in Nil(R)$. It implies $\alpha(\alpha^{k-4}(y)x) \in Nil(R)$. Again by using Remark 2.8 and Lemma 2.2, we get $x\alpha^{k-4}(y) \in Nil(R)$. Now continuing this procedure, we obtain $xy \in Nil(R)$.

Lemma 2.4. [8] *If R is semicommutative and $f(x) = r_0 + r_1x + r_2x^2 + ... + r_nx^n \in R[x]$. Then $f(x) \in Nil(R[x]) \Leftrightarrow r_0, r_1, \dots, r_n \in Nil(R)$.*

Let us define $\bar{\alpha} : R[x] \longrightarrow R[x]$ such that $\bar{\alpha}(r_0 + r_1x + r_2x^2 + ... + r_nx^n) = \alpha(r_0) + \alpha(r_1)x + ... + \alpha(r_n)x^n$ for all $r(x) = r_0 + r_1x + r_2x^2 + ... + r_nx^n \in R[x]$. Then $\bar{\alpha}$ is a ring endomorphism of $R[x]$.

Proposition 2.9. *Let R be semicommutative, then R is weakly α-shifting iff $R[x]$ is weakly $\bar{\alpha}$-shifting whereas $\bar{\alpha}(r_0+r_1x+r_2x^2+...+r_nx^n)=\alpha(r_0)+\alpha(r_1)x+...+\alpha(r_n)x^n$.*

Proof. Let us consider that R be weakly α-shifting ring. Now let $r(x)=r_0+r_1x+r_2x^2+...+r_nx^n$ and $s(x)=s_0+s_1x+s_2x^2+...+s_mx^m$ in $R[x]$ so that $r(x)\bar{\alpha}(s(x))\in Nil(R[x])$. We know that

$$r(x)\bar{\alpha}(s(x))=\Sigma_{k=0}^{m+n}\left(\Sigma_{i+j=k}r_i\alpha(s_j)\right)x^k \tag{2.1}$$

Now by using Lemma 2.23, we have

$$\Sigma_{i+j=k}r_i\alpha(s_j)\in Nil(R) \tag{2.2}$$

For $k=0$, (2) implies $r_0\alpha(s_0)\in Nil(R)$ and it implies $\alpha(s_0)r_0\in Nil(R)$ by using Lemma 2.2. Now for $k=1$, $r_0\alpha(s_1)+r_1\alpha(s_0)\in Nil(R)$ from Eq(2). Again it implies $(r_0\alpha(s_1)+r_1\alpha(s_0))r_0\in Nil(R)$ by using Lemma 2.19. By using the same Lemma 2.19, we have $r_0\alpha(s_1)\in Nil(R)$. Similarly we can show that$(r_0\alpha(s_1)+r_1\alpha(s_0))r_1\in Nil(R)$ implies $r_1\alpha(s_0)\in Nil(R)$. So $r_i\alpha(s_j)\in Nil(R)$ for $k=i+j=0,1$.

Now let us assume that there exists some positive integer $p>1$ such that $r_i\alpha(s_j)\in Nil(R)$ where $i+j\leq p$. Therefore $r_0\alpha(s_p),r_1\alpha(s_{p-1}),...,r_p\alpha(s_0)\in Nil(R)$. Then we have

$\alpha(s_p)r_0,\alpha(s_{p-1})r_1,...,\alpha(s_0)r_p\in Nil(R)$ by using Lemma 2.2.

Now we will show that $r_i\alpha(s_j)\in Nil(R)$ for $i+j=p+1$. From Eq. (2) for $k=p+1$, we have

$$r_0\alpha(s_{p+1})+r_1\alpha(s_p)+...+r_{p+1}\alpha(s_0)\in Nil(R) \tag{2.3}$$

Now multiplying Eq. (3) by r_0 from the right hand side, we have

$$(r_0\alpha(s_{p+1})+r_1\alpha(s_p)+...+r_{p+1}\alpha(s_0))r_0\in Nil(R). \tag{2.4}$$

Again by using our assumption that $r_i\alpha(s_j) \in Nil(R)$ for $i+j \leq p$ and Lemma 2.19, we have $r_0\alpha(s_{p+1})r_0 \in Nil(R)$ and it leads to $r_0\alpha(s_{p+1}) \in Nil(R)$.

Again multiplying Eq.(3) by r_1 from the right hand side and continuing with the same procedure as above, we can show that $r_1\alpha(s_p) \in Nil(R)$. Similarly we can get $r_2\alpha(s_{p-1}),...,r_{p+1}\alpha(s_0) \in Nil(R)$. Thus $r_i\alpha(s_j) \in Nil(R)$ for $i+j = p+1$. Now by induction hypothesis we can conclude that $r_i\alpha(s_j) \in Nil(R)$ for any $k = i+j$ where $k = 0,1,...,m+n$. Again $r_i\alpha(s_j) \in Nil(R) \Rightarrow s_j\alpha(r_i) \in Nil(R)$ as R is weakly α-shifting ring.

Therefore it can be easily shown that

$$s(x)\bar{\alpha}(r(x)) = \Sigma_{k=0}^{m+n}\left(\Sigma_{i+j=k}s_j\alpha(r_i)\right)x^k \in Nil(R[x])$$

by using Lemma 2.23 and hence $R[x]$ is weakly $\bar{\alpha}$-shifting. Converse part is trivial.

Let I be an ideal and α be a ring endomorphism of a ring R. Then the map $\bar{\alpha} : R/I \longrightarrow R/I$ defined by $\bar{\alpha}(x+I) = \alpha(x)+I$ for all $x+I \in R/I$ is a ring endomorphism of quotient ring R/I.

Proposition 2.10. *If $I \subseteq Nil(R)$. Then R is weakly α-shifting $\Leftrightarrow$ R/I is weakly $\bar{\alpha}$-shifting.*

Proof. Let R be weakly α-shifting ring. Now let us consider that $(x+I)\bar{\alpha}(y+I) \in Nil(R/I)$ for any $x+I, y+I \in R/I$. It implies clearly that $(x\alpha(y)+I)^m = I$ for some positive integer m. It implies $(x\alpha(y))^m + I = I$. So we have $(x\alpha(y))^m \in Nil(R)$ by using the condition that $I \subseteq Nil(R)$. So there exists some positive integer k such that $(x\alpha(y))^{mk} = 0$. Clearly, $x\alpha(y) \in Nil(R)$. Since R is weakly α-shifting, therefore $x\alpha(y) \in Nil(R) \Rightarrow y\alpha(x) \in Nil(R)$. Thus $(y\alpha(x))^n = 0$ for some positive integer n. It implies $(y\alpha(x))^n + I = I \Rightarrow ((y+I)\bar{\alpha}(x+I))^n = I \Rightarrow (y+I)\bar{\alpha}(x+I) \in Nil(R/I)$. Thus R/I is weakly $\bar{\alpha}$-shifting.

Conversely let us consider R/I is weakly $\bar{\alpha}$-shifting ring. Now we have to prove that R is weakly α-shifting ring. Let $x\alpha(y) \in Nil(R)$ for any $x,y \in R$. So we have $(x\alpha(y))^t = 0$ for some $t \in \mathbb{N}$. Then $(x\alpha(y))^t + I = I$. Therefore $((x+I)\bar{\alpha}(y+I))^t = I$. It implies $(x+I)\bar{\alpha}(y+I) \in Nil(R/I)$. Since R/I is weakly $\bar{\alpha}$-shifting ring, so $(y+I)\bar{\alpha}(x+I) \in Nil(R/I)$. It implies $(y\alpha(x)+I)^r = I$ for some $r \in \mathbb{N}$. Thus $(y\alpha(x))^r \in I \subseteq Nil(R)$. Now it leads to $y\alpha(x) \in Nil(R)$. Therefore R is weakly α-shifting.

Proposition 2.11. *If R is weakly α-shifting for a monomorphism α, then $\alpha(1) = 1$.*

Proof. Let R be weakly α-shifting ring for a monomorphism α. Here $(1-\alpha(1))\alpha(1) = 0 \in Nil(R)$ as $\alpha(1)$ is an idempotent element of R. Now by using the definition of weakly α-shifting of R, $\alpha(1-\alpha(1)) = 1.\alpha(1-\alpha(1)) \in Nil(R)$. It implies $1-\alpha(1) \in Nil(R)$ by using Remark **??**. Therefore $(1-\alpha(1))^m = 0$ for some integer m. It implies $1-\alpha(1) = 0$ as $1-\alpha(1)$ is an idempotent element. Thus $\alpha(1) = 1$.

Proposition 2.12. *Let $\sigma : R \longrightarrow S$ be a ring isomorphism. Then R is a weakly α-shifting ring $\Leftrightarrow$ S is weakly $\sigma\alpha\sigma^{-1}$-shifting ring.*

Proof. Let R be a weakly α-shifting ring. Let $\bar{x}, \bar{y} \in S$ so that $\bar{x}(\sigma\alpha\sigma^{-1})(\bar{y}) \in Nil(S)$. Since σ is onto, therefore there exist x and y in R such that $\sigma(x) = \bar{x}$ and $\sigma(y) = \bar{y}$. It implies $\sigma(x)(\sigma\alpha\sigma^{-1})(\sigma(y)) \in Nil(S)$. It leads to $\sigma(x\alpha(y)) \in Nil(S)$. Now by using the Remark 2.8, $x\alpha(y) \in Nil(R)$. Since R is weakly α-shifting ring, therefore $y\alpha(x) \in Nil(R)$. Again by using Lemma 2.7, we have $\sigma(y\alpha(x)) \in Nil(S)$. It leads to $\sigma(y)(\sigma\alpha\sigma^{-1})(\sigma(x)) \in Nil(S) \Rightarrow \bar{y}(\sigma\alpha\sigma^{-1})(\bar{x}) \in Nil(S)$. Thus we can conclude that S is weakly $\sigma\alpha\sigma^{-1}$-shifting ring.

Conversely let S be a weakly $\sigma\alpha\sigma^{-1}$-shifting ring. Let $r\alpha(s) \in Nil(R)$ for any $r, s \in R$. Then $\sigma(r\alpha(s)) \in Nil(S)$ by Lemma 2.7. It implies $\sigma(r)(\sigma\alpha\sigma^{-1})(\sigma(s)) \in Nil(S) \Rightarrow \bar{r}(\sigma\alpha\sigma^{-1})(\bar{s}) \in Nil(S)$ where $\sigma(r) = \bar{r}$ and $\sigma(s) = \bar{s}$. Since S is weakly $\sigma\alpha\sigma^{-1}$-shifting, so $\bar{s}(\sigma\alpha\sigma^{-1})(\bar{r}) \in Nil(S)$. It implies $\sigma(s\alpha(r)) \in Nil(S)$. Now using Remark 2.8, we get $s\alpha(r) \in Nil(R)$. Thus R is weakly α-shifting.

Conflict of Interests

The author(s) declare that there is no conflict of interests.

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