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# CHARACTERIZE WEAKLY REGULAR SEMIGROUPS BY USING INTERVAL VALUED Q-FUZZY IDEALS WITH THRESHOLDS $(\overline{\alpha},\overline{\beta})$

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**Abstract.** In this paper, we characterize of weakly regular semigroups by using the properties of these interval-valued Q-fuzzy subsemigroups with thresholds $(\overline{\alpha},\overline{\beta})$ of semigroups.

**Keywords:** interval-valued Q-fuzzy subsemigroups with thresholds $(\overline{\alpha};\overline{\beta})$; weakly regular semigroup.

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## 1. Introduction

The theory rudimentary of interval valued fuzzy sets was due to Zadeh in 1975 [21], as a generalization of the notion of fuzzy sets, where the values of the membership functions are intervals of the numbers instead of the numbers. The notion of interval valued fuzzy sets have wide-ranging applications like medical science [5], image processing [3], decision making method [23], etc. After that time, in 1994, Biswas [4] applied the notion of an interval valued fuzzy set to group theory. The study of an interval valued fuzzy subsemigroup and various interval valued fuzzy ideals in semigroups were initiated by Narayanan and Manikantan in 2006 [14]. In 2013, Sezgin et al. [19] characterized regular, intra-regular, completely regular, weakly regular and quasi-regular semigroups in terms of soft ideals of semigroups. In 2017, Murugads

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et al. [12] studied interval valued Q-fuzzy subsemigroup of ordered semigroup. In 2014, Abdullah et al. [1] recommended definitions of $(\overline{\alpha},\overline{\beta})$-interval valued fuzzy subsemigroups where $\overline{\alpha} \prec \overline{\beta}$ and characterized regular semigroups in terms of $(\overline{\alpha},\overline{\beta})$-interval valued fuzzy subsemigroups. In 2019, Murugads and Arikrishnan [13] gave concept of interval valued Q-fuzzy ideal with thresholds $(\overline{\alpha},\overline{\beta})$ where $\overline{\alpha} \prec \overline{\beta}$ and characterized regular semigroups in terms of interval valued Q-fuzzy ideal with thresholds $(\overline{\alpha},\overline{\beta})$.

In this article, we characterize of weakly regular semigroups by using the properties of these interval-valued Q-fuzzy subsemigroups with thresholds $(\overline{\alpha},\overline{\beta})$ of semigroups.

## 2. Preliminaries

Next we review some basic concepts, necessary to helpuseful in next the part.

A non-empty subset $K$ of a semigroup $S$ is called a *subsemigroup* of $S$ if $K^2 \subseteq K$. A non-empty subset $K$ of a semigroup $S$ is called a *left* (right) ideal of $S$ if $SK \subseteq K$ $(KS \subseteq K)$. An *ideal* $K$ of $S$ is a non-empty subset which is both a left ideal and a right ideal of $S$. A non-empty subset $K$ of $S$ is called a *generalized bi-ideal* of $S$ if $KSK \subseteq K$. A subsemigroup $K$ of a semigroup $S$ is called a *bi-ideal* of $S$ if $KSK \subseteq K$. A subsemigroup $K$ of a semigroup $S$ is called a *interior ideal* of $S$ if $SKS \subseteq K$. A subsemigroup $K$ of a semigroup $S$ is called a *quasi-ideal* of $S$ if $SK \cap KS \subseteq K$.

For any $p_i \in [0,1]$ where $i \in \mathscr{A}$ define

$$\bigvee_{i\in\mathscr{A}} p_i := \sup_{i\in\mathscr{A}}\{p_i\} \quad \text{and} \quad \bigwedge_{i\in\mathscr{A}} p_i := \inf_{i\in\mathscr{A}}\{p_i\}.$$

We see that for any $p,q \in [0,1]$, we have

$$p \vee q = \max\{p,q\} \quad \text{and} \quad p \wedge q = \min\{p,q\}.$$

Let $\mathscr{C}$ be the set of all closed subintervals $[0,1]$, i.e.,

$$\mathscr{C} = \{\overline{p} = [p^-,p^+] \mid 0 \leq p^- \leq p^+ \leq 1\}.$$

We note that $[p,p] = \{p\}$ for all $p \in [0,1]$. For $p = 0$ or $1$ we shall denote $\overline{0} = [0,0] = \{0\}$ and $\overline{1} = [1,1] = \{1\}$.

Let $\overline{p} = [p^-,p^+]$ and $\overline{q} = [q^-,q^+]$ in $\mathscr{C}$. Define the operations "$\preceq$", "$=$", "$\curlywedge$" "$\curlyvee$" as follows:

(1) $\overline{p} \preceq \overline{q}$ if and only if $p^- \leq q^-$ and $p^+ \leq q^+$,

(2) $\overline{p} = \overline{q}$ if and only if $p^- = q^-$ and $p^+ = q^+$,

(3) $\overline{p} \curlywedge \overline{q} = [(p^- \wedge q^-), (p^+ \wedge q^+)]$,

(4) $\overline{p} \curlyvee \overline{q} = [(p^- \vee q^-), (p^+ \vee q^+)]$.

If $\overline{p} \succeq \overline{q}$, we mean $\overline{q} \preceq \overline{p}$.

**Proposition 2.1.** [6] Let $\overline{p}, \overline{q}, \overline{r} \in \mathscr{C}$. Then the following properties are true:

(1) $\overline{p} \curlywedge \overline{p} = \overline{p}$ and $\overline{p} \curlyvee \overline{p} = \overline{p}$,

(2) $\overline{p} \curlywedge \overline{q} = \overline{q} \curlywedge \overline{p}$ and $\overline{p} \curlyvee \overline{q} = \overline{q} \curlyvee \overline{p}$,

(3) $(\overline{p} \curlywedge \overline{q}) \curlywedge \overline{r} = \overline{p} \curlywedge (\overline{q} \curlywedge \overline{r})$ and
$(\overline{p} \curlyvee \overline{q}) \curlyvee \overline{r} = \overline{p} \curlyvee (\overline{q} \curlyvee \overline{r})$,

(4) $(\overline{p} \curlywedge \overline{q}) \curlyvee \overline{r} = (\overline{p} \curlyvee \overline{r}) \curlywedge (\overline{q} \curlyvee \overline{r})$ and
$(\overline{p} \curlyvee \overline{q}) \curlywedge \overline{r} = (\overline{p} \curlywedge \overline{r}) \curlyvee (\overline{q} \curlywedge \overline{r})$,

(5) If $\overline{p} \preceq \overline{q}$, then $\overline{p} \curlywedge \overline{r} \preceq \overline{q} \curlywedge \overline{r}$ and $\overline{p} \curlyvee \overline{r} \preceq \overline{q} \curlyvee \overline{r}$.

For each interval $\overline{p}_i = [p_i^-, p_i^+] \in \mathscr{C}$, $i \in \mathscr{A}$ where $\mathscr{A}$ is an index set, we define

$$\underset{i\in\mathscr{A}}{\curlywedge} \overline{p}_i = [\underset{i\in\mathscr{A}}{\wedge} p_i^-, \underset{i\in\mathscr{A}}{\wedge} p_i^+] \quad \text{and} \quad \underset{i\in\mathscr{A}}{\curlyvee} \overline{p}_i = [\underset{i\in\mathscr{A}}{\vee} p_i^-, \underset{i\in\mathscr{A}}{\vee} p_i^+].$$

A fuzzy subset (fuzzy set) of a set $T$ is a function $f : T \rightarrow [0,1]$.

**Definition 2.1.** Let $S$ be a semigroup and $Q$ be a non-empty set. A Q-fuzzy subset (Q-fuzzy set) of a set $T$ is a function $f : S \times Q \rightarrow [0,1]$

**Definition 2.2.** [17] Let $T$ be a non-empty set. An interval valued fuzzy subset (shortly, IVF subset) of $T$ is a function $\overline{f} : T \rightarrow \mathscr{C}$

**Definition 2.3.** [12] Let $S$ be a semigroup and $Q$ be a non-empty set. An interval valued Q-fuzzy subset (shortly, IVQF subset) of $T$ is a function $\overline{f} : S \times Q \rightarrow \mathscr{C}$

**Definition 2.4.** [12] Let $K$ be a non-empty subset of a semigroup $S$ and $Q$ be a non-empty set . An interval valued characteristic function $\overline{\lambda}_K$ of $K$ is defined to be a function $\overline{\lambda}_K : S \times Q \rightarrow \mathscr{C}$

by

$$\overline{\lambda}_K(u,q) = \begin{cases} \overline{1} & \text{if} \quad u \in K \\ \overline{0} & \text{if} \quad u \notin K \end{cases}$$

for all $u \in T$.

For two IVQF subsets $\overline{f}$ and $\overline{g}$ of a semigroups $S$, define

(1) $\overline{f} \sqsubseteq \overline{g} \Leftrightarrow \overline{f}(u,q) \preceq \overline{g}(u,q)$ for all $u \in S$ and $q \in Q$,

(2) $\overline{f} = \overline{g} \Leftrightarrow \overline{f} \sqsubseteq \overline{g}$ and $\overline{g} \sqsubseteq \overline{f}$,

(3) $(\overline{f} \sqcap \overline{g})(u,q) = \overline{f}(u,q) \curlywedge \overline{g}(u,q)$ for all $u \in S$ and $q \in Q$.

For two IVQF subsets $\overline{f}$ and $\overline{g}$ of a semigroup $S$. Then the product $\overline{f} \circ \overline{g}$ is defined as follows for all $u \in S$ and $q \in Q$,

$$(\overline{f} \circ \overline{g})(u,q) = \begin{cases} \underset{(y,z)\in F_u}{\curlyvee} \{\overline{f}(y,q) \curlywedge \overline{g}(z,q)\} & \text{if} \quad F_u \neq \emptyset, \\ \overline{0} & \text{if} \quad F_u = \emptyset, \end{cases}$$

where $F_u := \{(y,z) \in S \times S \mid u = yz\}$ [12].

Next, we shall give definitions of various types of IVQF subsemigroup of a semigroups.

**Definition 2.5.** [13] An IVF subset $\overline{f}$ of a semigroup $S$ is said to be

(1) an *IVQF subsemigroup* of $S$ if $\overline{f}(uv,q) \succeq \overline{f}(u,q) \curlywedge \overline{f}(v,q)$ for all $u,v \in S$ and $q \in Q$,

(2) an *IVQF left (right) ideal* of $S$ if $\overline{f}(uv,q) \succeq \overline{f}(v,q)$ $(\overline{f}(uv,q) \succeq \overline{f}(u,q))$ for all $u,v \in S$ and $q \in Q$. An *IVQF ideal* of $S$ if it is both an IVQF left ideal and an IVQF right ideal of $S$,

(3) an *IVQF generalized bi-ideal* of $S$ if $\overline{f}(uvw,q) \succeq \overline{f}(u,q) \curlywedge \overline{f}(w,q)$ for all $u,v,w \in S$ and $q \in Q$,

(4) an *IVQF bi-ideal* of $S$ if $\overline{f}$ is an IVQF subsemigroup of $S$ and $\overline{f}(uvw,q) \succeq \overline{f}(u,q) \curlywedge \overline{f}(w,q)$ for all $u,v,w \in S$ and $q \in Q$,

(5) an *IVQF interior ideal* of $S$ if if $\overline{f}$ is an IVQF subsemigroup of $S$ and $\overline{f}(uav,q) \succeq \overline{f}(a,q)$ for all $a,u,v \in S$ and $q \in Q$,

(6) an *IVQF quasi-ideal* of $S$ if $\overline{f}(u,q) \succeq (\overline{\mathscr{S}} \circ \overline{f})(u,q) \curlywedge (\overline{f} \circ \overline{\mathscr{S}})(u,q)$, for all $u \in S$ and $q \in Q$ where $\overline{S}$ is an IVQF subset of $S$ mapping every element of $S$ on $\overline{1}$.

The thought of an IVQF subsemigroup with thresholds $(\overline{\alpha},\overline{\beta})$ where $\overline{\alpha} \prec \overline{\beta}$ as follows:

**Definition 2.6.** [13] An IVF subset $\overline{f}$ of a semigroup $S$ and $\overline{\alpha} \prec \overline{\beta}$ and $\overline{\alpha}, \overline{\beta} \in \mathscr{C}$ is said to be

(1) an *IVQF subsemigroup with thresholds* $(\overline{\alpha}, \overline{\beta})$ of $S$ if $\overline{f}(uv,q) \curlyvee \overline{\alpha} \succeq \overline{f}(u,q) \curlywedge \overline{f}(v,q) \curlywedge \overline{\beta}$ for all $u, v \in S$ and $q \in Q$,

(2) an *IVQF left (right) ideal with thresholds* $(\overline{\alpha}, \overline{\beta})$ of $S$ if $\overline{f}(uv,q) \curlyvee \overline{\alpha} \succeq \overline{f}(v,q) \curlywedge \overline{\beta}$ $(\overline{f}(uv,q) \curlyvee \overline{\alpha} \succeq \overline{f}(u,q) \curlywedge \overline{\beta})$ for all $u, v \in S$ and $q \in Q$. An *IVQF ideal with thresholds* $(\overline{\alpha}, \overline{\beta})$ of $S$ if it is both an IVF left ideal and an IVF right ideal of $S$,

(3) an *IVQF generalized bi-ideal with thresholds* $(\overline{\alpha}, \overline{\beta})$ of $S$ if $\overline{f}(uvw,q) \curlyvee \overline{\alpha} \succeq \overline{f}(u,q) \curlywedge \overline{f}(w,q) \curlywedge \overline{\beta}$ for all $u, v, w \in S$ and $q \in Q$,

(4) an *IVQF bi-ideal with thresholds* $(\overline{\alpha}, \overline{\beta})$ of $S$ if $\overline{f}$ is an IVQF subsemigroup with thresholds $(\overline{\alpha}, \overline{\beta})$ of $S$ and $\overline{f}(uvw,q) \curlyvee \overline{\alpha} \succeq \overline{f}(u,q) \curlywedge \overline{f}(w,q) \curlywedge \overline{\beta}$ for all $u, v, w \in S$ and $q \in Q$,

(5) an *IVQF interior ideal with thresholds* $(\overline{\alpha}, \overline{\beta})$ of $S$ if $\overline{f}$ is an IVQF subsemigroup with thresholds $(\overline{\alpha}, \overline{\beta})$ of $S$ and $\overline{f}(uav,q) \curlyvee \overline{\alpha} \succeq \overline{f}(a,q) \curlywedge \overline{\beta}$ for all $a, u, v \in S$ and $q \in Q$,

(6) an *IVQF quasi-ideal with thresholds* $(\overline{\alpha}, \overline{\beta})$ of $S$ if $\overline{f}(u,q) \curlyvee \overline{\alpha} \succeq (\overline{\mathscr{S}} \circ \overline{f})(u,q) \curlywedge (\overline{f} \circ \overline{\mathscr{S}})(u,q) \curlywedge \overline{\beta}$, for all $u \in S$ and $q \in Q$.

**Remark 2.2.** [13] It is clear to see that every IVQF bi-ideal with thresholds $(\overline{\alpha}, \overline{\beta})$ is an IVQF generalized bi-ideal with thresholds $(\overline{\alpha}, \overline{\beta})$ of $S$, every IVQF ideal with thresholds $(\overline{\alpha}, \overline{\beta})$ is an IVQF interior ideal with thresholds $(\overline{\alpha}, \overline{\beta})$ of $S$ and IVQF quasi-ideal with thresholds $(\overline{\alpha}, \overline{\beta})$ is an IVQF bi-ideal with thresholds $(\overline{\alpha}, \overline{\beta})$ of $S$.

In this ensuing theorem is present relationship between types ideals of a semigroup $S$ and the interval valued characteristic function.

**Theorem 2.3.** [13] If $K$ is a left ideal (right ideal generalized bi-ideal, bi-ideal, interior ideal, quasi-ideal) of $S$, then characteristic function $\overline{\chi}$ is an IVQF left ideal (right ideal, generalized bi-ideal, bi-ideal, interior ideal, quasi-ideal) with thresholds $(\overline{\alpha}, \overline{\beta})$ of $S$ for all $\overline{\alpha} \prec \overline{\beta}$ and $\overline{\alpha}, \overline{\beta} \in \mathscr{C}$.

## 3. Characterizations of Weakly Regular Semigroup by Their Interval Valued Q-Fuzzy Ideals With Thresholds $(\overline{\alpha},\overline{\beta})$

In this part, we review symbols of IVQF ideals with thresholds $(\overline{\alpha},\overline{\beta})$ for use characterizes a semigroup in terms IVQF ideals with thresholds $(\overline{\alpha},\overline{\beta})$ of semigroup. And we characterized a wearkly reugular semigroup in terms IVQF ideals with thresholds $(\overline{\alpha},\overline{\beta})$ of semigroup.

In 2019, [13] Murugads and Arikrishnan propose symbols of IVQF ideals with thresholds $(\overline{\alpha},\overline{\beta})$ for use characterizes a semigroup in terms IVQF ideals with thresholds $(\overline{\alpha},\overline{\beta})$ of semigroup.

For any IVQF subset $\overline{f}$ of a semigroup $S$ with $\overline{\alpha} \prec \overline{\beta}$ and $\overline{\alpha},\overline{\beta} \in \mathscr{C}$., define

$$\overline{f}_{(\overline{\alpha},\overline{\beta})}(u,q) = (\overline{f}(u,q) \curlywedge \overline{\alpha}) \curlyvee \overline{\beta}$$

for all $u \in S$ and $q \in Q$.

For any IVQF subsets $\overline{f}$ and $\overline{g}$ of a semigroup $S$ with $\overline{\alpha} \prec \overline{\beta}$ and $\overline{\alpha},\overline{\beta} \in \mathscr{C}$, define the operation "$\curlywedge_{\overline{\beta}}^{\overline{\alpha}}$" as follows:

$$(\overline{f} \curlywedge_{\overline{\beta}}^{\overline{\alpha}} \overline{g})(u,q) = (\overline{f}(u,q) \curlywedge \overline{g}(u,q) \curlywedge \overline{\alpha}) \curlyvee \overline{\beta}$$

for all $u \in S$ and $q \in Q$. And define the product $\overline{f} \circ_{\overline{\beta}}^{\overline{\alpha}} \overline{g}$ as follows: for all $u \in S$ and $q \in Q$,

$$(\overline{f} \circ_{\overline{\beta}}^{\overline{\alpha}} \overline{g})(u,q) = ((\overline{f} \circ \overline{g})(u,q) \curlywedge \overline{\alpha}) \curlyvee \overline{\beta}$$

where

$$(\overline{f} \circ \overline{g})(u,q) = \begin{cases} \curlyvee_{(x,y) \in F_u} \{\overline{f}(x,q) \curlywedge \overline{g}(y,q)\} & \text{if} \quad F_u \neq \emptyset, \\ \overline{0} & \text{if} \quad F_u = \emptyset, \end{cases}$$

where $F_u := \{(x,y) \in S \times S \mid u = xy\}$.

**Remark 3.1.** Since $\overline{\chi}$ is an interval valued characteristic, we have

$$(\overline{\chi})_{(\overline{\alpha},\overline{\beta})}(u,q) := \begin{cases} \overline{\beta} & \text{if} \quad u \in K, \\ \overline{\alpha} & \text{if} \quad u \notin K. \end{cases}$$

**Lemma 3.2.** [13] Let $K$ and $L$ be non-empty subsets of a semigroup $S$ with $\overline{\alpha} \prec \overline{\beta}$ and $\overline{\alpha},\overline{\beta} \in \mathscr{C}$. Then the following assertions hold:

(1) $(\overline{\chi}_K) \curlywedge_{\overline{\beta}}^{\overline{\alpha}} (\overline{\chi}_L) = (\overline{\chi}_{K\cap L})_{(\overline{\alpha},\overline{\beta})}$.

(2) $(\overline{\chi}_K) \circ_{\overline{\beta}}^{\overline{\alpha}} (\overline{\chi}_L) = (\overline{\chi}_{KL})_{(\overline{\alpha},\overline{\beta})}$.

On the basis of Lemma 3.3, we can prove Theorem 3.5.

**Lemma 3.3.** [13] Let $S$ be a semigroup. If $\overline{f}$ is a $(\overline{\alpha},\overline{\beta})$-IVQF right ideal and $\overline{g}$ is a $(\overline{\alpha},\overline{\beta})$-IVQF left ideal of $S$, then $\overline{f} \circ_{\overline{\beta}}^{\overline{\alpha}} \overline{g} \sqsubseteq \overline{f} \curlywedge_{\overline{\beta}}^{\overline{\alpha}} \overline{g}$.

The following definition and lemma are tools in proving Theorem 3.5.

**Definition 3.1.** [20] A semigroup $S$ is called *weakly regular* if for every
$u \in S, u \in (uS)^2$.

**Lemma 3.4.** [20] A monoid $S$ is weakly regular if and only if $I \cap J = IJ$ for every right ideal $I$ and every ideal $J$ of $S$.

Now we characterize weakly regular semigroups in terms of generalized IVQF ideals.

**Theorem 3.5.** A monoid $S$ is weakly regular if and only if $\overline{f} \curlywedge_{\overline{\beta}}^{\overline{\alpha}} \overline{g} = \overline{f} \circ_{\overline{\beta}}^{\overline{\alpha}} \overline{g}$ for every IVQF right ideal $\overline{f}$ and every IVQF ideal $\overline{g}$ with thresholds $(\overline{\alpha},\overline{\beta})$ of $S$.

*Proof.* Assume that $\overline{f}$ is an IVQF right ideal with thresholds $(\overline{\alpha},\overline{\beta})$ and $\overline{g}$ is an IVQF ideal with thresholds $(\overline{\alpha},\overline{\beta})$ of $S$. Let $u \in S$ and $q \in Q$. Since $S$ is weakly regular, there exist $p,r \in S$ such that $u = upur$. Thus,

$$\begin{aligned}
(\overline{f} \circ_{\overline{\beta}}^{\overline{\alpha}} \overline{g})(u,q) &= ((\overline{f} \circ \overline{g})(u,q) \curlywedge \overline{\beta}) \curlyvee \overline{\alpha} = (\curlyvee_{(a,b)\in F_u} \{\overline{f}(a,q) \curlywedge \overline{g}(b,q)\} \curlywedge \overline{\beta}) \curlyvee \overline{\alpha} \\
&= (\curlyvee_{(a,b)\in F_{upur}} \{\overline{f}(a,q) \curlywedge \overline{g}(b,q)\} \curlywedge \overline{\beta}) \curlyvee \overline{\alpha} \\
&\succeq ((\overline{f}(up,q) \curlywedge \overline{g}(ur,q)) \curlywedge \overline{\beta}) \curlyvee \overline{\alpha} \\
&= ((\overline{f}(up,q) \curlyvee \overline{\alpha}) \curlywedge (\overline{g}(ur,q) \curlyvee \overline{\alpha})) \curlywedge (\overline{\beta} \curlyvee \overline{\alpha}) \\
&= ((\overline{f}(up,q) \curlyvee \overline{\alpha} \curlyvee \overline{\alpha}) \curlywedge (\overline{g}(ur,q) \curlyvee \overline{\alpha} \curlyvee \overline{\alpha})) \curlywedge (\overline{\beta} \curlyvee \overline{\alpha}) \\
&= ((\overline{f}(up,q) \curlyvee \overline{\alpha}) \curlywedge (\overline{g}(ur,q) \curlyvee \overline{\alpha}) \curlywedge \overline{\beta}) \curlyvee \overline{\alpha} \\
&\succeq ((\overline{f}(u,q) \curlywedge \overline{\beta}) \curlywedge (\overline{g}(u,q) \curlywedge \overline{\beta}) \curlywedge \overline{\beta}) \curlyvee \overline{\alpha} \\
&= ((\overline{f}(u,q) \curlywedge \overline{g}(u) \curlywedge \overline{\beta}) \curlywedge \overline{\beta}) \curlyvee \overline{\alpha} \\
&= (\overline{f}(u,q) \curlywedge \overline{g}(u,q) \curlywedge \overline{\beta}) \curlyvee \overline{\alpha} = (\overline{f} \curlywedge_{\overline{\beta}}^{\overline{\alpha}} \overline{g})(u,q).
\end{aligned}$$

Hence, $(\overline{f} \curlywedge^{\overline{\alpha}}_{\overline{\beta}} \overline{g})(u,q) \preceq (\overline{f} \circ^{\overline{\alpha}}_{\overline{\beta}} \overline{g})(u,q)$. Therefore, $\overline{f} \curlywedge^{\overline{\alpha}}_{\overline{\beta}} \overline{g} \sqsubseteq \overline{f} \circ^{\overline{\alpha}}_{\overline{\beta}} \overline{g}$.

On the other hand, since $\overline{g}$ is an IVQF ideal with thresholds $(\overline{\alpha}, \overline{\beta})$ of $S$ we see that $\overline{g}$ is an IVQF left ideal with thresholds $(\overline{\alpha}, \overline{\beta})$ of $S$. Thus by Lemma 3.3, $\overline{f} \circ^{\overline{\alpha}}_{\overline{\beta}} \overline{g} \sqsubseteq \overline{f} \curlywedge^{\overline{\alpha}}_{\overline{\beta}} \overline{g}$. Hence, $\overline{f} \curlywedge^{\overline{\alpha}}_{\overline{\beta}} \overline{g} = \overline{f} \circ^{\overline{\alpha}}_{\overline{\beta}} \overline{g}$.

Conversely, let $I$ be a right ideal and $J$ be an ideal of $S$. Then, by Theorem 2.3, $\overline{\chi}_I$ is an IVQF right ideal and $\overline{\chi}_J$ is an IVQF ideal with thresholds $(\overline{\alpha}, \overline{\beta})$ of $S$.

By supposition and Lemma 3.2, we have

$$\overline{\beta} = (\overline{\chi}_{I \cap J})_{(\overline{\alpha},\overline{\beta})}(u,q) = ((\overline{\chi}_I) \curlywedge^{\overline{\alpha}}_{\overline{\beta}} (\overline{\chi}_J))(u,q) = ((\overline{\chi}_I) \circ^{\overline{\alpha}}_{\overline{\beta}} (\overline{\chi}_J))(u,q) = (\overline{\chi}_{IJ})_{(\overline{\alpha},\overline{\beta})}(u,q).$$

Thus we have $u \in IJ$. Hence, $I \cap J = IJ$. Therefore, by Lemma 3.4, $S$ is weakly regular. □

The following lemma will be used in the proof of in Theorem 3.7.

**Lemma 3.6.** [19] Let $S$ be a monoid. Then the following statements are equivalent:

(1) $S$ is weakly regular.

(2) $Q \cap I \subseteq QI$ for every quasi-ideal $Q$ and every ideal $I$ of $S$.

**Theorem 3.7.** For a monoid $S$, the following statements are equivalent.

(1) $S$ is weakly regular.

(2) $\overline{f} \curlywedge^{\overline{\alpha}}_{\overline{\beta}} \overline{g} \sqsubseteq \overline{f} \circ^{\overline{\alpha}}_{\overline{\beta}} \overline{g}$ for every IVQF quasi-ideal $\overline{f}$ and every IVQF ideal $\overline{g}$ with thresholds $(\overline{\alpha}, \overline{\beta})$ of $S$.

(3) $\overline{f} \curlywedge^{\overline{\alpha}}_{\overline{\beta}} \overline{g} \sqsubseteq \overline{f} \circ^{\overline{\alpha}}_{\overline{\beta}} \overline{g}$ for every IVQF bi-ideal $\overline{f}$ and every IVQF ideal $\overline{g}$ with thresholds $(\overline{\alpha}, \overline{\beta})$ of $S$.

(4) $\overline{f} \curlywedge^{\overline{\alpha}}_{\overline{\beta}} \overline{g} \sqsubseteq \overline{f} \circ^{\overline{\alpha}}_{\overline{\beta}} \overline{g}$ for every IVQF generalized bi-ideal $\overline{f}$ and every IVQF interior ideal $\overline{g}$ with thresholds $(\overline{\alpha}, \overline{\beta})$ of $S$.

*Proof.* (1) $\Rightarrow$ (4) Assume that $\overline{f}$ is an IVQF generalized bi-ideal with thresholds $(\overline{\alpha}, \overline{\beta})$ and $\overline{g}$ is an IVQF interior ideal with thresholds $(\overline{\alpha}, \overline{\beta})$ of $S$. Let $u \in S$ and $q \in Q$. Then there exist

$p,r \in S$ such that $u = upur$. Thus,

$$\begin{aligned}
(\overline{f} \circ^{\overline{\alpha}}_{\overline{\beta}} \overline{g})(u,q) &= ((\overline{f} \circ \overline{g})(u,q) \curlywedge \overline{\beta}) \curlyvee \overline{\alpha} = (\curlyvee_{(i,j)\in F_u} \{\overline{f}(i,q) \curlywedge \overline{g}(j,q)\} \curlywedge \overline{\beta}) \curlyvee \overline{\alpha} \\
&= (\curlyvee_{(i,j)\in F_{upur}} \{\overline{f}(i) \curlywedge \overline{g}(j)\} \curlywedge \overline{\beta}) \curlyvee \overline{\alpha}) \\
&\succeq ((\overline{f}(u,q) \curlywedge \overline{g}(pur,q)) \curlywedge \overline{\beta}) \curlyvee \overline{\alpha} = (\overline{f}(u) \curlywedge (\overline{g}(pur) \curlyvee \overline{\alpha}) \curlywedge \overline{\beta}) \curlyvee \overline{\alpha} \\
&\succeq (\overline{f}(u,q) \curlywedge (\overline{g}(u,q) \curlywedge \overline{\beta}) \curlywedge \overline{\beta}) \curlyvee \overline{\alpha} = (\overline{f}(u,q) \curlywedge \overline{g}(u,q) \curlywedge \overline{\beta}) \curlyvee \overline{\alpha} \\
&= (\overline{f} \curlywedge^{\overline{\alpha}}_{\overline{\beta}} \overline{g})(u,q).
\end{aligned}$$

Hence, $(\overline{f} \curlywedge^{\overline{\alpha}}_{\overline{\beta}} \overline{g})(u,q) \preceq (\overline{f} \circ^{\overline{\alpha}}_{\overline{\beta}} \overline{g})(u,q)$. Therefore, $\overline{f} \curlywedge^{\overline{\alpha}}_{\overline{\beta}} \overline{g} \sqsubseteq \overline{f} \circ^{\overline{\alpha}}_{\overline{\beta}} \overline{g}$.

(4) $\Rightarrow$ (3) $\Rightarrow$ (2) This is obvious because every IVQF bi-ideal with thresholds $(\overline{\alpha},\overline{\beta})$ is an IVQF generalized bi-ideal with thresholds $(\overline{\alpha},\overline{\beta})$ of $S$, every IVQF ideal is an IVQF interior ideal with thresholds $(\overline{\alpha},\overline{\beta})$ of $S$ and every IVQF quasi-ideal with thresholds $(\overline{\alpha},\overline{\beta})$ is an IVQF bi-ideal with thresholds $(\overline{\alpha},\overline{\beta})$ of $S$.

(2) $\Rightarrow$ (1) Let $Q$ be a quasi-ideal and $I$ be an ideal of $S$. Then, by Theorem 2.3, $\overline{\chi}_Q$ is an IVQF quasi-ideal with thresholds $(\overline{\alpha},\overline{\beta})$ and $\overline{\chi}_I$ is an IVQF ideal with thresholds $(\overline{\alpha},\overline{\beta})$ of $S$. By supposition and Lemma 3.2, we have

$$\overline{\beta} = (\overline{\chi}_{Q\cap I})_{(\overline{\alpha},\overline{\beta})}(u,q) = ((\overline{\chi}_Q) \curlywedge^{\overline{\alpha}}_{\overline{\beta}} (\overline{\chi}_I))(u,q) \sqsubseteq ((\overline{\chi}_Q) \circ^{\overline{\alpha}}_{\overline{\beta}} (\overline{\chi}_I))(u,q) = (\overline{\chi}_{QI})_{(\overline{\alpha},\overline{\beta})}(u,q).$$

Thus, $u \in QI$. Hence, $Q \cap I \sqsubseteq QI$. Therefore, by Lemma 3.6, $S$ is weakly regular. □

The following lemma will be used in the proof of in Theorem3.9.

**Lemma 3.8.** [19] Let $S$ be a monoid. Then the following statements are equivalent:

(1) $S$ is weakly regular.

(2) $Q \cap I \cap R \subseteq QIR$ for every quasi-ideal $Q$, every ideal $I$ and every right ideal $R$ of $S$.

**Theorem 3.9.** Let $S$ be a monoid. Then the following statements are equivalent:

(1) $S$ is weakly regular.

(2) $\overline{f} \curlywedge^{\overline{\alpha}}_{\overline{\beta}} \overline{g} \curlywedge^{\overline{\alpha}}_{\overline{\beta}} \overline{h} \sqsubseteq \overline{f} \circ^{\overline{\alpha}}_{\overline{\beta}} \overline{g} \circ^{\overline{\alpha}}_{\overline{\beta}} \overline{h}$, for every IVQF quasi-ideal $\overline{f}$, every IVQF ideal $\overline{g}$ and every IVQF right ideal $\overline{h}$ with thresholds $(\overline{\alpha},\overline{\beta})$ of $S$.

(3) $\overline{f} \curlywedge^{\overline{\alpha}}_{\overline{\beta}} \overline{g} \curlywedge^{\overline{\alpha}}_{\overline{\beta}} \overline{h} \sqsubseteq \overline{f} \circ^{\overline{\alpha}}_{\overline{\beta}} \overline{g} \circ^{\overline{\alpha}}_{\overline{\beta}} \overline{h}$, for every IVQF bi-ideal $\overline{f}$, every IVQF ideal $\overline{g}$ and every IVQF right ideal $\overline{h}$ with thresholds $(\overline{\alpha},\overline{\beta})$ of $S$.

(4) $\overline{f} \curlywedge^{\overline{\alpha}}_{\overline{\beta}} \overline{g} \curlywedge^{\overline{\alpha}}_{\overline{\beta}} \overline{h} \sqsubseteq \overline{f} \circ^{\overline{\alpha}}_{\overline{\beta}} \overline{g} \circ^{\overline{\alpha}}_{\overline{\beta}} \overline{h}$, for every IVQF generalized bi-ideal $\overline{f}$, every IVQF interior ideal $\overline{g}$ and every IVQF right ideal $\overline{h}$ with thresholds $(\overline{\alpha}, \overline{\beta})$ of $S$.

*Proof.* $(1) \Rightarrow (4)$ Assume that $\overline{f}$ is an IVQF generalized bi-ideal with thresholds $(\overline{\alpha}, \overline{\beta})$, $\overline{g}$ is an IVQF interior ideal with thresholds $(\overline{\alpha}, \overline{\beta})$ and $\overline{h}$ is an IVQF right ideal with thresholds $(\overline{\alpha}, \overline{\beta})$ of $S$. Let $u \in S$ and $q \in Q$. Then there exist $p, r \in S$ such that $u = upur$. Thus,

$$
\begin{aligned}
(\overline{f} \circ^{\overline{\alpha}}_{\overline{\beta}} \overline{g} \circ^{\overline{\alpha}}_{\overline{\beta}} \overline{h})(u,q) &= (\overline{f} \circ (\overline{g} \circ^{\overline{\alpha}}_{\overline{\beta}} \overline{h})(u,q) \curlywedge \overline{\beta}) \curlyvee \overline{\alpha} \\
&= (\underset{(i,j)\in F_u}{\curlyvee} \{\overline{f}(i,q) \curlywedge (\overline{g} \circ^{\overline{\alpha}}_{\overline{\beta}} \overline{h})(j,q)\} \curlywedge \overline{\beta}) \curlyvee \overline{\alpha} \\
&= ((\underset{(i,j)\in F_{upur}}{\curlyvee} \{\overline{f}(i,q) \curlywedge (\overline{g} \circ^{\overline{\alpha}}_{\overline{\beta}} \overline{h})(j,q)\} \curlywedge \overline{\beta}) \curlyvee \overline{\alpha} \\
&\succeq (\overline{f}(u,q) \curlywedge (\overline{g} \circ^{\overline{\alpha}}_{\overline{\beta}} \overline{h})(pur,q) \curlywedge \overline{\beta}) \curlyvee \overline{\alpha} \\
&= (\overline{f}(u,q) \curlywedge ((\underset{(y,z)\in F_{pur}}{\curlyvee} \{\overline{g}(y,q) \curlywedge \overline{h}(z,q)\} \curlywedge \overline{\beta}) \curlyvee \overline{\alpha}) \curlywedge \overline{\beta}) \curlyvee \overline{\alpha} \\
&= (\overline{f}(u,q) \curlywedge ((\underset{(y,z)\in F_{pupur^2}}{\curlyvee} \{\overline{g}(y,q) \curlywedge \overline{h}(z,q)\} \curlywedge \overline{\beta}) \curlyvee \overline{\alpha}) \curlywedge \overline{\beta}) \curlyvee \overline{\alpha} \\
&\succeq (\overline{f}(u,q) \curlywedge (((\overline{g}(pup,q) \curlywedge \overline{h}(ur^2,q)) \curlywedge \overline{\beta}) \curlyvee \overline{\alpha}) \curlywedge \overline{\beta}) \curlyvee \overline{\alpha} \\
&= (\overline{f}(u,q) \curlywedge (((\overline{g}(pup,q) \curlyvee \overline{\alpha}) \curlywedge (\overline{h}(ur^2,q) \curlyvee \overline{\alpha}) \curlywedge \overline{\beta}) \curlyvee \overline{\alpha}) \curlywedge \overline{\beta}) \curlyvee \overline{\alpha} \\
&\succeq (\overline{f}(u,q) \curlywedge (((\overline{g}(u,q) \curlywedge \overline{\beta}) \curlywedge (\overline{h}(u,q) \curlywedge \overline{\beta}) \curlywedge \overline{\beta}) \curlyvee \overline{\alpha}) \curlywedge \overline{\beta}) \curlyvee \overline{\alpha} \\
&= (\overline{f}(u,q) \curlywedge ((\overline{g}(u,q) \curlywedge \overline{h}(u,q) \curlywedge \overline{\beta}) \curlywedge \overline{\beta}) \curlyvee \overline{\alpha}) \curlywedge \overline{\beta}) \curlyvee \overline{\alpha} \\
&= (\overline{f}(u,q) \curlywedge (\overline{g}(u,q) \curlywedge \overline{h}(u,q) \curlywedge \overline{\beta} \curlyvee \overline{\alpha}) \curlywedge \overline{\beta}) \curlyvee \overline{\alpha} \\
&= (\overline{f}(u,q) \curlywedge (\overline{g} \curlywedge^{\overline{\alpha}}_{\overline{\beta}} \overline{h})(u,q) \curlywedge \overline{\beta}) \curlyvee \overline{\alpha} = (\overline{f} \curlywedge^{\overline{\alpha}}_{\overline{\beta}} \overline{g} \curlywedge^{\overline{\alpha}}_{\overline{\beta}} \overline{h})(u,q).
\end{aligned}
$$

Hence, $(\overline{f} \curlywedge^{\overline{\alpha}}_{\overline{\beta}} \overline{g} \curlywedge^{\overline{\alpha}}_{\overline{\beta}} \overline{h})(u,q) \preceq (\overline{f} \circ^{\overline{\alpha}}_{\overline{\beta}} \overline{g} \circ^{\overline{\alpha}}_{\overline{\beta}} \overline{h})(u,q)$. Therefore, $\overline{f} \curlywedge^{\overline{\alpha}}_{\overline{\beta}} \overline{g} \curlywedge^{\overline{\alpha}}_{\overline{\beta}} \overline{h} \sqsubseteq \overline{f} \circ^{\overline{\alpha}}_{\overline{\beta}} \overline{g} \circ^{\overline{\alpha}}_{\overline{\beta}} \overline{h}$.

It is obvious that $(4) \Rightarrow (3) \Rightarrow (2)$.

$(2) \Rightarrow (1)$ Let $Q$ be a quasi-ideal, $I$ be an ideal and $R$ be a right ideal of $S$. Then, by Theorem 2.3, $\overline{\chi}_Q$ is an IVQF quasi-ideal with thresholds $(\overline{\alpha}, \overline{\beta})$, $\overline{\chi}_I$ is an IVQF ideal with thresholds $(\overline{\alpha}, \overline{\beta})$ and $\overline{\chi}_R$ is an IVQF right ideal with thresholds $(\overline{\alpha}, \overline{\beta})$
of $S$. By supposition and Lemma 3.2, we have

$$
\begin{aligned}
\overline{\beta} &= (\overline{\chi}_{Q\cap I\cap R})_{(\overline{\alpha},\overline{\beta})}(u,q) = ((\overline{\chi}_Q) \curlywedge^{\overline{\alpha}}_{\overline{\beta}} (\overline{\chi}_I) \curlywedge^{\overline{\alpha}}_{\overline{\beta}} (\overline{\chi}_R))(u,q) \\
&\sqsubseteq ((\overline{\chi}_Q) \circ^{\overline{\alpha}}_{\overline{\beta}} (\overline{\chi}_I) \circ^{\overline{\alpha}}_{\overline{\beta}} (\overline{\chi}_R))(u,q) = (\overline{\chi}_{QIR})_{(\overline{\alpha},\overline{\beta})}(u,q).
\end{aligned}
$$

Thus, $u \in QIR$. Hence, $Q \cap I \cap R \sqsubseteq QIR$. Therefore, by Lemma 3.8, $S$ is weakly regular. □

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## CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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