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CHARACTERIZATION OF INTRA-REGULAR SEMIGROUPS IN TERMS OF INTERVAL VALUED Q-FUZZY SUBSEMIGROUPS WITH THRESHOLDS $(\overline{\alpha},\overline{\beta})$

JUKKRID SUWANNARAT, THITI GAKETEM*

Department of Mathematics, School of Science, University of Phayao, Phayao 56000, Thailand

Abstract. In this paper, we characterize of intra-regular semigroups by using the properties of these interval-valued Q-fuzzy subsemigroups with thresholds $(\overline{\alpha},\overline{\beta})$ of semigroups.

Keywords: interval-valued Q-fuzzy subsemigroups with thresholds $(\overline{\alpha},\overline{\beta})$; intra-regular semigroup.

2010 AMS Subject Classification: 03E72, 18B40.

1. INTRODUCTION

The concept of fuzzy sets is published by Zadeh in 1965, [17]. In 1971, Rosenfeld [13] inspired the fuzzification of algebraic structures and introduced the notion of fuzzy subgroups. In 1979, Kuroki [8] was first developed the concept of fuzzy subsemigroups and various kinds of fuzzy ideals in semigroups.

In 1975, Zadeh [16] initiated the theory of interval valued fuzzy sets as the generalization of fuzzy sets, where the values of the membership functions are intervals of the numbers instead of the numbers. The notion of interval valued fuzzy sets have several applications like medical science [4], image processing [2], etc. In 1994, Biswas [3] firstly initiated the concept of interval valued fuzzy subgroups of the same nature of Rosenfeld's fuzzy subgroups and investigated

*Corresponding author
E-mail address: thiti.ga@up.ac.th
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some elementary properties. In 2006, Narayanan and Manikantan [12] initiated the concept of interval valued fuzzy subsemigroups and presented various interval valued fuzzy ideals in semigroups. In 2013, Singaram and Kandasamy [14] characterized intra-regular semigroups in terms of interval valued fuzzy left (right) ideals in semigroups. In 2014, Abdullah et al. [1] instituted extension the definition of interval valued fuzzy subsemigroups to $(\overline{\alpha},\overline{\beta})$-interval valued fuzzy subsemigroups where $\overline{\alpha} \prec \overline{\beta}$, which are generalization of interval valued fuzzy subsemigroups and characterized intra-regular semigroups in terms of $(\overline{\alpha},\overline{\beta})$-interval valued fuzzy subsemigroups.

In 2019, Murugads and Arikrishnan [11] gave concept of interval valued Q-fuzzy ideal with thresholds $(\overline{\alpha},\overline{\beta})$ where $\overline{\alpha} \prec \overline{\beta}$ and characterized regular semigroups in terms of interval valued Q-fuzzy ideal with thresholds $(\overline{\alpha},\overline{\beta})$.

This paper is divided in the following sections, in section 2, we give some basic definitions and results which are helpful in the study of section 3 and section 4. In section 3, we discuss characterization intra-regular semigroups by using of interval valued Q-fuzzy ideal with thresholds $(\overline{\alpha},\overline{\beta})$.

2. PRELIMINARIES

Before we begin our work, we give some basic definitions and results which are helpful in the study of section 3 and section 4.

A non-empty subset H of a semigroup S is called a *subsemigroup* of S if $H^2 \subseteq H$. A non-empty subset H of a semigroup S is called a *left* (right) ideal of S if $SH \subseteq H$ $(HS \subseteq H)$. By an *ideal* K of a semigroup S we mean a left ideal and a right ideal of S A non-empty subset H of S is called a *generalized bi-ideal* of S if $HSH \subseteq H$. A subsemigroup H of a semigroup S is called a *bi-ideal* of S if $HSH \subseteq H$. A subsemigroup H of a semigroup S is called an *interior ideal* of S if $SHS \subseteq H$. A subsemigroup H of a semigroup S is called a *quasi-ideal* of S if $SH \cap HS \subseteq H$.

For any $p_i \in [0,1]$ where $i \in \mathscr{A}$ define

$$\bigvee_{i\in\mathscr{A}} p_i := \sup_{i\in\mathscr{A}}\{p_i\} \quad \text{and} \quad \bigwedge_{i\in\mathscr{A}} p_i := \inf_{i\in\mathscr{A}}\{p_i\}.$$

We see that for any $p,q \in [0,1]$, we have

$$p \vee q = \max\{p,q\} \quad \text{and} \quad p \wedge q = \min\{p,q\}.$$

Let $\mathscr{C}$ be the set of all closed subintervals $[0,1]$, i.e.,

$$\mathscr{C} = \{\overline{p} = [p^-, p^+] \mid 0 \leq p^- \leq p^+ \leq 1\}.$$

Let $\overline{p} := [p^-, p^+]$ and $\overline{q} := [q^-, q^+]$ in $\mathscr{C}$. Define the operations $\preceq$, $=$, $\curlywedge$, $\curlyvee$ as follows:

(1) $\overline{p} \preceq \overline{q}$ if and only if $p^- \leq q^-$ and $p^+ \leq q^+$

(2) $\overline{p} = \overline{q}$ if and only if $p^- = q^-$ and $p^+ = q^+$

(3) $\overline{p} \curlywedge \overline{q} = [(p^- \wedge q^-), (p^+ \wedge q^+)]$

(4) $\overline{p} \curlyvee \overline{q} = [(p^- \vee q^-), (p^+ \vee q^+)]$.

If $\overline{p} \succeq \overline{q}$, we mean $\overline{q} \preceq \overline{p}$.

Proposition 2.1. [5] Let $\overline{p}, \overline{q}, \overline{r} \in \mathscr{C}$. Then the following properties are true:

(1) $\overline{p} \curlywedge \overline{p} = \overline{p}$ and $\overline{p} \curlyvee \overline{p} = \overline{p}$,

(2) $\overline{p} \curlywedge \overline{q} = \overline{q} \curlywedge \overline{p}$ and $\overline{p} \curlyvee \overline{q} = \overline{q} \curlyvee \overline{p}$,

(3) $(\overline{p} \curlywedge \overline{q}) \curlywedge \overline{r} = \overline{p} \curlywedge (\overline{q} \curlywedge \overline{r})$ and $(\overline{p} \curlyvee \overline{q}) \curlyvee \overline{r} = \overline{p} \curlyvee (\overline{q} \curlyvee \overline{r})$,

(4) $(\overline{p} \curlywedge \overline{q}) \curlyvee \overline{r} = (\overline{a} \curlyvee \overline{c}) \curlywedge (\overline{q} \curlyvee \overline{r})$ and $(\overline{a} \curlyvee \overline{q}) \curlywedge \overline{r} = (\overline{p} \curlywedge \overline{r}) \curlyvee (\overline{q} \curlywedge \overline{r})$,

(5) If $\overline{p} \preceq \overline{q}$, then $\overline{p} \curlywedge \overline{r} \preceq \overline{q} \curlywedge \overline{r}$ and $\overline{p} \curlyvee \overline{r} \preceq \overline{q} \curlyvee \overline{r}$.

For each interval $\overline{p}_i = [p_i^-, p_i^+] \in \mathscr{C}$, $i \in \mathscr{A}$ where $\mathscr{A}$ is an index set, we define

$$\underset{i\in\mathscr{A}}{\curlywedge} \overline{p}_i = [\underset{i\in\mathscr{A}}{\wedge} p_i^-, \underset{i\in\mathscr{A}}{\wedge} p_i^+] \quad \text{and} \quad \underset{i\in\mathscr{A}}{\curlyvee} \overline{p}_i = [\underset{i\in\mathscr{A}}{\vee} p_i^-, \underset{i\in\mathscr{A}}{\vee} p_i^+].$$

A fuzzy subset (fuzzy set) of a set T is a function $f : T \to [0,1]$.

Definition 2.1. Let S be a semigroup and Q be a non-empty set. A Q-fuzzy subset (Q-fuzzy set) of a set T is a function $f : S \times Q \to [0,1]$

Definition 2.2. [14] Let T be a non-empty set. An interval valued fuzzy subset (shortly, IVF subset) of T is a function $\overline{f} : T \to \mathscr{C}$

Definition 2.3. [10] Let S be a semigroup and Q be a non-empty set. An interval valued Q-fuzzy subset (shortly, IVQF subset) of T is a function

$\overline{f} : S \times Q \to \mathscr{C}$

Definition 2.4. [10] Let K be a non-empty subset of a semigroup S and Q be a non-empty set . An interval valued characteristic function $\overline{\chi}_K$ of K is defined to be a function $\overline{\chi}_K : S \times Q \to \mathscr{C}$ by

$$\overline{\chi}_K(u,q) = \begin{cases} [1,1] & \text{if} \quad u \in K \\ [0,0] & \text{if} \quad u \notin K \end{cases}$$

for all $u \in T$.

For two IVQF subsets $\overline{f}$ and $\overline{g}$ of a semigroups S, define

(1) $\overline{f} \sqsubseteq \overline{g} \Leftrightarrow \overline{f}(u,q) \preceq \overline{g}(u,q)$ for all $u \in S$ and $q \in Q$,
(2) $\overline{f} = \overline{g} \Leftrightarrow \overline{f} \sqsubseteq \overline{g}$ and $\overline{g} \sqsubseteq \overline{f}$,
(3) $(\overline{f} \curlywedge \overline{g})(u,q) = \overline{f}(u,q) \curlywedge \overline{g}(u,q)$ for all $u \in S$ and $q \in Q$.

For two IVQF subsets $\overline{f}$ and $\overline{g}$ of a semigroup S. Then the product $\overline{f} \circ \overline{g}$ is defined as follows for all $u \in S$ and $q \in Q$,

$$(\overline{f} \circ \overline{g})(u,q) = \begin{cases} \underset{(y,z) \in F_u}{\curlyvee} \{\overline{f}(y,q) \curlywedge \overline{g}(z,q)\} & \text{if} \quad F_u \neq \emptyset, \\ \overline{0} & \text{if} \quad F_u = \emptyset, \end{cases}$$

where $F_u := \{(y,z) \in S \times S \mid u = yz\}$ [10].

Next, we shall give definitions of various types of IVQF subsemigroup of a semigroups.

Definition 2.5. [11] An IVF subset $\overline{f}$ of a semigroup S is said to be

(1) an *IVQF subsemigroup* of S if $\overline{f}(uv,q) \succeq \overline{f}(u,q) \curlywedge \overline{f}(v,q)$ for all $u, v \in S$ and $q \in Q$,
(2) an *IVQF left (right) ideal* of S if $\overline{f}(uv,q) \succeq \overline{f}(v,q)$ $(\overline{f}(uv,q) \succeq \overline{f}(u,q))$ for all $u, v \in S$ and $q \in Q$. An *IVQF ideal* of S if it is both an IVQF left ideal and an IVQF right ideal of S,
(3) an *IVQF generalized bi-ideal* of S if $\overline{f}(uvw,q) \succeq \overline{f}(u,q) \curlywedge \overline{f}(w,q)$ for all $u, v, w \in S$ and $q \in Q$,
(4) an *IVQF bi-ideal* of S if $\overline{f}$ is an IVQF subsemigroup of S and $\overline{f}(uvw,q) \succeq \overline{f}(u,q) \curlywedge \overline{f}(w,q)$ for all $u, v, w \in S$ and $q \in Q$,
(5) an *IVQF interior ideal* of S if if $\overline{f}$ is an IVQF subsemigroup of S and $\overline{f}(uav,q) \succeq \overline{f}(a,q)$ for all $a, u, v \in S$ and $q \in Q$,

(6) an *IVQF quasi-ideal* of S if $\overline{f}(u,q) \succeq (\overline{\mathscr{S}} \circ \overline{f})(u,q) \curlywedge (\overline{f} \circ \overline{\mathscr{S}})(u,q)$, for all $u \in S$ and $q \in Q$ where $\overline{S}$ is an IVQF subset of S mapping every element of S on $\overline{1}$.

The thought of an IVQF subsemigroup with thresholds $(\overline{\alpha}, \overline{\beta})$ where $\overline{\alpha} \prec \overline{\beta}$ as follows:

Definition 2.6. [11] An IVF subset $\overline{f}$ of a semigroup S and $\overline{\alpha} \prec \overline{\beta}$ and $\overline{\alpha}, \overline{\beta} \in \mathscr{C}$ is said to be

(1) an *IVQF subsemigroup with thresholds* $(\overline{\alpha}, \overline{\beta})$ of S if $\overline{f}(uv,q) \curlyvee \overline{\alpha} \succeq \overline{f}(u,q) \curlywedge \overline{f}(v,q) \curlywedge \overline{\beta}$ for all $u, v \in S$ and $q \in Q$,

(2) an *IVQF left (right) ideal with thresholds* $(\overline{\alpha}, \overline{\beta})$ of S if $\overline{f}(uv,q) \curlyvee \overline{\alpha} \succeq \overline{f}(v,q) \curlywedge \overline{\beta}$ $(\overline{f}(uv,q) \curlyvee \overline{\alpha} \succeq \overline{f}(u,q) \curlywedge \overline{\beta})$ for all $u, v \in S$ and $q \in Q$. An *IVQF ideal with thresholds* $(\overline{\alpha}, \overline{\beta})$ of S if it is both an IVF left ideal and an IVF right ideal of S,

(3) an *IVQF generalized bi-ideal with thresholds* $(\overline{\alpha}, \overline{\beta})$ of S if $\overline{f}(uvw,q) \curlyvee \overline{\alpha} \succeq \overline{f}(u,q) \curlywedge \overline{f}(w,q) \curlywedge \overline{\beta}$ for all $u, v, w \in S$ and $q \in Q$,

(4) an *IVQF bi-ideal with thresholds* $(\overline{\alpha}, \overline{\beta})$ of S if $\overline{f}$ is an IVQF subsemigroup with thresholds $(\overline{\alpha}, \overline{\beta})$ of S and $\overline{f}(uvw,q) \curlyvee \overline{\alpha} \succeq \overline{f}(u,q) \curlywedge \overline{f}(w,q) \curlywedge \overline{\beta}$ for all $u, v, w \in S$ and $q \in Q$,

(5) an *IVQF interior ideal with thresholds* $(\overline{\alpha}, \overline{\beta})$ of S if $\overline{f}$ is an IVQF subsemigroup with thresholds $(\overline{\alpha}, \overline{\beta})$ of S and $\overline{f}(uav,q) \curlyvee \overline{\alpha} \succeq \overline{f}(a,q) \curlywedge \overline{\beta}$ for all $a, u, v \in S$ and $q \in Q$,

(6) an *IVQF quasi-ideal with thresholds* $(\overline{\alpha}, \overline{\beta})$ of S if $\overline{f}(u,q) \curlyvee \overline{\alpha} \succeq (\overline{\mathscr{S}} \circ \overline{f})(u,q) \curlywedge (\overline{f} \circ \overline{\mathscr{S}})(u,q) \curlywedge \overline{\beta}$, for all $u \in S$ and $q \in Q$.

Remark 2.2. [11] It is clear to see that every IVQF bi-ideal with thresholds $(\overline{\alpha}, \overline{\beta})$ is an IVQF generalized bi-ideal with thresholds $(\overline{\alpha}, \overline{\beta})$ of S, every IVQF ideal with thresholds $(\overline{\alpha}, \overline{\beta})$ is an IVQF interior ideal with thresholds $(\overline{\alpha}, \overline{\beta})$ of S and IVQF quasi-ideal with thresholds $(\overline{\alpha}, \overline{\beta})$ is an IVQF bi-ideal with thresholds $(\overline{\alpha}, \overline{\beta})$ of S.

In this ensuing theorem is present relationship between types ideals of a semigroup S and the interval valued characteristic function.

Theorem 2.3. [11] If K is a left ideal (right ideal generalized bi-ideal, bi-ideal, interior ideal, quasi-ideal) of S, then characteristic function $\overline{\chi}_K$ is an IVQF left ideal (right ideal, generalized bi-ideal, bi-ideal, interior ideal, quasi-ideal) with thresholds $(\overline{\alpha}, \overline{\beta})$ of S for all $\overline{\alpha} \prec \overline{\beta}$ and $\overline{\alpha}, \overline{\beta} \in \mathscr{C}$.

3. Characterizations of Intra-Regular Semigroups in Terms Interval Valued Q-Fuzzy Ideals With Thresholds $(\overline{\alpha},\overline{\beta})$

In this part, we review symbols of IVQF ideals with thresholds $(\overline{\alpha},\overline{\beta})$ for use characterizes a semigroup in terms IVQF ideals with thresholds $(\overline{\alpha},\overline{\beta})$ of semigroup. And we characterized a wearkly reugular semigroup in terms IVQF ideals with thresholds $(\overline{\alpha},\overline{\beta})$ of semigroup.

In 2019, [11] Murugads and Arikrishnan propose symbols of IVQF ideals with thresholds $(\overline{\alpha},\overline{\beta})$ for use characterizes a semigroup in terms IVQF ideals with thresholds $(\overline{\alpha},\overline{\beta})$ of semigroup.

For any IVQF subset $\overline{f}$ of a semigroup S with $\overline{\alpha} \prec \overline{\beta}$ and $\overline{\alpha},\overline{\beta} \in \mathscr{C}$., define

$$\overline{f}_{(\overline{\alpha},\overline{\beta})}(u,q) = (\overline{f}(u,q) \curlywedge \overline{\alpha}) \curlyvee \overline{\beta}$$

for all $u \in S$ and $q \in Q$.

For any IVQF subsets $\overline{f}$ and $\overline{g}$ of a semigroup S with $\overline{\alpha} \prec \overline{\beta}$ and $\overline{\alpha},\overline{\beta} \in \mathscr{C}$, define the operation "$\curlywedge^{\overline{\alpha}}_{\overline{\beta}}$" as follows:

$$(\overline{f} \curlywedge^{\overline{\alpha}}_{\overline{\beta}} \overline{g})(u,q) = (\overline{f}(u,q) \curlywedge \overline{g}(u,q) \curlywedge \overline{\alpha}) \curlyvee \overline{\beta}$$

for all $u \in S$ and $q \in Q$. And define the product $\overline{f} \circ^{\overline{\alpha}}_{\overline{\beta}} \overline{g}$ as follows: for all $u \in S$ and $q \in Q$,

$$(\overline{f} \circ^{\overline{\alpha}}_{\overline{\beta}} \overline{g})(u,q) = ((\overline{f} \circ \overline{g})(u,q) \curlywedge \overline{\alpha}) \curlyvee \overline{\beta}$$

where

$$(\overline{f} \circ \overline{g})(u,q) = \begin{cases} \underset{(x,y)\in F_u}{\curlyvee} \{\overline{f}(x,q) \curlywedge \overline{g}(y,q)\} & \text{if} \quad F_u \neq \emptyset, \\ \overline{0} & \text{if} \quad F_u = \emptyset, \end{cases}$$

where $F_u := \{(x,y) \in S \times S \mid u = xy\}$.

Remark 3.1. Since $\overline{\chi}$ is an interval valued characteristic, we have

$$\overline{\chi}_{(\overline{\alpha},\overline{\beta})}(u,q) := \begin{cases} \overline{\beta} & \text{if} \quad u \in K, \\ \overline{\alpha} & \text{if} \quad u \notin K. \end{cases}$$

Lemma 3.2. [11] Let K and L be non-empty subsets of a semigroup S with $\overline{\alpha} \prec \overline{\beta}$ and $\overline{\alpha},\overline{\beta} \in \mathscr{C}$. Then the following assertions hold:

(1) $(\overline{\chi}_K) \curlywedge_{\overline{\beta}}^{\overline{\alpha}} (\overline{\chi}_L) = (\overline{\chi}_{K\cap L})_{(\overline{\alpha},\overline{\beta})}$.

(2) $(\overline{\chi}_K) \circ_{\overline{\beta}}^{\overline{\alpha}} (\overline{\chi}_L) = (\overline{\chi}_{KL})_{(\overline{\alpha},\overline{\beta})}$.

The following definition and lemma will be used to prove in Theorem 3.4.

Definition 3.1. [14] A semigroup S is called an *intra-regular* if for each $u \in S$, there exist $a, b \in S$ such that $u = au^2b$.

Lemma 3.3. [9] A semigroup S is intra-regular if and only if $L \cap R \subseteq LR$, for every left ideal L and every right ideal R of S.

Theorem 3.4. A semigroup S is intra-regular if and only if $\overline{f} \curlywedge_{\overline{\beta}}^{\overline{\alpha}} \overline{g} \sqsubseteq \overline{f} \circ_{\overline{\beta}}^{\overline{\alpha}} \overline{g}$, for every IVQF left ideal $\overline{f}$ with thresholds $(\overline{\alpha},\overline{\beta})$ and every IVQF right ideal $\overline{g}$ with thresholds $(\overline{\alpha},\overline{\beta})$ of S.

Proof. $(\Rightarrow)$ Assume that $\overline{f}$ and $\overline{g}$ is an IVQF left ideal with thresholds $(\overline{\alpha},\overline{\beta})$ and an IVQF right ideal with thresholds $(\overline{\alpha},\overline{\beta})$ of S respectively. Let $u \in S$ and $q \in Q$. Since S is an intra-regular semigroup we have there exist $a, b \in S$ such that $u = au^2b$. Thus

$$\begin{aligned}
(\overline{f} \circ_{\overline{\beta}}^{\overline{\alpha}} \overline{g})(u,q) &= \Big(\curlyvee_{(i,j)\in F_u} \{\overline{f}(i,q) \curlywedge \overline{g}(j,q)\} \curlywedge \overline{\beta}\Big) \curlyvee \overline{\alpha} \\
&= \Big(\curlyvee_{(i,j)\in F_{auub}} \{\overline{f}(i,q) \curlywedge \overline{g}(j,q)\} \curlywedge \overline{\beta} \curlyvee \overline{\alpha} \\
&\succeq ((\overline{f}(auq) \curlywedge \overline{g}(ub)) \curlywedge \overline{\beta})) \curlyvee \overline{\alpha} \\
&= ((\overline{f}(au,q) \curlyvee \overline{\alpha}) \curlywedge (\overline{g}(ub,q) \curlyvee \overline{\alpha}) \curlywedge \overline{\beta}) \curlyvee \overline{\alpha} \\
&\succeq ((\overline{f}(u,q) \curlywedge \overline{\beta}) \curlywedge (\overline{g}(u,q) \curlywedge \overline{\beta}) \curlywedge \overline{\beta}) \curlyvee \overline{\alpha} \\
&= (\overline{f}(u,q) \curlywedge \overline{g}(u,q) \curlywedge \overline{\beta}) \curlyvee \overline{\alpha} = (\overline{f} \curlywedge_{\overline{\beta}}^{\overline{\alpha}} \overline{g})(u,q).
\end{aligned}$$

It implies that, $(\overline{f} \curlywedge_{\overline{\beta}}^{\overline{\alpha}} \overline{g})(u,q) \preceq (\overline{f} \circ_{\overline{\beta}}^{\overline{\alpha}} \overline{g})(u,q)$. Hence, $\overline{f} \curlywedge_{\overline{\beta}}^{\overline{\alpha}} \overline{g} \sqsubseteq \overline{f} \circ_{\overline{\beta}}^{\overline{\alpha}} \overline{g}$.

$(\Leftarrow)$ Let R and L be a right ideal and a left ideal of S respectively. Then by Theorem 2.3, $\overline{\chi}_R$ and $\overline{\chi}_L$ is a IVQF right ideal with thresholds $(\overline{\alpha},\overline{\beta})$ and a IVQF left ideal with thresholds $(\overline{\alpha},\overline{\beta})$ of S respectively. By supposition and Lemma 3.2, we have

$$\overline{\beta} = (\overline{\chi}_{L\cap R})_{(\overline{\alpha},\overline{\beta})}(u,q) = ((\overline{\chi}_L) \curlywedge_{\overline{\beta}}^{\overline{\alpha}} (\overline{\chi}_R)_(u,q) \sqsubseteq ((\overline{\chi}_L) \circ_{\overline{\beta}}^{\overline{\alpha}} (\overline{\chi}_R))(u,q) = (\overline{\chi}_{LR})_{(\overline{\alpha},\overline{\beta})}(u,q)$$

Thus, $u \in LR$. Hence, $L \cap R \subseteq LR$. Therefore by Lemma 3.3, S is intra-regular. □

The following lemma will be used to prove in Theorem 3.6.

Lemma 3.5. [7] Let S be a semigroup. Then the following are equivalent:

(1) S is intra-regular,

(2) $I\cap B\cap R\subseteq IBR$, for every interior ideal I, every bi-ideal B and every right ideal R of S.

Theorem 3.6. Let S be a semigroup. Then the following equivalent:

(1) S is intra-regular,

(2) $\overline{f}\curlywedge\frac{\overline{\alpha}}{\overline{\beta}}\overline{g}\curlywedge\frac{\overline{\alpha}}{\overline{\beta}}\overline{h}\sqsubseteq\overline{f}\circ\frac{\overline{\alpha}}{\overline{\beta}}\overline{g}\circ\frac{\overline{\alpha}}{\overline{\beta}}\overline{h}$, for every IVQF interior ideal $\overline{f}$ with thresholds $(\overline{\alpha},\overline{\beta})$, every IVQF bi-ideal $\overline{g}$ with thresholds $(\overline{\alpha},\overline{\beta})$ and every IVQF right ideal $\overline{h}$ with thresholds $(\overline{\alpha},\overline{\beta})$ of S,

(3) $\overline{f}\curlywedge\frac{\overline{\alpha}}{\overline{\beta}}\overline{g}\curlywedge\frac{\overline{\alpha}}{\overline{\beta}}\overline{h}\sqsubseteq\overline{f}\circ\frac{\overline{\alpha}}{\overline{\beta}}\overline{g}\circ\frac{\overline{\alpha}}{\overline{\beta}}\overline{h}$, for every IVQF interior ideal $\overline{f}$ with thresholds $(\overline{\alpha},\overline{\beta})$, every IVQF generalized bi-ideal $\overline{g}$ with thresholds $(\overline{\alpha},\overline{\beta})$ and every IVQF right ideal $\overline{h}$ with thresholds $(\overline{\alpha},\overline{\beta})$ of S.

Proof. $(1)\Rightarrow(3)$ Let $\overline{f},\overline{g}$ and $\overline{h}$ be an IVQF interior ideals with thresholds $(\overline{\alpha},\overline{\beta})$, an IVQF genralized bi-ideal with thresholds $(\overline{\alpha},\overline{\beta})$, an IVQF right ideal with thresholds $(\overline{\alpha},\overline{\beta})$ of S respectively. Let $u\in S$ and $q\in Q$. Then there exist $x,y\in S$ such that $u=xu^2y=xuuy=xu(xu^2y)y=xuxu^2yy=xuxuuyy=(xux)u(uy^2)$. Thus

$$\begin{aligned}
(\overline{f}\circ\tfrac{\overline{\alpha}}{\overline{\beta}}\overline{g}\circ\tfrac{\overline{\alpha}}{\overline{\beta}}\overline{h})(u,q) &= \Big(\curlyvee_{(i,j)\in F_u}(\overline{f}(i,q)\curlywedge(\overline{g}\underset{(\overline{\beta},\overline{\alpha})}{\circ}\overline{h})(j,q))\curlywedge\overline{\beta}\Big)\curlyvee\overline{\alpha}\\
&= \Big(\curlyvee_{(i,j)\in F_{(xux)u(uy^2)}}(\overline{f}(i,q)\curlywedge(\overline{g}\circ\tfrac{\overline{\alpha}}{\overline{\beta}}\overline{h})(j,q))\curlywedge\overline{\beta}\Big)\curlyvee\overline{\alpha}\\
&\succeq (\overline{f}(xux,q)\curlywedge(\overline{g}\circ\tfrac{\overline{\alpha}}{\overline{\beta}}\overline{h})(u(uy^2),q)\curlywedge\overline{\beta})\curlyvee\overline{\alpha}\\
&= (\overline{f}(xux)\curlywedge((\curlyvee_{(m,n)\in F_{u(uy^2,q)}}(\overline{g}(m,q)\curlywedge\overline{h}(n,q))\curlywedge\overline{\beta})\curlyvee\overline{\alpha})\curlywedge\overline{\beta})\curlyvee\overline{\alpha}\\
&\succeq (\overline{f}(xux,q)\curlywedge(((\overline{g}(u,q)\curlywedge\overline{h}(uy^2,q))\curlywedge\overline{\beta})\curlyvee\overline{\alpha})\curlywedge\overline{\beta})\curlyvee\overline{\alpha}\\
&= (\overline{f}(xux,q)\curlyvee\overline{\alpha}\curlywedge(\overline{g}(u,q)\curlywedge((\overline{h}(uy^2,q)\curlyvee\overline{\alpha})\curlywedge\overline{\beta})\curlyvee\overline{\alpha})\curlywedge\overline{\beta})\curlyvee\overline{\alpha}\\
&\succeq (\overline{f}(u,q)\curlywedge\overline{\beta}\curlywedge(\overline{g}(u,q)\curlywedge((\overline{h}(u,q)\curlywedge\overline{\beta})\curlywedge\overline{\beta})\curlyvee\overline{\alpha})\curlywedge\overline{\beta})\curlyvee\overline{\alpha}\\
&= (\overline{f}(u,q)\curlywedge\overline{\beta}\curlywedge(\overline{g}(u,q)\curlywedge\overline{\beta}\curlyvee\overline{\alpha})\curlywedge(\overline{h}(u,q)\curlywedge\overline{\beta}\curlywedge\overline{\beta}\curlyvee\overline{\alpha})\curlywedge\overline{\beta})\curlyvee\overline{\alpha}\\
&= (\overline{f}(u,q)\curlywedge\overline{\beta}\curlywedge(\overline{g}(u,q)\curlywedge\overline{\beta}\curlyvee\overline{\alpha})\curlywedge(\overline{h}(u,q)\curlywedge\overline{\beta}\curlyvee\overline{\alpha})\curlywedge\overline{\beta})\curlyvee\overline{\alpha}\\
&= (\overline{f}(u,q)\curlywedge\overline{\beta}\curlywedge((\overline{g}(u,q)\curlywedge\overline{h}(u,q)\curlywedge\overline{\beta})\curlyvee\overline{\alpha})\curlywedge\overline{\beta})\curlyvee\overline{\alpha}
\end{aligned}$$

$$= (\overline{f}(u,q) \curlywedge \overline{\beta} \curlywedge (\overline{g} \curlywedge_{\beta}^{\alpha} \overline{h})(u,q) \curlywedge \overline{\beta}) \curlyvee \overline{\alpha}$$
$$= (\overline{f}(u,q) \curlywedge (\overline{g} \curlywedge_{\beta}^{\alpha} \overline{h})(u,q) \curlywedge \overline{\beta}) \curlyvee \overline{\alpha} = (\overline{f} \curlywedge_{\beta}^{\alpha} \overline{g} \curlywedge_{\beta}^{\alpha} \overline{h})(u,q).$$

Hence, $(\overline{f} \curlywedge_{\beta}^{\alpha} \overline{g} \curlywedge_{\beta}^{\alpha} \overline{h})(u,q) \preceq (\overline{f} \circ_{\beta}^{\alpha} \overline{g} \circ_{\beta}^{\alpha} \overline{h})(u,q)$. Therefore, $\overline{f} \curlywedge_{\beta}^{\alpha} \overline{g} \curlywedge_{\beta}^{\alpha} \overline{h} \sqsubseteq \overline{f} \underset{(\overline{\alpha},\overline{\beta})}{\circ} \overline{g} \underset{(\overline{\alpha},\overline{\beta})}{\circ} \overline{h}$.

$(3) \Rightarrow (2)$ This is obvious because every IVQF bi-ideal with thresholds $(\overline{\alpha},\overline{\beta})$ is an IVQF generalized bi-ideal with thresholds $(\overline{\alpha},\overline{\beta})$ of S.

$(2) \Rightarrow (1)$ Let I, B and R be an interior ideal, a bi-ideal and a right ideal of S respectively. Then by Theorem 2.3, $\overline{\chi}_I, \overline{\chi}_B$ and $\overline{\chi}_R$ is a IVQF interior ideal with thresholds $(\overline{\alpha},\overline{\beta})$, an IVQF bi-ideals with thresholds $(\overline{\alpha},\overline{\beta})$ and an IVQF right ideal with thresholds $(\overline{\alpha},\overline{\beta})$ of S respectively. By supposition and Lemma 3.2, we have

$$\begin{aligned}
\overline{\beta} &= (\overline{\chi}_{I\cap B\cap R})_{((\overline{\alpha},\overline{\beta})}(u,q) = ((\overline{\chi}_I) \curlywedge_{\beta}^{\alpha} (\overline{\chi}_B \curlywedge_{\beta}^{\alpha} (\overline{\chi}_R))(u,q) \\
&\sqsubseteq ((\overline{\chi}_I) \circ_{\beta}^{\alpha} (\overline{\chi}_B) \circ_{\beta}^{\alpha} (\overline{\chi}_R))(u,q) = (\overline{\chi}_{IBR})_{(\overline{\alpha},\overline{\beta})}(u,q).
\end{aligned}$$

Thus, $u \in IBR$. Hence, $I \cap B \cap R \sqsubseteq IBR$. Therefore by Lemma 3.5, S is intra-regular. □

The following lemma will be used to prove in Theorem 3.8.

Lemma 3.7. [7] Let S be a semigroup. Then the following are equivalent:

(1) S is intra-regular,

(2) $L \cap B \cap I \subseteq LBI$, for every L is a left ideal, for each B is a bi-ideal and for each I is a interior ideal of S.

Theorem 3.8. Let S be a semigroup. Then the following equivalent:

(1) S is intra-regular,

(2) $\overline{f} \curlywedge_{\beta}^{\alpha} \overline{g} \curlywedge_{\beta}^{\alpha} \overline{h} \sqsubseteq \overline{f} \circ_{\beta}^{\alpha} \overline{g} \circ_{\beta}^{\alpha} \overline{h}$, for every IVQF left ideal with thresholds $(\overline{\alpha},\overline{\beta})$ $\overline{f}$, every IVQF bi-ideal with thresholds $(\overline{\alpha},\overline{\beta})$ $\overline{g}$ and every IVQF interior ideal with thresholds $(\overline{\alpha},\overline{\beta})$ $\overline{h}$ of S

(3) $\overline{f} \curlywedge_{\beta}^{\alpha} \overline{g} \curlywedge_{\beta}^{\alpha} \overline{h} \sqsubseteq \overline{f} \circ_{\beta}^{\alpha} \overline{g} \circ_{\beta}^{\alpha} \overline{h}$, for every every IVQF left ideal with thresholds $(\overline{\alpha},\overline{\beta})$ $\overline{f}$, every IVQF genralized bi-ideal with thresholds $(\overline{\alpha},\overline{\beta})$ $\overline{g}$ and every IVQF interior ideal with thresholds $(\overline{\alpha},\overline{\beta})$ $\overline{h}$ of S.

Proof. (1) $\Rightarrow$ (3) Let $\overline{f},\overline{g}$ and $\overline{h}$ be an IVQF left ideal with thresholds $(\overline{\alpha},\overline{\beta})$, an IVQF genralized bi-ideal with thresholds $(\overline{\alpha},\overline{\beta})$, an IVQF interior ideal with thresholds $(\overline{\alpha},\overline{\beta})$ of S respectively. Let $u\in S$ and $q\in Q$. Then there exist $x,y\in S$ such that $u=xu^2y=xuuy=x(xu^2y)uy=xxuuyuy=(x^2u)u(yuy)$. Thus

$$\begin{aligned}
(\overline{f}\circ^{\overline{\alpha}}_{\overline{\beta}}\overline{g}\circ^{\overline{\alpha}}_{\overline{\beta}}\overline{h})(u,q) &= (\curlyvee_{(i,j)\in F_u}(\overline{f}(i,q)\curlywedge(\overline{g}\circ^{\overline{\alpha}}_{\overline{\beta}}\overline{h})(j,q))\curlywedge\overline{\beta})\curlyvee\overline{\alpha}\\
&= (\curlyvee_{(i,j)\in F_{(x^2u)u(yauy)}}(\overline{f}(i,q)\curlywedge(\overline{g}\circ^{\overline{\alpha}}_{\overline{\beta}}\overline{h})(j,q))\curlywedge\overline{\beta})\curlyvee\overline{\alpha}\\
&\succeq (\overline{f}(x^2u,q)\curlywedge(\overline{g}\circ^{\overline{\alpha}}_{\overline{\beta}}\overline{h})(u(yuy),q)\curlywedge\overline{\beta})\curlyvee\overline{\alpha}\\
&= (\overline{f}(x^2u,q)\curlywedge((\curlyvee_{(m,n)\in F_{u(yuy,q)}}(\overline{g}(p)\curlywedge\overline{h}(q))\curlywedge\overline{\beta})\curlyvee\overline{\alpha})\curlywedge\overline{\beta})\curlyvee\overline{\alpha}\\
&\succeq (\overline{f}(x^2u,q)\curlywedge(((\overline{g}(u,q)\curlywedge\overline{h}(yuy,q))\curlywedge\overline{\beta})\curlyvee\overline{\alpha})\curlywedge\overline{\beta})\curlyvee\overline{\alpha}\\
&= (\overline{f}(x^2u,q)\curlyvee\overline{\alpha}\curlywedge(\overline{g}(u,q)\curlywedge(\overline{h}(yuy,q)\curlyvee\overline{\alpha}\curlywedge\overline{\beta})\curlyvee\overline{\alpha})\curlywedge\overline{\beta})\curlyvee\overline{\alpha}\\
&\succeq (\overline{f}(u,q)\curlywedge\overline{\beta}\curlywedge(\overline{g}(u,q)\curlywedge((\overline{h}(u,q)\curlywedge\overline{\beta})\curlywedge\overline{\beta})\curlyvee\overline{\alpha})\curlywedge\overline{\beta})\curlyvee\overline{\alpha}\\
&= (\overline{f}(u,q)\curlywedge\overline{\beta}\curlywedge(\overline{g}(u,q)\curlywedge\overline{\beta}\curlyvee\overline{\alpha})\curlywedge(\overline{h}(u,q)\curlywedge\overline{\beta}\curlywedge\overline{\beta}\curlyvee\overline{\alpha})\curlywedge\overline{\beta})\curlyvee\overline{\alpha}\\
&= (\overline{f}(u,q)\curlywedge\overline{\beta}\curlywedge(\overline{g}(u,q)\curlywedge\overline{\beta}\curlyvee\overline{\alpha})\curlywedge(\overline{h}(u,q)\curlywedge\overline{\beta}\curlyvee\overline{\alpha})\curlywedge\overline{\beta})\curlyvee\overline{\alpha}\\
&= (\overline{f}(u,q)\curlywedge\overline{\beta}\curlywedge((\overline{g}(u,q)\curlywedge\overline{h}(u,q)\curlywedge\overline{\beta})\curlyvee\overline{\alpha})\curlywedge\overline{\beta})\curlyvee\overline{\alpha}\\
&= (\overline{f}(u,q)\curlywedge\overline{\beta}\curlywedge(\overline{g}\underset{(\overline{\alpha},\overline{\beta})}{\sqcap}\overline{h})(u,q)\curlywedge\overline{\beta})\curlyvee\overline{\alpha}\\
&= (\overline{f}(u,q)\curlywedge(\overline{g}\curlywedge^{\overline{\alpha}}_{\overline{\beta}}\overline{h})(u,q)\curlywedge\overline{\beta})\curlyvee\overline{\alpha}=(\overline{f}\curlywedge^{\overline{\alpha}}_{\overline{\beta}}\overline{g}\curlywedge^{\overline{\alpha}}_{\overline{\beta}}\overline{h})(u,q).
\end{aligned}$$

Hence, $(\overline{f}\curlywedge^{\overline{\alpha}}_{\overline{\beta}}\overline{g}\curlywedge^{\overline{\alpha}}_{\overline{\beta}}\overline{h})(u,q)\preceq(\overline{f}\circ^{\overline{\alpha}}_{\overline{\beta}}\overline{g}\circ^{\overline{\alpha}}_{\overline{\beta}}\overline{h})(u,q)$. Therefore, $\overline{f}\curlywedge^{\overline{\alpha}}_{\overline{\beta}}\overline{g}\curlywedge^{\overline{\alpha}}_{\overline{\beta}}\overline{h}\sqsubseteq\overline{f}\underset{(\overline{\alpha},\overline{\beta})}{\circ}\overline{g}\underset{(\overline{\alpha},\overline{\beta})}{\circ}\overline{h}$.

It is obvious that (3) $\Rightarrow$ (2).

(2) $\Rightarrow$ (1) Let L,B and I be a left ideal, a bi-ideal and an interior ideal of S respectively. Then by Theorem 2.3, $\overline{\chi}_L,\overline{\chi}_B$ and $\overline{\chi}_I$ is a IVQF left ideal with thresholds $(\overline{\alpha},\overline{\beta})$, an IVQF bi-ideal with thresholds $(\overline{\alpha},\overline{\beta})$ and an IVQF interior ideal with thresholds $(\overline{\alpha},\overline{\beta})$ of S respectively.
By supposition and Lemma 3.2, we have

$$\begin{aligned}
\overline{\beta} &= (\overline{\chi}_{L\cap B\cap I})_{(\overline{\alpha},\overline{\beta})}(u,q)=((\overline{\chi}_L)\curlywedge^{\overline{\alpha}}_{\overline{\beta}}(\overline{\chi}_B)\curlywedge^{\overline{\alpha}}_{\overline{\beta}}(\overline{\chi}_I))(u,q)\\
&\sqsubseteq ((\overline{\chi}_L)\circ^{\overline{\alpha}}_{\overline{\beta}}(\overline{\chi}_B)\circ^{\overline{\alpha}}_{\overline{\beta}}(\overline{\chi}_I))(u,q)=(\overline{\chi}_{LBI})_{(\overline{\alpha},\overline{\beta})}(u,q).
\end{aligned}$$

Thus, $u\in LBI$. Hence, $L\cap B\cap I\sqsubseteq LBI$. Therefore by Lemma 3.7, S is intra-regular. $\square$

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CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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