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ON CONVOLUTION PROPERTY OF *HY* TRANSFORM AND ITS APPLICATIONS

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Abstract. In this paper, we have given a new application of *HY* transform. The convolution property for *HY* transform is obtained. We used this new result to solve integral equations and fractional integral equation. Few examples have been presented to illustrate the efficiency of the property.

Keywords: integral transform; convolution.

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1. Introduction

Integral transform methods are convenient mathematical methods for solving advance problems of engineering and sciences which are mathematically expressed in terms of differential equations, partial differential equations, integro differential equations, fractional differential equations, etc. During last two decades, many integral transforms are introduced such as Shehu [1], Sumudu [2], Elzaki [3], Natural [4], Aboodh [5], Pourreza [6], Mohand [7] and Sawi [8]. In 2019, Ahmadi [9] defined a new integral transform which is called *HY* transform. However, some properties of this integral transform are not given such as convolution property. Then, the aim of this paper is to prove convolution property of *HY* transform which is an important property used to solve integral equations. The basic definition of *HY* transform is given in

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Section 2. The convolution property is discussed in Section 3, several test examples to show the effectiveness of the proposed property are given in Section 4, and finally the conclusion is summarized in Section 5.

2. MATHEMATICAL PRELIMINARIES

In this section, we present some basic idea about HY transform [8].

Definition 2.1. The HY transform of the function $f(t)$ of exponential order is defined over the set of functions

$$A = \{\exists M > 0, |f(t)| < Me^{\alpha t}, t \in [0,\infty)\}$$

by the following integral

$$HY[f(t)] = F(v) = v\int_0^\infty e^{-v^2 t} f(t)dt. \tag{1}$$

where $HY[f(t)]$ is called the HY transform of time function. Variables v is the HY transform variable. It converges if the limit of the integral exists, and diverges if not. The HY^{-1} will be the inverse of the HY transform.

The following useful formulas follow directly from equation (1):

(i) $HY[1] = \frac{1}{v}$.
(ii) $HY[t^n] = \frac{n!}{v^{2n+1}}$, $n = 1,2,3,\ldots.$
(iii) $HY[t^p] = \frac{\Gamma(p+1)}{v^{2p+1}}$, $p > -1$.
(iv) $HY[f(t)+g(t)] = HY[f(t)] + HY[g(t)]$. (Linearity property).

Theorem 2.2. *Let $HY[f(t)] = F(v)$. Then*

$$HY[f^{(n)}(t)] = v^{2n}F(v) - \sum_{k=0}^{n-1} v^{2(n-k)-1} f^k(0), \ \ n \geq 1. \tag{2}$$

Definition 2.3. The function $f_1 * f_2 = \int_{\mathbb{R}} f_1(t-\tau)f_2(\tau)d\tau$ is called the convolution of both functions f_1 and f_2 defined on $\mathbb{R}$.

3. Convolution Property for *HY* Transform

Theorem 3.1. *Let $HY[f(t)] = F(v)$ and $HY[g(t)[= G(v)$. Then HY transform of $(f*g)(t)$ is*

$$HY[(f*g)(t)] = \frac{1}{v}F(v)G(v). \tag{3}$$

Proof. The convolution of two function $f(t)$ and $g(t)$ is

$$(f*g)(t) = \int_0^t f(t-\tau)g(\tau)d\tau.$$

Using HY transform of equation (1), we get

$$\begin{aligned} HY[(f*g)(t)] &= HY\left[\int_0^t f(\tau)g(t-\tau)d\tau\right], \\ &= v\int_0^\infty e^{-v^2t}\left(\int_0^t f(\tau)g(t-\tau)d\tau\right)dt, \\ &= v\int_0^\infty \int_0^t e^{-v^2t} f(\tau)g(t-\tau)d\tau dt, \\ &= v\int_0^\infty \int_\tau^\infty e^{-v^2t} f(\tau)g(t-\tau)dt d\tau. \end{aligned}$$

Now setting $b = t-\tau$, we have

$$\begin{aligned} HY[(f*g)(t)] &= v\int_0^\infty f(\tau)\int_0^\infty e^{-v^2(b+\tau)}g(b)dbd\tau, \\ &= v\int_0^\infty e^{-v^2\tau}f(\tau)d\tau\int_0^\infty e^{-v^2b}g(b)db, \\ &= F(v)\int_0^\infty e^{-v^2b}g(b)db. \qquad (4) \end{aligned}$$

Multiplying both sides of equation (4) by v, we obtain

$$vHY[(f*g)(t)] = F(v)G(v).$$

Thus

$$HY\{(f*g)(t)\} = \frac{1}{v}F(v)G(v).$$

This proves the theorem of convolution. □

Corollary 3.2. *Let $HY[f(t)] = F(v)$ and $HY[g(t)[= G(v)$. Then HY transform of $(f*g)'(t)$ is*

$$HY[(f*g)'(t)] = vF(v)G(v). \tag{5}$$

Proof. From equation (2), we have

$$HY[f'(t)] = v^2 HY[f(t)] - vf(0).$$

That is,

$$HY[(f*g)'(t)] = v^2 HY[(f*g)(t)] - v(f*g)(0).$$

By Theorem 3.1, we obtain

$$HY[(f*g)'(t)] = v^2 \left(\frac{1}{v}F(v)G(v)\right).$$

Therefore

$$HY[(f*g)'(t)] = vF(v)G(v).$$

□

Theorem 3.3. *Let $HY[f(t)] = F(v)$ and $HY[g(t)[= G(v)$. Then HY transform of $(f*g)^{(n)}(t)$ is*

$$HY[(f*g)^{(n)}(t)] = v^{2n-1}F(v)G(v). \tag{6}$$

Proof. Assuming that equation (6) is true for $n = k$. From equation (6) and by mathematical induction, we have that

$$\begin{aligned} HY[((f*g)^{(k)}(t))'] &= HY\left[\frac{d}{dt}(f*g)^{(k)}(t)\right], \\ &= v^2 HY[(f*g)^{(k)}(t)] - (f*g)^{(k)}(0), \\ &= v^{2k-1+2}F(v)G(v), \\ &= v^{2k+1}F(v)G(v). \end{aligned} \tag{7}$$

Therefore

$$HY[(f*g)^{(n)}(t)] = v^{2n-1}F(v)G(v).$$

□

4. Examples

Example 4.1 Consider the following Volterra integral equation of first kind

$$f(x) = \int_0^x h(x-t)g(t)dt. \tag{8}$$

Taking HY transform on both sides of equation (8), we have

$$\begin{aligned} HY[f(x)] &= HY\left[\int_0^x h(x-t)g(t)dt\right], \\ F(v) &= HY[h(x) * g(x)]. \end{aligned}$$

By Theorem 3.1, we obtain

$$\begin{aligned} F(v) &= \frac{1}{v}H(v)G(v), \\ G(v) &= v \cdot \frac{F(v)}{H(v)}. \end{aligned}$$

Thus,

$$g(x) = HY^{-1}\left[v \cdot \frac{F(v)}{H(v)}\right]. \tag{9}$$

Example 4.2 Consider the following Volterra integral equation of second kind

$$g(x) = f(x) + \int_0^x h(x-t)g(t)dt. \tag{10}$$

Taking HY transform on both sides of equation (10), we have

$$\begin{aligned} HY[g(x)] &= HY[f(x)] + HY\left[\int_0^x h(x-t)g(t)dt\right], \\ G(v) &= F(v) + HY[h(x) * g(x)]. \end{aligned}$$

By Theorem 3.1, we obtain

$$\begin{aligned} G(v) &= F(v) + \frac{1}{v}H(v)G(v), \\ G(v) &= \frac{vF(v)}{v - H(v)}. \end{aligned}$$

Then,

$$g(x) = HY^{-1}\left[\frac{vF(v)}{v - H(v)}\right]. \tag{11}$$

Example 4.3 Consider the following Volterra integral equation

$$\int_0^x e^{2(x-t)} f(t)dt = x. \tag{12}$$

Taking HY transform both sides of (12) and by Theorem 3.1, we obtain

$$\begin{aligned} HY\left[\int_0^x e^{2(x-t)} f(t)dt\right] &= HY[x], \\ \frac{1}{v}\left(\frac{v}{v^2+2}\right)F(v) &= \frac{1}{v^3}, \\ F(v) &= \frac{v^2+2}{v^3}. \end{aligned}$$

Then,

$$f(x) = 1 + 2x. \tag{13}$$

Example 4.4 Consider the following Volterra integral equation

$$g(x) = x + \int_0^x g(t)\sin(x-t)dt. \tag{14}$$

Taking HY transform on both sides of equation (14) and by Theorem 3.1, we obtain

$$\begin{aligned} G(v) &= HY[x] + HY[g(x) * \sin(x)], \\ &= \frac{1}{v^3} + \frac{1}{v}\left(\frac{v}{v^4+1}\right)G(v), \\ &= \frac{v^4+1}{v^7}. \end{aligned}$$

Then,

$$g(x) = x + \frac{1}{6}x^3. \tag{15}$$

Example 4.5 Consider the following integro-differential equation

$$f'(x) = 1 - \int_0^x f(t)dt, \quad f(0) = 0. \tag{16}$$

Taking HY transform on both sides of equation (16) and by Theorem 3.1, we obtain

$$\begin{aligned} HY[f'(x)] &= HY[1] - HY[1 * f(x)], \\ v^2F(v) - vf(0) &= \frac{1}{v} - \frac{1}{v}\left(\frac{1}{v}F(v)\right), \\ v^2F(v) &= \frac{1}{v} - \frac{1}{v^2}F(v), \\ F(v) &= \frac{v}{v^4+1}. \end{aligned}$$

Then,

$$f(x) = \sin x. \tag{17}$$

Example 4.6 Consider the following fractional integral equation

$$y(t) = g(t) + I^{\alpha}y(t),\ \alpha \in \mathbb{R}^{+}, \tag{18}$$

where I^{α} is the well known Riemann–Liouville fractional integral operator. It is defined by $I^{\alpha} = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}y(\tau)d\tau$.

By substituting $\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}y(\tau)d\tau$ instead of I^{α} in equation (18) and applying the convolution Theorem 3.1, we have

$$\begin{aligned} Y(v) &= G(v) + \frac{1}{\Gamma(\alpha)}HY[t^{\alpha-1}]HY[y(t)], \\ &= G(v) + \frac{1}{\Gamma(\alpha)}\frac{\Gamma(\alpha)}{v^{2\alpha}}Y(v), \\ Y(v) &= \frac{v^{2\alpha}}{v^{2\alpha}-1}G(v). \end{aligned}$$

Then,

$$y(t) = HY^{-1}\left[\frac{v^{2\alpha}}{v^{2\alpha}-1}G(v)\right] \tag{19}$$

where $Y(v)$ is $HY[y(t)]$.

5. Conclusion

In this paper, convolution property of HY transform of is obtained. We have successfully applied HY transform for the solution of integral equations and fractional integral equations. For further study, HY transform can be applied for solving other singular integral equations and their systems.

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Conflict of Interests

The author(s) declare that there is no conflict of interests.

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