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CONFORMABLE FRACTIONAL LOMAX PROBABILITY DISTRIBUTION

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Abstract. In this paper, we consider the conformable fractional Lomax distribution. The main functions associated to this new distribution are obtained, including conformable cumulative distribution function and hazard rate function. Further, we derive an exact expression of the r^{th} moment, the mean and the variance of such new distribution. The mode and the quantile function related to this distribution are also obtained. Some entropy measures, namely, Shannon entropy and Renyi entropy are derived. Moreover, we introduce the order statistics of a fractional random variable, the density of the k^{th} order statistic and the joint density of the k^{th} and m^{th} order statistics for the new distribution are obtained.

Key words: fractional probability distribution; Lomax distribution; entropy; order statistics.

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1. INTRODUCTION

Fractional probability distribution is a very recent topic, on which a probability distribution is expanded to a new generalized distribution based on fractional derivatives. Hammad et al. [6], proposed an approach to define a fractional probability distribution using conformable fractional derivative, this derivative has the following definition:

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Definition 1.1. Let $\alpha \in (0,1]$ and $f : E \subseteq [0,\infty) \to \mathbb{R}. For\ x \in E$, let

$$f^{(\alpha)}(x) = \lim_{\epsilon \to 0} \frac{f(x + \epsilon x^{1-\alpha}) - f(x)}{\epsilon}.$$

If the limit exists, then it is called the α-conformable fractional derivative of f at x. For $x = 0$, $f^{(\alpha)}(0) = \lim_{x \to 0^+} f^{(\alpha)}(x)$, if such limit exists.

The conformable derivative, as defined above, satisfies all the classical properties of the usual first derivative. Moreover, $f^{(\alpha)}(x) = f'(x)x^{1-\alpha}$ for all $x > 0$, $\alpha \in (0,1]$, where $f'(x)$ represents the usual first derivative. For more details on conformable fractional derivative one may refer to [8], [1] and [5].

Using a fractional differential equation, Hammad et al. [6] introduced a conformable probability distribution function (CPDF) for some fractional distributions, including fractional chi-square, Rayleigh, gamma, beta and Lomax distributions. Hammad et al. [7] discussed the fractional chi-square distribution and presented some properties of this new distribution. In this paper, we consider the fractional Lomax distribution (FLD) and discuss its main properties as the conformable cumulative distribution function (CCDF), mode, moments, some entropy measures, and the densities of the order statistics.

2. THE MAIN ASSOCIATED FUNCTIONS OF THE FLD

Lomax distribution is a special case of the second type of a Pareto distribution, it was proposed by Lomax [9]. It has been shifted from Pareto distribution so that its support begins at zero. Lomax distribution has been used in economics, insurance, queuing theory and engineering, more details on Lomax distribution can be found in [2] and [3]. Its PDF is given by:

$$f(x) = \frac{\theta}{\lambda}\left(1 + \frac{t}{\lambda}\right)^{-(\theta+1)}, x \geq 0\ , \theta, \lambda > 0,$$

where θ is the shape parameter and λ is the scale parameter. Hammad et al. [6] obtained the CPDF of the FLD and it is given by:

$$f_\alpha(x) = \frac{\theta - \alpha + 1}{\lambda}\left(1 + \frac{x^\alpha}{\lambda}\right)^{-\frac{(\theta+1)}{\alpha}}, x \geq 0, \theta > 0, \lambda > 0, \alpha \in (0,1]. \qquad (2.1)$$

The CCDF of a fractional distribution with CPDF $g_\alpha(x)$ is defined as:

$$G_\alpha(x) = \int_{-\infty}^{x} g_\alpha(t) d^\alpha t = \int_{-\infty}^{x} g_\alpha(t)\, t^{\alpha-1} dt.$$

So, the CCDF of the FLD is:

$$F_\alpha(x) = \int_0^x \frac{\theta - \alpha + 1}{\lambda}\left(1 + \frac{t^\alpha}{\lambda}\right)^{-\frac{(\theta+1)}{\alpha}} d^\alpha t. \qquad (2.2)$$

Precisely:

$$F_\alpha(x) = 1 - \left(1 + \frac{x^\alpha}{\lambda}\right)^{-\frac{(\theta+1)}{\alpha}}, x \geq 0. \qquad (2.3)$$

Consequently, the conformable survival function of the FLD is given by:

$$S_\alpha(x) = \left(1 + \frac{x^\alpha}{\lambda}\right)^{-\frac{(\theta+1)}{\alpha}}, x \geq 0. \qquad (2.4)$$

The conformable hazard function of the FLD is:

$$h_\alpha(x) = \frac{\theta - \alpha + 1}{\lambda + x^\alpha}, x \geq 0. \qquad (2.5)$$

It can be verified, as $\alpha \to 1^-$, that Eq.'s (2.3), (2.4) and (2.5) are reduced to be the CDF, survival function, and hazard function, respectively, of the usual Lomax distribution.

3. The Mode and Quantile of the FLD

In this section, we compute an exact expressions of the mode and the quantile of the FLD.

3.1. The mode.

It is known that the mode of the usual Lomax distribution is zero, see [3]. In the following theorem we generalize this fact for the FLD.

Theorem 3.1. The mode of the FLD is zero.

Proof. The logarithm of $f_\alpha(x)$ is given by:

$$\log(f_\alpha(x)) = \log\left(\frac{\theta-\alpha+1}{\lambda}\right) - \frac{(\theta+1)}{\alpha} log\left(1+\frac{x^\alpha}{\lambda}\right)$$

The conformable fraction derivative of $\log(f_\alpha(x))$is:

$$[\log(f_\alpha(x))]^{(\alpha)} = -\frac{\lambda(\theta+1)}{\lambda+x^\alpha}. \tag{3.1}$$

Using Eq. (3.1), it can be shown that $f_\alpha(x)$ has no critical points. Since

$$[\log(f_\alpha(x))]^{(\alpha)} < 0 \text{ for all } x \geq 0,$$

$f_\alpha(x)$ is decreasing function in x. Therefore, we conclude that $x_0 = 0$ is the mode of the FLD.

3.2. The quantile function.

Let $F_\alpha(x) = p$. Then the quantile function q can be obtained by solving the following equation:

$$1-\left(1+\frac{q^\alpha}{\lambda}\right)^{1-\frac{(\theta+1)}{\alpha}} = p. \tag{3.2}$$

So, q can be expressed as:

$$q = \left(\lambda\left[(1-p)^{\frac{\alpha}{\alpha-(\theta+1)}}-1\right]\right)^{\frac{1}{\alpha}}. \tag{3.3}$$

Consequently, the $median$ of the FLD is given by:

$$median = \left(\lambda\left[2^{\frac{\alpha}{\theta+1-\alpha}}-1\right]\right)^{\frac{1}{\alpha}}. \tag{3.4}$$

4. CONFORMABLE FRACTIONAL MOMENTS

Hammad et al. [7] defined the conformable fractional r^{th} moment of a random variable X with CPDF $g_\alpha(x)$ by:

$$\mu_\alpha(r) = E_\alpha(X^r) = \int_{-\infty}^{\infty} x^r g_\alpha(x) d^\alpha x. \tag{4.1}$$

Accordingly, the conformable fractional r^{th} moment of the FLD is:

$$E_\alpha(X^r) = \int_0^\infty x^r \frac{\theta-\alpha+1}{\lambda}\left(1+\frac{x^\alpha}{\lambda}\right)^{-\frac{(\theta+1)}{\alpha}} d^\alpha x.$$

$$= \frac{\theta-\alpha+1}{\lambda}\int_0^\infty x^{r+\alpha-1}\left(1+\frac{x^\alpha}{\lambda}\right)^{-\frac{(\theta+1)}{\alpha}} dx.$$

Using the substitution $z = \frac{x^\alpha}{\lambda}$ and the fact

$$\int_0^\infty x^{a-1}(1+x)^{-a-b}\, dx = B(a,b)$$

where $B(a,b)$ is the beta function, we get

$$E_\alpha(X^r) = \begin{cases} \lambda^{\frac{r}{\alpha}} \dfrac{\Gamma\left(\frac{r}{\alpha}+1\right)\Gamma\left(\frac{\theta+1-r}{\alpha}-1\right)}{\Gamma\left(\frac{\theta+1}{\alpha}-1\right)}, \theta > \alpha+r-1 \\ undefined, \qquad otherwise \end{cases} \tag{4.2}$$

where $\Gamma(.)$ is the gamma function.

So, the conformable fractional expected value of the FLD is:

$$E_\alpha(X) = \begin{cases} \lambda^{\frac{1}{\alpha}} \dfrac{\Gamma\left(\frac{1}{\alpha}+1\right)\Gamma\left(\frac{\theta}{\alpha}-1\right)}{\Gamma\left(\frac{\theta+1}{\alpha}-1\right)}, \theta > \alpha \\ undefined, \qquad otherwise \end{cases} \tag{4.3}$$

The conformable fractional r^{th} central moment is defined by $E_\alpha([X-\mu_\alpha(1)]^r)$. Consequently, the variance of the FLD is given by:

$$\sigma_\alpha^2 = E_\alpha([X-\mu_\alpha(1)]^2)$$

$$= \lambda^{\frac{2}{\alpha}}\left[\frac{\Gamma\left(\frac{2}{\alpha}+1\right)\Gamma\left(\frac{\theta-1}{\alpha}-1\right)}{\Gamma\left(\frac{\theta+1}{\alpha}-1\right)} - \left(\frac{\Gamma\left(\frac{1}{\alpha}+1\right)\Gamma\left(\frac{\theta}{\alpha}-1\right)}{\Gamma\left(\frac{\theta+1}{\alpha}-1\right)}\right)^2\right], \theta > \alpha+1. \tag{4.4}$$

Similarly, the conformable fractional skewness S_α and kurtosis K_α can be obtained as above by evaluating:

$$S_\alpha = E_\alpha\left(\left[\frac{X-\mu_\alpha(1)}{\sigma_\alpha}\right]^3\right),$$

and

$$K_\alpha = E_\alpha\left(\left[\frac{X-\mu_\alpha(1)}{\sigma_\alpha}\right]^4\right),$$

respectively, where $\sigma_\alpha = \sqrt{\sigma_\alpha^2}$ is the conformable fractional standard deviation of the FLD.

5. Conformable Fractional Entropy

The entropy is a well-known concept from information theory and provides a measure of uncertainty of a probability distribution. In this section, we obtain the Shannon and Renyi entropies for the FLD.

5.1. Shannon entropy.

If X is a continuous random variable with PDF g(x), then the Shannon entropy is defined as:

$$H(x) = -E(log[g(x)]),$$

as provided in [11]. So, the conformable fractional Shannon entropy of the FLD is given by:

$$H_\alpha(x) = -E_\alpha\left[log\left(\frac{\theta-\alpha+1}{\lambda}\right) - \frac{(\theta+1)}{\alpha}log\left(1+\frac{x^\alpha}{\lambda}\right)\right].$$

Therefore,

$$H_\alpha(x) = \frac{(\theta+1)}{\alpha}E_\alpha\left[log\left(1+\frac{x^\alpha}{\lambda}\right)\right] - \log\left(\frac{\theta-\alpha+1}{\lambda}\right). \tag{5.1}$$

Using the substitution $z = 1+\frac{x^\alpha}{\lambda}$ and integration by parts, Eq. (5.1) simplifies to:

$$H_\alpha(x) = \frac{\theta+1}{\theta-\alpha+1} - \log\left(\frac{\theta-\alpha+1}{\lambda}\right). \tag{5.2}$$

Further, as $\alpha \to 1^-$, the limit of $H_\alpha(x)$ is:

$$\lim_{\alpha\to 1^-} H_\alpha(x) = 1+\frac{1}{\theta} - \log\left(\frac{\theta}{\lambda}\right),$$

which represents the Shannon entropy of the usual Lomax distribution, see [3].

5.2. Renyi entropy

The Renyi entropy of a random variable X with PDF g(x) is defined as:

$$R(x) = \frac{1}{1-p}\log(E[g(x)]^{p-1}),$$

as given in [10]. So, the conformable fractional Renyi entropy for the FLD is:

$$R_\alpha(x) = \frac{1}{1-p}\log\left(E_\alpha\left[\frac{\theta-\alpha+1}{\lambda}\left(1+\frac{x^\alpha}{\lambda}\right)^{-\frac{(\theta+1)}{\alpha}}\right]^{p-1}\right),$$

$$= \log\left(\frac{\lambda}{\theta-\alpha+1}\right) + \frac{1}{1-p}\log\left(E_\alpha\left[\left(1+\frac{x^\alpha}{\lambda}\right)^{-\frac{(\theta+1)(p-1)}{\alpha}}\right]\right). \tag{5.3}$$

To proceed, we find $E_\alpha\left[\left(1+\frac{x^\alpha}{\lambda}\right)^{-\frac{(\theta+1)(p-1)}{\alpha}}\right]$ as follows:

$$E_\alpha\left[\left(1+\frac{x^\alpha}{\lambda}\right)^{-\frac{(\theta+1)(p-1)}{\alpha}}\right] = \frac{\theta-\alpha+1}{\lambda}\int_0^\infty \left(1+\frac{x^\alpha}{\lambda}\right)^{-\frac{(\theta+1)(p-1)}{\alpha}}\left(1+\frac{x^\alpha}{\lambda}\right)^{-\frac{(\theta+1)}{\alpha}} d^\alpha x.$$

Using the substitution $z = 1+\frac{x^\alpha}{\lambda}$, we get:

$$E_\alpha\left[\left(1+\frac{x^\alpha}{\lambda}\right)^{-\frac{(\theta+1)(p-1)}{\alpha}}\right] = \frac{\theta-\alpha+1}{p(\theta+1)-\alpha}. \tag{5.4}$$

Therefore, Eq. (5.3) is reduced to:

$$R_\alpha(x) = \log\left(\frac{\lambda}{\theta-\alpha+1}\right) + \frac{1}{1-p}\log\left(\frac{\theta-\alpha+1}{p(\theta+1)-\alpha}\right). \tag{5.5}$$

It can be shown using L'Hopital rule and some algebraic simplifications that as $p \to 1$, the limit of the conformable fractional Renyi entropy is equivalent to the conformable fractional Shannon entropy as expected, see [10].

6. CONFORMABLE FRACTIONAL ORDER STATISTICS

In this section, we introduce the order statistics of fractional distributions and compute the CPDF of the k^{th} order statistic Y_k from FLD. Also, we introduce the conformable fractional joint density of the k^{th} and m^{th} order statistics.

Definition 6.1. Let $X_1, X_2, \dots, X_n$ be a random sample of size n from a fractional distribution with CPDF $f_\alpha(x)$ and CCDF $F_\alpha(x)$. If $Y_1, Y_2, \dots, Y_n$ denotes the order statistics corresponding to this sample, then the CPDF of the k^{th} order statistic $Y_k, 1 \leq k \leq n$ is defined as:

$$g_k(y) = \frac{n!}{(k-1)!\,(n-k)!}\left(F_\alpha(y)\right)^{k-1}\left(1 - F_\alpha(y)\right)^{n-k} f_\alpha(y), -\infty < y < \infty. \qquad (6.1)$$

Therefore, if $X_1, X_2, \dots, X_n$ is a random sample of size n from FLD, it can be shown, that the CPDFs of the order statistics Y_1 and Y_n are given by:

$$g_1(y) = n\left(\frac{\theta-\alpha+1}{\lambda}\right)\left(1+\frac{y^\alpha}{\lambda}\right)^{n\left(1-\frac{\theta+1}{\alpha}\right)-1}, y > 0, \qquad (6.2)$$

and

$$g_n(y) = n\left(\frac{\theta-\alpha+1}{\lambda}\right)\left(1+\frac{y^\alpha}{\lambda}\right)^{-\frac{(\theta+1)}{\alpha}}\left(1-\left(1+\frac{y^\alpha}{\lambda}\right)^{1-\frac{\theta+1}{\alpha}}\right)^{n-1}, y > 0, \quad (6.3)$$

respectively. The following theorem gives the CPDF of the k^{th} order statistic Y_k from FLD.

Theorem 6.2. Let $X_1, X_2, \dots, X_n$ is a random sample of size n from FLD. Then the CPDF of the k^{th} order statistic Y_k is given by:

$$\begin{aligned} g_k(y) &= \frac{n!}{(k-1)!\,(n-k)!}(\theta-\alpha+1) \\ &\times \sum_{i=0}^{n-k}\sum_{j=0}^{k+i-1}\binom{n-k}{i}\binom{k+i-1}{j}(-1)^{i+j}\,\lambda^{\left(\frac{\theta+1}{\alpha}-1\right)(j+1)}(y^\alpha+\lambda)^{-\frac{(\theta+1)(j+1)}{\alpha}}. \end{aligned} \qquad (6.4)$$

Proof. It can be shown that Eq. (6.1) may be expressed as

$$g_k(y) = \frac{n!}{(k-1)!\,(n-k)!} f_\alpha(y)\sum_{i=0}^{n-k}\binom{n-k}{i}(-1)^i\left(F_\alpha(y)\right)^{k+i-1}, -\infty < y < \infty. \qquad (6.5)$$

By substituting Eq..s (2.1) and (2.3) in Eq. (6.5) we get:

$$g_k(y) = \frac{n!}{(k-1)!\,(n-k)!} n \left(\frac{\theta-\alpha+1}{\lambda}\right)\left(1+\frac{y^\alpha}{\lambda}\right)^{-\frac{(\theta+1)}{\alpha}}$$

$$\times \sum_{i=0}^{n-k} \binom{n-k}{i} (-1)^i \left(1-\left(1+\frac{y^\alpha}{\lambda}\right)^{1-\frac{\theta+1}{\alpha}}\right)^{k+i-1}, y > 0.$$

By setting

$$1-\left(1+\frac{y^\alpha}{\lambda}\right)^{1-\frac{\theta+1}{\alpha}} = \lambda^{\frac{\theta+1}{\alpha}-1}\left(\lambda^{1-\frac{\theta+1}{\alpha}} - (\lambda+y^\alpha)^{-\frac{\theta+1}{\alpha}}\right), \tag{6.6}$$

and using the binomial expansion in the right hand side of Eq. (6.6), we conclude the result.

Remark 6.3. In theorem (6.2), taking the limit of $g_k(y)$ as $\alpha \to 1^-$, we obtain the PDF of the k^{th} order statistic of the usual Lomax distribution, see [4].

Definition 6.4. Let $X_1, X_2, \dots, X_n$ be a random sample of size n from any fractional distribution with CPDF $f_\alpha(x)$ and CCDF $F_\alpha(x)$, and let $Y_1, Y_2, \dots, Y_n$ denotes the corresponding order statistics of this sample. The conformable fractional joint CPDF of the order statistics Y_k and Y_m, $1 \le k \le m \le n$ is given by:

$$g_{k,m}(x,y) = C_{k,m}\, f_\alpha(x) f_\alpha(y) \big(F_\alpha(x)\big)^{k-1} \big(F_\alpha(y) - F_\alpha(x)\big)^{m-k-1} \big(1-F_\alpha(y)\big)^{n-m},$$

$$-\infty < x < y < \infty, \tag{6.7}$$

where $C_{k,m} = \frac{n!}{(k-1)!(m-k-1)!(n-m)!}$.

Theorem 6.5. The conformable fractional joint CPDF of the k^{th} and m^{th} order statistics from the FLD is given by:

$$g_{k,m}(x,y) = C_{k,m} (\theta-\alpha+1)^2 (\lambda+x^\alpha)^{-\frac{(\theta+1)}{\alpha}} (\lambda+y^\alpha)^{-\frac{(\theta+1)}{\alpha}}$$

$$\times \sum_{i=0}^{m-k-1} \sum_{j=0}^{n-m} \left\{ \binom{m-k-1}{i} \binom{n-m}{j} (-1)^{i+j} \lambda^{\left(\frac{\theta+1}{\alpha}-1\right)(m+j)} \right.$$

$$\left. \times \left[\lambda^{1-\frac{\theta+1}{\alpha}} - (\lambda+y^\alpha)^{1-\frac{(\theta+1)}{\alpha}}\right]^{m-k-1-i+j} \left[\lambda^{1-\frac{\theta+1}{\alpha}} - (\lambda+x^\alpha)^{1-\frac{(\theta+1)}{\alpha}}\right]^{k+i-1} \right\} \tag{6.8}$$

Proof. It can be shown that Eq. (6.7) can be written as:

$$g_{k,m}(x,y) = C_{k,m}\, f_\alpha(x) f_\alpha(y)$$

$$\times \sum_{i=0}^{m-k}\sum_{j=0}^{n-m}\binom{m-k-1}{i}\binom{n-m}{j}(-1)^{i+j}\left(F_\alpha(y)\right)^{m-k-i+j}\left(F_\alpha(x)\right)^{k+i-1}, -\infty < y < \infty. \quad (6.9)$$

Now, if we substitute Eq.'s (2.1) and (2.3) in Eq. (6.9) and do some algebraic simplifications, the result follows.

Remark 6.6. Taking the limit of $g_{k,m}(x,y)$ as $\alpha \to 1^-$ in Eq. (6.8), we get the joint PDF of the order statistics Y_k and Y_m of the usual Lomax distribution, see [4].

Conflict of Interests

The author(s) declare that there is no conflict of interests.

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