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ON WEAKLY REGULAR SEMIGROUPS CHARACTERIZED IN TERMS OF INTERVAL VALUED Q-FUZZY SUBSEMIGROUPS WITH THRESHOLDS $(\overline{\alpha},\overline{\beta})$

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Abstract. In this article, we provide relationship between interval valued Q-fuzzy interior ideals with thresholds $(\overline{\alpha},\overline{\beta})$ and interval valued Q-fuzzy ideals with thresholds $(\overline{\alpha},\overline{\beta})$. In the goal results, we proceed to characterize the simisimple semigroup by using interval valued Q-fuzzy interior ideals with thresholds $(\overline{\alpha},\overline{\beta})$.

Keywords: interval-valued Q-fuzzy ideals with thresholds $(\overline{\alpha};\overline{\beta})$; interval-valued Q-fuzzy interior ideals with thresholds $(\overline{\alpha};\overline{\beta})$; left regular; regular; intra-regular; semisimple semigroup.

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1. Introduction

As a generalization of fuzzy set interval valued fuzzy set was conceptualized by Zadeh in 1975[19]. This concept is not only used in mathematics and logic but also in medical science [5], image processing [3] and decision making method [22] etc. In 1994, Biswas [4] used the ideal of interval valued fuzzy sets to interval valued subgroups. In 2006, Narayanan and Manikantan [14] were studied interval valued fuzzy subsemigroups and types interval valued fuzzy ideals in semigroups. In 2014, Aslam et al. [2], gave the concept interval valued

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$(\overline{\alpha},\overline{\beta})$-fuzzy ideals of LA-semigroups where $\overline{\alpha},\overline{\beta} \in \{\overline{\varepsilon},\overline{\varepsilon},\vee\overline{q}\}$ and he characterized regular LA-semigroups by using interval valued $(\overline{\alpha},\overline{\beta})$-fuzzy ideals. In 2017, Murugads et al. [12] studied interval valued Q-fuzzy subsemigroup of ordered semigroup.

In the same year Abdullah et al. [1] gave the definition of $(\overline{\alpha},\overline{\beta})$-interval valued fuzzy subsemigroups where $\overline{\alpha} \prec \overline{\beta}$, which are generalization of interval valued fuzzy subsemigroups and they characterized regular semigroups in terms of $(\overline{\alpha},\overline{\beta})$-interval valued fuzzy subsemigroups. In 2019, Murugads and Arikrishnan [13] gave concept of interval valued Q-fuzzy ideal with thresholds $(\overline{\alpha},\overline{\beta})$ where $\overline{\alpha} \prec \overline{\beta}$ and characterized regular semigroups in terms of interval valued Q-fuzzy ideal with thresholds $(\overline{\alpha},\overline{\beta})$.

In this article, we provide relationship between interval valued Q-fuzzy interior ideals with thresholds $(\overline{\alpha},\overline{\beta})$ and interval valued Q-fuzzy ideals with thresholds $(\overline{\alpha},\overline{\beta})$. In the goal results, we proceed to characterize the simisimple semigroup by using interval valued Q-fuzzy interior ideals with thresholds $(\overline{\alpha},\overline{\beta})$.

2. Preliminaries

In this topic, we give some basic definitions which will be helpful in next topic.

By a subsemigroup of a semigroup S we mean a non-empty subset K of S such that $K^2 \subseteq K$. A non-empty subset K of a semigroup S is called a *left* (right) ideal of S if $SK \subseteq K$ $(KS \subseteq K)$. By an *ideal* K of a semigroup S we mean a left ideal and a right ideal of S. A subsemigroup K of a semigroup S is called an *interior ideal* of S if $SKS \subseteq K$. A semigroup S is called *left* (right) regular if for each $u \in S$, there exists $a \in S$ such that $u = au^2$ $(u = u^2a)$. A semigroup S is said to be *intra-regular* if for each $u \in S$, there exist $a,b \in S$ such that $u = au^2b$. A semigroup S is called *semisimple* if every ideal of S is an idempotent. It is evident that S is semisimple if and only if $u \in (SuS)(SuS)$ for every $u \in S$, that is there exist $w,y,z \in S$ such that $u = wuyuz$.

For any $m_i \in [0,1]$, $i \in \mathscr{A}$, define

$$\bigvee_{i\in\mathscr{A}} m_i := \sup_{i\in\mathscr{A}}\{m_i\} \quad \text{and} \quad \bigwedge_{i\in\mathscr{A}} m_i := \inf_{i\in\mathscr{A}}\{m_i\}.$$

We see that for any $m,n \in [0,1]$, we have

$$m \vee n = \max\{m,n\} \quad \text{and} \quad m \wedge n = \min\{m,n\}.$$

We use $\mathscr{C}$ to denote the set of all closed subintervals in $[0,1]$, i.e.,

$$\mathscr{C} = \{\overline{m} := [m^-, m^+] \mid 0 \leq m^- \leq m^+ \leq 1\}.$$

We note that $[m,m] = \{m\}$ for all $m \in [0,1]$. For $m = 0$ or 1 we shall denote $\overline{0} = [0,0] = \{0\}$ and $\overline{1} = [1,1] = \{1\}$.

For any two interval numbers $\overline{m}$ and $\overline{n}$ in $\mathscr{C}$, define the operations "$\preceq$", "$=$", "$\curlywedge$" "$\curlyvee$" as follows:

(1) $\overline{m} \preceq \overline{n}$ if and only if $m^- \leq n^-$ and $m^+ \leq n^+$

(2) $\overline{m} = \overline{n}$ if and only if $m^- = n^-$ and $m^+ = n^+$

(3) $\overline{m} \curlywedge \overline{n} = [(m^- \wedge n^-), (m^+ \wedge n^+)]$

(4) $\overline{m} \curlyvee \overline{n} = [(m^- \vee n^-), (m^+ \vee n^+)]$.

If $\overline{m} \succeq \overline{n}$, we mean $\overline{n} \preceq \overline{m}$.

The following proposition is a tool used to prove the section 4 and 5.

Proposition 2.1. [6] For any elements $\overline{m}, \overline{n}$ and $\overline{p}$ in $\mathscr{C}$, the following properties are true:

(1) $\overline{m} \curlywedge \overline{m} = \overline{m}$ and $\overline{m} \curlyvee \overline{m} = \overline{m}$,

(2) $\overline{m} \curlywedge \overline{n} = \overline{n} \curlywedge \overline{m}$ and $\overline{m} \curlyvee \overline{n} = \overline{n} \curlyvee \overline{m}$,

(3) $(\overline{m} \curlywedge \overline{n}) \curlywedge \overline{p} = \overline{m} \curlywedge (\overline{n} \curlywedge \overline{p})$ and $(\overline{m} \curlyvee \overline{n}) \curlyvee \overline{p} = \overline{m} \curlyvee (\overline{n} \curlyvee \overline{p})$,

(4) $(\overline{m} \curlywedge \overline{n}) \curlyvee \overline{p} = (\overline{m} \curlyvee \overline{p}) \curlywedge (\overline{n} \curlyvee \overline{p})$ and $(\overline{m} \curlyvee \overline{n}) \curlywedge \overline{p} = (\overline{m} \curlywedge \overline{p}) \curlyvee (\overline{n} \curlywedge \overline{p})$,

(5) If $\overline{m} \preceq \overline{n}$, then $\overline{m} \curlywedge \overline{p} \preceq \overline{n} \curlywedge \overline{p}$ and $\overline{m} \curlyvee \overline{p} \preceq \overline{n} \curlyvee \overline{p}$.

For each interval $\{\overline{m}_i := [m_i^-, m_i^+] \mid i \in \mathscr{A}\}$ be a family of closed subintervals of $[0,1]$. Define $\underset{i \in \mathscr{A}}{\curlywedge} \overline{m}_i = [\underset{i \in \mathscr{A}}{\wedge} m_i^-, \underset{i \in \mathscr{A}}{\wedge} m_i^+]$ and $\underset{i \in \mathscr{A}}{\curlyvee} \overline{m}_i = [\underset{i \in \mathscr{A}}{\vee} m_i^-, \underset{i \in \mathscr{A}}{\vee} m_i^+]$.

Definition 2.1. Let S be a semigroup and Q be a non-empty set. A Q-fuzzy subset (Q-fuzzy set) of a set T is a function $f : S \times Q \to [0,1]$

Definition 2.2. [17] Let T be a non-empty set. An interval valued fuzzy subset (shortly, IVF subset) of T is a function $\overline{f} : T \to \mathscr{C}$

Definition 2.3. [12] Let S be a semigroup and Q be a non-empty set. An interval valued Q-fuzzy subset (shortly, IVQF subset) of T is a function $\overline{f} : S \times Q \to \mathscr{C}$

Definition 2.4. [12] Let K be a non-empty subset of a semigroup S and Q be a non-empty set . An interval valued characteristic function $\overline{\lambda}_K$ of K is defined to be a function $\overline{\lambda}_K : S \times Q \to \mathscr{C}$ by

$$\overline{\lambda}_K(u,q) = \begin{cases} \overline{1} & \text{if} \quad u \in K \\ \overline{0} & \text{if} \quad u \notin K \end{cases}$$

for all $u \in T$.

For two IVQF subsets $\overline{f}$ and $\overline{g}$ of a semigroups S, define

(1) $\overline{f} \sqsubseteq \overline{g} \Leftrightarrow \overline{f}(u,q) \preceq \overline{g}(u,q)$ for all $u \in S$ and $q \in Q$,

(2) $\overline{f} = \overline{g} \Leftrightarrow \overline{f} \sqsubseteq \overline{g}$ and $\overline{g} \sqsubseteq \overline{f}$,

(3) $(\overline{f} \sqcap \overline{g})(u,q) = \overline{f}(u,q) \curlywedge \overline{g}(u,q)$ for all $u \in S$ and $q \in Q$.

For two IVQF subsets $\overline{f}$ and $\overline{g}$ of a semigroup S. Then the product $\overline{f} \circ \overline{g}$ is defined as follows for all $u \in S$ and $q \in Q$,

$$(\overline{f} \circ \overline{g})(u,q) = \begin{cases} \underset{(y,z) \in F_u}{\curlyvee} \{\overline{f}(y,q) \curlywedge \overline{g}(z,q)\} & \text{if} \quad F_u \neq \emptyset, \\ \overline{0} & \text{if} \quad F_u = \emptyset, \end{cases}$$

where $F_u := \{(y,z) \in S \times S \mid u = yz\}$.

Next, we shall give definitions of various types of IVQF subsemigroup of a semigroups.

Definition 2.5. [13] An IVF subset $\overline{f}$ of a semigroup S is said to be

(1) an *IVQF subsemigroup* of S if $\overline{f}(uv,q) \succeq \overline{f}(u,q) \curlywedge \overline{f}(v,q)$ for all $u,v \in S$ and $q \in Q$,

(2) an *IVQF left (right) ideal* of S if $\overline{f}(uv,q) \succeq \overline{f}(v,q)$ $(\overline{f}(uv,q) \succeq \overline{f}(u,q))$ for all $u,v \in S$ and $q \in Q$. An *IVQF ideal* of S if it is both an IVQF left ideal and an IVQF right ideal of S,

(3) an *IVQF generalized bi-ideal* of S if $\overline{f}(uvw,q) \succeq \overline{f}(u,q) \curlywedge \overline{f}(w,q)$ for all $u,v,w \in S$ and $q \in Q$,

(4) an *IVQF bi-ideal* of S if $\overline{f}$ is an IVQF subsemigroup of S and $\overline{f}(uvw,q) \succeq \overline{f}(u,q) \curlywedge \overline{f}(w,q)$ for all $u,v,w \in S$ and $q \in Q$,

(5) an *IVQF interior ideal* of S if if $\overline{f}$ is an IVQF subsemigroup of S and $\overline{f}(uav,q) \succeq \overline{f}(a,q)$ for all $a,u,v \in S$ and $q \in Q$,

(6) an *IVQF quasi-ideal* of S if $\overline{f}(u,q) \succeq (\overline{\mathscr{S}} \circ \overline{f})(u,q) \curlywedge (\overline{f} \circ \overline{\mathscr{S}})(u,q)$, for all $u \in S$ and $q \in Q$ where $\overline{S}$ is an IVQF subset of S mapping every element of S on $\overline{1}$.

The thought of an IVQF subsemigroup with thresholds $(\overline{\alpha},\overline{\beta})$ where $\overline{\alpha} \prec \overline{\beta}$ as follows:

Definition 2.6. [13] An IVF subset $\overline{f}$ of a semigroup S and $\overline{\alpha} \prec \overline{\beta}$ and $\overline{\alpha},\overline{\beta} \in \mathscr{C}$ is said to be

(1) an *IVQF subsemigroup with thresholds* $(\overline{\alpha},\overline{\beta})$ of S if $\overline{f}(uv,q) \curlyvee \overline{\alpha} \succeq \overline{f}(u,q) \curlywedge \overline{f}(v,q) \curlywedge \overline{\beta}$ for all $u,v \in S$ and $q \in Q$,

(2) an *IVQF left (right) ideal with thresholds* $(\overline{\alpha},\overline{\beta})$ of S if $\overline{f}(uv,q) \curlyvee \overline{\alpha} \succeq \overline{f}(v,q) \curlywedge \overline{\beta}$ $(\overline{f}(uv,q) \curlyvee \overline{\alpha} \succeq \overline{f}(u,q) \curlywedge \overline{\beta})$ for all $u,v \in S$ and $q \in Q$. An *IVQF ideal with thresholds* $(\overline{\alpha},\overline{\beta})$ of S if it is both an IVF left ideal and an IVF right ideal of S,

(3) an *IVQF generalized bi-ideal with thresholds* $(\overline{\alpha},\overline{\beta})$ of S if $\overline{f}(uvw,q) \curlyvee \overline{\alpha} \succeq \overline{f}(u,q) \curlywedge \overline{f}(w,q) \curlywedge \overline{\beta}$ for all $u,v,w \in S$ and $q \in Q$,

(4) an *IVQF bi-ideal with thresholds* $(\overline{\alpha},\overline{\beta})$ of S if $\overline{f}$ is an IVQF subsemigroup with thresholds $(\overline{\alpha},\overline{\beta})$ of S and $\overline{f}(uvw,q) \curlyvee \overline{\alpha} \succeq \overline{f}(u,q) \curlywedge \overline{f}(w,q) \curlywedge \overline{\beta}$ for all $u,v,w \in S$ and $q \in Q$,

(5) an *IVQF interior ideal with thresholds* $(\overline{\alpha},\overline{\beta})$ of S if $\overline{f}$ is an IVQF subsemigroup with thresholds $(\overline{\alpha},\overline{\beta})$ of S and $\overline{f}(uav,q) \curlyvee \overline{\alpha} \succeq \overline{f}(a,q) \curlywedge \overline{\beta}$ for all $a,u,v \in S$ and $q \in Q$,

(6) an *IVQF quasi-ideal with thresholds* $(\overline{\alpha},\overline{\beta})$ of S if $\overline{f}(u,q) \curlyvee \overline{\alpha} \succeq (\overline{\mathscr{S}} \circ \overline{f})(u,q) \curlywedge (\overline{f} \circ \overline{\mathscr{S}})(u,q) \curlywedge \overline{\beta}$, for all $u \in S$ and $q \in Q$.

Remark 2.2. [13] It is clear to see that every IVQF bi-ideal with thresholds $(\overline{\alpha},\overline{\beta})$ is an IVQF generalized bi-ideal with thresholds $(\overline{\alpha},\overline{\beta})$ of S, every IVQF ideal with thresholds $(\overline{\alpha},\overline{\beta})$ is an IVQF interior ideal with thresholds $(\overline{\alpha},\overline{\beta})$ of S and IVQF quasi-ideal with thresholds $(\overline{\alpha},\overline{\beta})$ is an IVQF bi-ideal with thresholds $(\overline{\alpha},\overline{\beta})$ of S.

In this ensuing theorem is present relationship between types ideals of a semigroup S and the interval valued characteristic function.

Theorem 2.3. [13] If K is a left ideal (right ideal generalized bi-ideal, bi-ideal, interior ideal, quasi-ideal) of S, then characteristic function $\overline{\chi}_K$ is an IVQF left ideal (right ideal, generalized bi-ideal, bi-ideal, interior ideal, quasi-ideal) with thresholds $(\overline{\alpha},\overline{\beta})$ of S for all $\overline{\alpha} \prec \overline{\beta}$ and $\overline{\alpha},\overline{\beta} \in \mathscr{C}$.

The following theorem is easy to prove.

Theorem 2.4. [13] Every IVQF ideal with thresholds $(\overline{\alpha},\overline{\beta})$ of a semigroup S is an IVQF interior ideal with thresholds $(\overline{\alpha},\overline{\beta})$ of S.

Example 2.1. Consider a semigroup $S = \{0, a, b, c\}$ and Q be any non-empty set

$\cdot$	0	a	b	c
0	0	0	0	0
a	0	0	0	0
b	0	0	0	a
c	0	0	a	b

Let $\overline{f}$ be an IVQF subset of S such that $\overline{f}(0,q) = [0.7,0.8], \overline{f}(a,q) = [0.4,0.5], \overline{f}(b,q) = [0.6,0.7]$ $\overline{f}(c,q) = \overline{0}$ and let $\overline{\alpha} = [0.3,0.3]$, $\overline{\beta} = [0.5,0.5]$. Then $\overline{f}$ is not an IVQF interior ideal with $(\overline{\alpha},\overline{\beta})$ of S . But the $\overline{\lambda}$ is an IVQF ideal with $(\overline{\alpha},\overline{\beta})$ of S, because $\overline{f}(bc,q) \curlyvee \overline{\alpha} = \overline{f}(a,q) \curlyvee \overline{\alpha} = [0.4,0.5] \not\succeq [0.5,0.5] = \overline{f}(b,q) \curlywedge \overline{\beta}$. Thus $\overline{f}$ is not an IVQF right ideal subsemigroup with $(\overline{\alpha},\overline{\beta})$ of S.

The following theorem show that the IVQF interior ideals with thresholds $(\overline{\alpha},\overline{\beta})$ and IVQF ideals with thresholds $(\overline{\alpha},\overline{\beta})$ coincide for some types of semigroups. The proof of this theorem is straightforward and simple.

Lemma 2.5. Let S be a semigroup. If S is left (right) regular, then every IVQF interior ideals with thresholds $(\overline{\alpha},\overline{\beta})$ of S is thresholds $(\overline{\alpha},\overline{\beta})$-IVF ideal of S.

Proof. Suppose that $\overline{f}$ is an IVQF interior ideals with thresholds $(\overline{\alpha},\overline{\beta})$ of S and let $u, v \in S$ and $q \in Q$. Since S is left regular, there exists $k \in S$ such that $u = ku^2$. Thus, $\overline{f}(uv,q) \curlyvee \overline{\alpha} = \overline{f}((ku^2)v,q) \curlyvee \overline{\alpha} = \overline{f}(kuuv,q) \curlyvee \overline{\alpha} = \overline{f}((ku)uv,q) \curlyvee \overline{\alpha} \succeq \overline{f}(u,q) \curlywedge \overline{\beta}$. Hence $\overline{f}$ is an IVQF right ideals with $(\overline{\alpha},\overline{\beta})$ of S. Similarly, we can show that $\overline{f}$ is an IVQF left ideals with $(\overline{\alpha},\overline{\beta})$ of S. Thus $\overline{f}$ is an IVQF ideals with $(\overline{\alpha},\overline{\beta})$ of S. □

Lemma 2.6. Let S be a semigroup. If S is intra-regular, then every IVQF interior ideals with $(\overline{\alpha},\overline{\beta})$ of S is an IVQF ideal thresholds $(\overline{\alpha},\overline{\beta})$ of S.

Proof. Suppose that $\overline{f}$ is an IVQF interior ideals with $(\overline{\alpha},\overline{\beta})$ of semigroup S and let $u, v \in S$ and $q \in Q$. Since S is intra-regular, there exist $x, y \in S$ such that $u = xu^2y$. Thus, $\overline{f}(uv,q) \curlyvee \overline{\alpha} = \overline{f}((xu^2y)v,q) \curlyvee \overline{\alpha} = \overline{f}((xuuy)v,q) \curlyvee \overline{\alpha} = \overline{f}((xu)u(yv),q) \curlyvee \overline{\alpha} \succeq \overline{f}(u,q) \curlywedge \overline{\beta}$. Hence $\overline{f}$ is an IVQF right ideals with $(\overline{\alpha},\overline{\beta})$ of S. Similarly, we can show that $\overline{f}$ is an IVQF left ideals with $(\overline{\alpha},\overline{\beta})$ of S. Thus $\overline{f}$ is an IVQF ideals with $(\overline{\alpha},\overline{\beta})$ of S. □

Lemma 2.7. Let S be a semigroup. If S is semisimple, then every IVQF interior ideals with $(\overline{\alpha},\overline{\beta})$ of S is is an IVQF ideal thresholds $(\overline{\alpha},\overline{\beta})$ of S.

Proof. Suppose that $\overline{f}$ is an IVQF interior ideals with $(\overline{\alpha},\overline{\beta})$ of S and let $u,v\in S$ and $q\in Q$. Since S is semisimple, there exist $x,y,z\in S$ such that $u=xuyuz$. Thus, $\overline{f}(uv,q)\curlyvee\overline{\alpha}=\overline{f}((xuyuz)v,q)\curlyvee\overline{\alpha}=\overline{f}((xuy)u(zv),q)\curlyvee\overline{\alpha}\succeq\overline{f}(u,q)\curlywedge\overline{\beta}$. Hence $\overline{f}$ is an IVQF right ideals with $(\overline{\alpha},\overline{\beta})$ ideal of S. Similarly, we can show that $\overline{f}$ is an IVQF left ideals with $(\overline{\alpha},\overline{\beta})$ of S. Thus $\overline{f}$ is an IVQF ideals with $(\overline{\alpha},\overline{\beta})$ of S. □

By Lemma 2.5, 2.6 and 2.7 we have Theorem 2.8.

Theorem 2.8. In left (right) regular, intra-regular and semisimple semigroup, the IVQF interior ideals with $(\overline{\alpha},\overline{\beta})$ and the is an IVQF ideal thresholds $(\overline{\alpha},\overline{\beta})$ coincide.

3. Characterize Semsimiple Semigroups in Terms IVQF Interior Ideal With Thresholds $(\overline{\alpha},\overline{\beta})$ and IVQF Ideals With Thresholds $(\overline{\alpha},\overline{\beta})$.

In this topic, we will characterize a semsimiple semigroup in terms of IVQF interior ideals with thresholds $(\overline{\alpha},\overline{\beta})$ and IVQF ideals with thresholds $(\overline{\alpha},\overline{\beta})$.

In 2019, [13] Murugads and Arikrishnan propose symbols of IVQF ideals with thresholds $(\overline{\alpha},\overline{\beta})$ for use characterizes a semigroup in terms IVQF ideals with thresholds $(\overline{\alpha},\overline{\beta})$ of semigroup.

For any IVQF subset $\overline{f}$ of a semigroup S with $\overline{\alpha}\prec\overline{\beta}$ and $\overline{\alpha},\overline{\beta}\in\mathscr{C}$., define

$$\overline{f}_{(\overline{\alpha},\overline{\beta})}(u,q)=(\overline{f}(u,q)\curlywedge\overline{\alpha})\curlyvee\overline{\beta}$$

for all $u\in S$ and $q\in Q$.

For any IVQF subsets $\overline{f}$ and $\overline{g}$ of a semigroup S with $\overline{\alpha}\prec\overline{\beta}$ and $\overline{\alpha},\overline{\beta}\in\mathscr{C}$, define the operation "$\curlywedge_{\overline{\beta}}^{\overline{\alpha}}$" as follows:

$$(\overline{f}\curlywedge_{\overline{\beta}}^{\overline{\alpha}}\overline{g})(u,q)=(\overline{f}(u,q)\curlywedge\overline{g}(u,q)\curlywedge\overline{\alpha})\curlyvee\overline{\beta}$$

for all $u\in S$ and $q\in Q$. And define the product $\overline{f}\circ_{\overline{\beta}}^{\overline{\alpha}}\overline{g}$ as follows: for all $u\in S$ and $q\in Q$,

$$(\overline{f}\circ_{\overline{\beta}}^{\overline{\alpha}}\overline{g})(u,q)=((\overline{f}\circ\overline{g})(u,q)\curlywedge\overline{\alpha})\curlyvee\overline{\beta}$$

where

$$(\overline{f}\circ\overline{g})(u,q)=\begin{cases}\curlyvee_{(x,y)\in F_u}\{\overline{f}(x,q)\curlywedge\overline{g}(y,q)\} & \text{if} \quad F_u\neq\emptyset,\\ \overline{0} & \text{if} \quad F_u=\emptyset,\end{cases}$$

where $F_u:=\{(x,y)\in S\times S\mid u=xy\}$.

Remark 3.1. Since $\overline{\chi}$ is an interval valued characteristic, we have

$$\overline{\lambda}_{(\overline{\alpha},\overline{\beta})}(u,q) := \begin{cases} \overline{\beta} & \text{if} \quad u \in K, \\ \overline{\alpha} & \text{if} \quad u \notin K. \end{cases}$$

Lemma 3.2. [13] Let K and L be non-empty subsets of a semigroup S with $\overline{\alpha} \prec \overline{\beta}$ and $\overline{\alpha},\overline{\beta} \in \mathscr{C}$. Then the following assertions hold:

(1) $(\overline{\lambda}_K) \curlywedge^{\overline{\alpha}}_{\overline{\beta}} (\overline{\lambda}_L) = (\overline{\lambda}_{K\cap L})_{(\overline{\alpha},\overline{\beta})}$.
(2) $(\overline{\lambda}_K) \circ^{\overline{\alpha}}_{\overline{\beta}} (\overline{\lambda}_L) = (\overline{\lambda}_{KL})_{(\overline{\alpha},\overline{\beta})}$.

On the basis of Lemma 3.3, we can prove Theorem 3.5.

Lemma 3.3. [13] Let S be a semigroup. If $\overline{f}$ is a $(\overline{\alpha},\overline{\beta})$-IVQF right ideal and $\overline{g}$ is a $(\overline{\alpha},\overline{\beta})$-IVQF left ideal of S, then $\overline{f} \underset{(\overline{\beta},\overline{\alpha})}{\circ} \overline{g} \sqsubseteq \overline{f} \curlywedge^{\overline{\alpha}}_{\overline{\beta}} \overline{g}$.

Lemma 3.4. [11] For a semigroup S, the following statements are equivalent.

(1) S is semisimple,

(2) Every interior ideal K of S is idempotent,

(3) Every ideal K of S is idempotent,

(4) For any ideals K and L of S, $K \cap L = KL$

(5) For any ideal K and any interior ideal L of S, $K \cap L = KL$

(6) For any interior K and any ideal L of S, $K \cap L = KL$

(7) For any interior ideals K and L of S, $K \cap L = KL$.

The following Theorem show an equivalent conditional statement for a semisimple semigroup.

Theorem 3.5. Let S be a semigroup. Then the following are equivalent:

(1) S is semisimple,
(2) $\overline{f} \circ^{\overline{\alpha}}_{\overline{\beta}} \overline{f} = \overline{f}$, for every IVQF interior ideals with thresholds $(\overline{\alpha},\overline{\beta})$ $\overline{f}$ of S,
(3) $\overline{f} \circ^{\overline{\alpha}}_{\overline{\beta}} \overline{f} = \overline{f}$, for every IVQF ideals with thresholds $(\overline{\alpha},\overline{\beta})$ $\overline{f}$ of S,
(4) $\overline{f} \circ^{\overline{\alpha}}_{\overline{\beta}} \overline{g} = \overline{f} \curlywedge^{\overline{\alpha}}_{\overline{\beta}} \overline{g}$, for every IVQF interior ideals with thresholds $(\overline{\alpha},\overline{\beta})$ $\overline{f}$ and $\overline{g}$ of S,
(5) $\overline{f} \circ^{\overline{\alpha}}_{\overline{\beta}} \overline{g} = \overline{f} \curlywedge^{\overline{\alpha}}_{\overline{\beta}} \overline{g}$, for every IVQF ideals with thresholds $(\overline{\alpha},\overline{\beta})$ $\overline{f}$ and $\overline{g}$ of S,
(6) $\overline{f} \circ^{\overline{\alpha}}_{\overline{\beta}} \overline{g} = \overline{f} \curlywedge^{\overline{\alpha}}_{\overline{\beta}} \overline{g}$, for every IVQF interior ideal with thresholds $(\overline{\alpha},\overline{\beta})$ $\overline{f}$ of S and every IVQF ideal with thresholds $(\overline{\alpha},\overline{\beta})$ $\overline{g}$ of S,

(7) $\overline{f}\circ_{\overline{\beta}}^{\overline{\alpha}}\overline{g}=\overline{f}\curlywedge_{\overline{\beta}}^{\overline{\alpha}}\overline{g}$, for every IVQF ideal with thresholds $(\overline{\alpha},\overline{\beta})$ $\overline{f}$ of S and every IVQF interior ideal with thresholds $(\overline{\alpha},\overline{\beta})$ $\overline{g}$ of S.

Proof. (1) $\Rightarrow$ (2) Suppose that $\overline{f}$ is a IVQF interior ideal with thresholds $(\overline{\alpha},\overline{\beta})$ of S. Then $\overline{f}$ is a IVQF subsemigroup with thresholds $(\overline{\alpha},\overline{\beta})$. We will show that $\overline{f}\underset{(\bar{t},\bar{s})}{\circ}\overline{f}=\overline{f}_{(\bar{t},\bar{s})}$. Let $u\in S$ and $q\in Q$.

If $F_u=\emptyset$, then it is easy to verify that $(\overline{f}\circ_{\overline{\beta}}^{\overline{\alpha}}\overline{f})(u,q)\preceq\overline{f}_{(\overline{\alpha},\overline{\beta})}(u,q)$.

If $F_u\neq\emptyset$, then

$$\begin{aligned}
(\overline{f}\circ_{\overline{\beta}}^{\overline{\alpha}}\overline{f})(u,q) &= \Big(\curlyvee_{(x,y)\in F_u}\{\overline{f}(x,q)\curlywedge\overline{f}(y,q)\}\curlywedge\overline{\beta}\Big)\curlyvee\overline{\alpha}\\
&= \Big(\curlyvee_{(x,y)\in F_u}\{\overline{f}(x,q)\curlywedge\overline{f}(y,q)\curlywedge\overline{\beta}\}\curlywedge\overline{\beta}\Big)\curlyvee\overline{\alpha}\\
&\preceq \Big(\curlyvee_{(x,y)\in F_u}\{\overline{f}(xy,q)\curlyvee\overline{\alpha}\}\curlywedge\overline{\beta}\Big)\curlyvee\overline{\alpha}\\
&= ((\overline{f}(u,q)\curlyvee\overline{\alpha})\curlywedge\overline{\beta})\curlyvee\overline{\alpha}=((\overline{f}(u,q)\curlyvee\overline{\alpha})\curlyvee\overline{\alpha})\curlywedge(\overline{\beta}\curlyvee\overline{\alpha})\\
&= (\overline{f}(u,q)\curlyvee\overline{\alpha})\curlywedge(\overline{\beta}\curlyvee\overline{\alpha})=(\overline{f}(u,q)\curlywedge\overline{\beta})\curlyvee\overline{\alpha}=\overline{f}_{(\overline{\alpha},\overline{\beta})}(u,q).
\end{aligned}$$

Thus, $(\overline{f}\circ_{\overline{\beta}}^{\overline{\alpha}}\overline{f})(u,q)\preceq\overline{f}_{(\overline{\alpha},\overline{\beta})}(u,q)$. Hence, $\overline{f}\circ_{\overline{\beta}}^{\overline{\alpha}}\overline{f}\sqsubseteq\overline{f}_{(\overline{\alpha},\overline{\beta})}$.

Since S is semisimple, we have there exist $w,x,y,z\in S$ such that $u=(xuy)(zuw)$. Thus

$$\begin{aligned}
(\overline{f}\circ_{\overline{\beta}}^{\overline{\alpha}}\overline{f})(u,q) &= \Big(\curlyvee_{(i,j)\in F_u}\{\overline{f}(i,q)\curlywedge\overline{f}(j,q)\}\curlywedge\overline{\beta}\Big)\curlyvee\overline{\alpha}\\
&= \Big(\curlyvee_{(i,j)\in F_{(xuy)(wuz)}}\{\overline{f}(i,q)\curlywedge\overline{f}(j,q)\}\curlywedge\overline{\beta}\Big)\curlyvee\overline{\alpha}\\
&\succeq ((\overline{f}(xuy,q)\curlywedge\overline{f}(wuz,q))\curlywedge\overline{\beta})\curlyvee\overline{\alpha}\\
&= ((\overline{f}(xuy,q)\curlyvee\overline{\alpha})\curlywedge(\overline{f}(wuz,q)\curlyvee\overline{\alpha})\curlywedge\overline{\beta})\curlyvee\overline{\alpha}\\
&\succeq ((\overline{f}(u,q)\curlywedge\overline{\beta})\curlywedge(\overline{f}(u,q)\curlywedge\overline{\beta})\curlywedge\overline{\beta})\curlyvee\overline{\alpha}\\
&= ((\overline{f}(u,q)\curlywedge\overline{\beta})\curlywedge\overline{\beta})\curlyvee\overline{\alpha}=(\overline{f}(u,q)\curlywedge\overline{\beta})\curlyvee\overline{\alpha}=\overline{f}_{(\overline{\alpha},\overline{\beta})}(u,q).
\end{aligned}$$

Hence, $(\overline{f}\circ_{\overline{\beta}}^{\overline{\alpha}}\overline{f})(u,q)\succeq\overline{f}_{(\overline{\alpha},\overline{\beta})}(u,q)$, and so $\overline{f}_{(\overline{\alpha},\overline{\beta})}\sqsubseteq\overline{f}\circ_{\overline{\beta}}^{\overline{\alpha}}\overline{f}$. Therefore, $\overline{f}\circ_{\overline{\beta}}^{\overline{\alpha}}\overline{f}=\overline{f}_{(\overline{\alpha},\overline{\beta})}$.

(2) $\Rightarrow$ (1) Let K be an interior ideal of S. Then by Theorem 2.3, $\overline{\lambda}_K$ is a IVQF interior ideal with thresholds $(\overline{\alpha},\overline{\beta})$ of S. By supposition and Lemma 3.2, we have

$$(\overline{\lambda}_{K^2})_{(\overline{\alpha},\overline{\beta})}(u,q)=((\overline{\lambda}_K)\circ_{\overline{\beta}}^{\overline{\alpha}}(\overline{\lambda}_K))(u,q)=(\overline{\lambda}_K)_{(\overline{\alpha},\overline{\beta})}(u,q)=\overline{\beta}.$$

Thus $u\in K^2$. Hence $K^2=K$. By Lemma 3.4, we have S is semisimple.

(1) $\Rightarrow$ (4) Let $\overline{f}$ and $\overline{g}$ be IVQF interior ideal with thresholds $(\overline{\alpha},\overline{\beta})$ of S. Then by Theorem 2.7, $\overline{f}$ and $\overline{g}$ are IVQF ideal with thresholds $(\overline{\alpha},\overline{\beta})$ of S. Thus by Lemma 3.3, $\overline{f}\circ_{\overline{\beta}}^{\overline{\alpha}}\overline{g}\sqsubseteq\overline{f}\curlywedge_{\overline{\beta}}^{\overline{\alpha}}\overline{g}$. On other hand, let $u\in S$ and $q\in Q$. Then there exist $w,x,y,z\in S$ such that $u=(xuy)(zuw)$. Thus

$$\begin{aligned}
(\overline{f}\circ^{\overline{\alpha}}_{\overline{\beta}}\overline{g})(u,q) &= (\curlyvee_{(i,j)\in F_u}\{\overline{f}(i,q)\curlywedge\overline{g}(j,q)\}\curlywedge\overline{\beta})\curlyvee\overline{\alpha}\\
&= (\curlyvee_{(i,j)\in F_{(xuy)(wuz)}}\{\overline{f}(i,q)\curlywedge\overline{g}(j,q)\}\curlywedge\overline{\beta})\curlyvee\overline{\alpha}\\
&\succeq ((\overline{f}(xuy,q)\curlywedge\overline{g}(wuz,q))\curlywedge\overline{\beta})\curlyvee\overline{\alpha}\\
&= ((\overline{f}(xuy,q)\curlyvee\overline{\alpha})\curlywedge(\overline{g}(wuz,q)\curlyvee\overline{\alpha})\curlywedge\overline{\beta})\curlyvee\overline{\alpha}\\
&\succeq ((\overline{f}(u,q)\curlywedge\overline{\beta})\curlywedge(\overline{g}(u,q)\curlywedge\overline{\beta})\curlywedge\overline{\beta})\curlyvee\overline{\alpha}\\
&= (((\overline{f}(u,q)\curlywedge\overline{g}(u,q))\curlywedge\overline{\beta})\curlywedge\overline{\beta})\curlyvee\overline{\alpha}\\
&= ((\overline{f}(u,q)\curlywedge\overline{g}(u,q))\curlywedge\overline{\beta})\curlyvee\overline{\alpha} = (\overline{f}\curlywedge^{\overline{\alpha}}_{\overline{\beta}}\overline{g})(u,q).
\end{aligned}$$

Hence, $(\overline{f}\circ^{\overline{\alpha}}_{\overline{\beta}}\overline{g})(u,q)\succeq(\overline{f}\curlywedge^{\overline{\alpha}}_{\overline{\beta}}\overline{g})(u,q)$ and so $\overline{f}\curlywedge^{\overline{\alpha}}_{\overline{\beta}}\overline{g}\sqsubseteq\overline{f}\circ^{\overline{\alpha}}_{\overline{\beta}}\overline{g}$. Therefore, $\overline{f}\circ^{\overline{\alpha}}_{\overline{\beta}}\overline{g}=\overline{f}\curlywedge^{\overline{\alpha}}_{\overline{\beta}}\overline{g}$.

(4) $\Rightarrow$ (1) Let K and L be interior ideals of S. Then by Theorem 2.3, $\overline{\lambda}_K$ and $\overline{\lambda}_L$ are IVQF interior ideals with thresholds $(\overline{\alpha},\overline{\beta})$ of S. By supposition and Lemma 3.2, we have

$$(\overline{\lambda}_{KL})_{(\overline{\alpha},\overline{\beta})}(u,q)=((\overline{\lambda}_K)\circ^{\overline{\alpha}}_{\overline{\beta}}(\overline{\lambda}_L))(u,q)=((\overline{\lambda}_K)\curlywedge^{\overline{\alpha}}_{\overline{\beta}}(\overline{\lambda}_L))(u,q)=(\overline{\lambda}_{K\cap L})_{(\overline{\alpha},\overline{\beta})}(uq)=\overline{\beta}.$$

Thus, $u\in KL$. Hence, $KL=K\cap L$. By Lemma 3.4, S is semisimple.

(1) $\Rightarrow$ (6) Let $\overline{f}$ and $\overline{g}$ be an IVQF interior ideal with thresholds $(\overline{\alpha},\overline{\beta})$ and an IVQF ideal with thresholds $(\overline{\alpha},\overline{\beta})$of S respectively. Then $\overline{g}$ is an IVQF interior ideal with thresholds $(\overline{\alpha},\overline{\beta})$ of S. Thus by (4), $\overline{f}\circ^{\overline{\alpha}}_{\overline{\beta}}\overline{g}=\overline{f}\curlywedge^{\overline{\alpha}}_{\overline{\beta}}\overline{g}$.

(6) $\Rightarrow$ (1) Let K,L be an interior ideal and ideal of S respectively. Then by Theorem 2.3, $\overline{\lambda}_K$ and $\overline{\lambda}_L$ is a $\overline{\lambda}_L$ are IVQF interior ideal with thresholds $(\overline{\alpha},\overline{\beta})$ and $\overline{\lambda}_L$ are IVQF ideals with thresholds $(\overline{\alpha},\overline{\beta})$ of S respectively. Then by Theorem 2.4, $\overline{\lambda}_L$ is an IVQF interior ideal with thresholds $(\overline{\alpha},\overline{\beta})$ of S. Similarly from (4) $\Rightarrow$ (1), we have S is semisimple.

So, (1) $\Leftrightarrow$ (3), (1) $\Leftrightarrow$ (5) and (1) $\Leftrightarrow$ (7) are Straightforward. □

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CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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