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DIVISOR PRIME GRAPH

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Abstract. Let n be an integer. The divisor prime graph $G_{Dp(n)}$ is a graph whose vertices are divisors of n and two vertices d_i and d_j are adjacent if and only if $\gcd(d_i, d_j) = 1$. In this paper we introduce and investigate the structural properties of divisor prime graph.

Keywords: divisor function graph; prime graph; divisor prime graphs.

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1. INTRODUCTION

Here we consider only finite, simple and undirected graph. Singh and Santhosh [7] introduced the concept of divisor graphs in 2000. A divisor graph G is an ordered pair (V, E) where V is a subset of Z and $uv \in V$ if and only if either u/v or v/u for all $u \neq v$. A graph which is isomorphic to a divisor graph is also called a divisor graph. Furthermore, they revealed that every odd cycle with a length greater than or equal to five is not a divisor graph, whereas all even cycles, complete graphs, and caterpillars are. Chartrand, Muntean, Saenpholphant, and Zhang [1] investigated more attributes. In addition, Le Anh Vinh [4] shown that a divisor graph of order n and size m exists, where m is an integer between 0 and any positive integer n. Yu-ping Tsao[9]

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analyse the vertex-chromatic number, clique number, clique cover number, and independence number of divisor graphs and its complement, where $n = \{i : 1 \leq i \leq n,\ n \in N\}$. K. Kannan, D. Narasimhan and S.Shanmugavelan introduced a new concept called divisor function graph [5] and studied its properties. For an integer n, the graph of divisor function $G_{D(n)}$ is graph with vertex set are set of all divisors d_i of n, and two vertices d_i and d_j are adjacent if either $d_i|d_j$ or $d_j|d_i$. Moreover they analysed the divisor function subgraph, complete divisor function graph and its Eulerian properties.

The concept of prime labeling was introduced by Roger Entringer and was discussed Tout et al. [8]. A graph with n vertices is said to be a prime graph if for any two vertices u and v, $\gcd(u,v) = 1$. Many researchers have studied prime graphs. Fu and Huany [3] proved that the path P_n on n vertices is a prime graph. Deresky et al.[2] proved that the cycle C_n on n vertices is a prime graph. The concept of Euler phi function graph was introduced by S. shanmugavelan [6]. For any natural number n, Euler function graph $G(\phi(n))$ is a simple graph G with vertex set $V(G(\phi(n))) = \{a/\gcd(a,n) = 1 \textit{ and } a < n\}$ and
$E(G(\phi(n))) = \{am/\gcd(a,m) = 1 \textit{ and } a < m \textit{ or } m < a\}$.

In this sequel we combine the ideas of divisor function graphs and prime graphs.

Corollary 1.1 [7] For any positive integer k the divisor function graph $G_D(p^k)$ is complete.

2. Main Results

Definition 2.1

For any positive integer $n \geq 1$ with r divisors $d_1, d_2, \ldots.d_r$, the divisor prime graph $G_{Dp(n)}$ is a graph with vertex set $\{d_1, d_2, \ldots\ldots d_r\}$ such that two vertices d_i and d_j are adjacent if and only if $\gcd(d_i, d_j) = 1$.

Since we consider only simple graph the possible loop at 1 is neglected.

Observations

- In any divisor prime graph the vertex 1 is adjacent to all other vertices.
- The divisor prime graph $G_{Dp(n)}$ is connected for every integer n.
- $E(G_{Dp(n)}) \cap E(G_{D(n)}) = \{(1,d_1),(1,d_2),(1,d_3) \ldots\ldots (1,d_r)\}$
- If $n = p^k$, then $G_{Dp(n)}$ has exactly k edges.

Theorem 2.1. *For any $G_{Dp(n)}$ the maximum degree among the vertices of G, $\Delta(G_{Dp(n)}) = (n_1+1)(n_2+1).....(n_r+1)-1$, where $n_1, n_2, ...n_r$ are the powers of prime factors of n. The minimum degree among the vertices of G, $\delta(G_{Dp(n)}) = 1$. For n is a prime number $\Delta(G_{Dp(n)}) = \delta(G_{Dp(n)}) = 1$.*

Proof. Suppose $n = p_1^{n_1} p_2^{n_2} ......p_r^{n_r}$. By the definition of Divisor prime graph $|V(G_{Dp(n)})| =$ total number of divisors of $n = (n_1+1)(n_2+1).....(n_r+1)$. Since 1 divides all other divisors of n, degree of 1 is $(n_1+1)(n_2+1).....(n_r+1)-1$. Since $\gcd(n,d_i) \neq 1\ \forall i$, degree of n is 1. For n is a prime number $G_{Dp(n)}$ is K_2. Then $\Delta(G_{Dp(n)}) = \delta(G_{Dp(n)}) = 1$. □

Theorem 2.2. *$G_{Dp(n)} - 1$ is a null graph if and only if $n = p^k$.*

Proof. Suppose $n = p^k$. Then $D(n) = \left\{1, p, p^2, ...p^k\right\}$. Then $G_{Dp(n)}$ has only k edges such that $\left\{(1,p),(1,p^2).......(1,p^k)\right\}$. Hence $G_{Dp(n)} - 1$ is a null graph.
Conversly prove that if $G_{Dp(n)} - 1$ is a null graph then $n = p^k$. Here we use the method of contraposition.
Suppose $n = p^k q$.
Case 1:
If q is a number such that $\gcd(p,q) = 1$. Then there exist an edge (p,q) in $G_{Dp(n)}$. Hence $G_{Dp(n)} - 1$ is not a null graph.
Case 2:
If q is a number such that $\gcd(p,q) \neq 1$ and $q \neq p^i$. Then q has a prime factor other than p say q_p. Then $\gcd(p,q_p) = 1$, there exist an edge (p,q_p) in $G_{Dp(n)}$. Hence $G_{Dp(n)} - 1$ is not a null graph. □

Theorem 2.3. *Divisor prime graph is not an Euler graph.*

Proof. A graph G is Eulerian if and only if all of its vertices have even degree. Here $degree(n) = 1$ for all cases of n. Hence Divisor prime graph is not an Euler graph. □

Theorem 2.4. *For any positive integer $n \neq p^k$, the divisor prime graph $G_{Dp(n)}$ has atleast one cycle of length* 3.

Proof. If $n \neq p^k$, then there exist atleast two prime factors of n say p_1 and p_2. Then $\gcd(1, p_1) = \gcd(p_1, p_2) = \gcd(p_2, 1) = 1$. Therefore there exist a cycle $1p_1p_21$ of length 3. □

Theorem 2.5. *If $n = p_1p_2.....p_r$. Then $G_{Dp(n)} - n$ is a complete graph.*

Proof. Let $n = p_1p_2.....p_r$. Then $D(n) = 1, p_1, p_2, ...p_r, n$. Then $V(G_{Dp}(n) - n)$ is $1, p_1, p_2, ...p_r$. Since $p_1, p_2, ...p_r$ are primes each pair of them constitute an edge. Also 1 is connected to every other vertices. Hence $G_{Dp(n)} - n$ is a complete graph with $r+1$ vertices. □

Theorem 2.6. *$G_{Dp(n)}$ is a bipartite graph if and only if $n = p^k$.*

Proof. Let $n = p^k$. Then $d(n) = \left\{1, p, p^2, ...p^k\right\}$. Then 1 is connected to every other vertices. For all remaining vertices

$$\gcd(p^i, p^j) = \begin{cases} p^i & for\, i < j \\ p^j & for\, j < i \end{cases} \tag{1}$$

Hence by the definition of divisor prime graph there is no edge connecting p^i and p^j $\forall i, j$. Then there exist a bi-partition $X = 1$ and $Y = \left\{p^1, p^2, ...p^k\right\}$.

Conversely let $G_{Dp(n)}$ be a bipartite graph with partite sets X and Y. Since 1 is connected to every other vertices $1 \in X$ and all other devisors $d_1, d_2, ...d_r \in Y$. Since $G_{Dp(n)}$ is bipartite, there exist no edge connecting d_i and d_j $\forall\, i, j$. That is $\gcd(d_i, d_j) \neq 1$ $\forall\, i, j$. Then $d_i = p^i$, for some prime number p. If there exist another number c which is a divisor of n and not a power of p with $\gcd(c, p) \neq 1$. Then there exist a prime devisor of c, say $q \in V(G_{Dp(n)})$ and $\gcd(p, q) = 1$. Hence there exist an edge from p to q. Which is a contradiction. Then $n = p^k$. □

Conflict of Interests

The author(s) declare that there is no conflict of interests.

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