Available online at http://scik.org
J. Math. Comput. Sci. 2022, 12:109
https://doi.org/10.28919/jmcs/7195
ISSN: 1927-5307

EXISTENCE AND UNIQUENESS OF SOLUTION OF FRACTIONAL ORDER DIFFERENTIAL EQUATION OF FINITE DELAY IN CONE METRIC SPACE

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Abstract. In this paper, we use Caputo sence to prove the existence and uniqueness of solutions to fractional differential equations with finite delay and nonlocal conditions in cone metric space. The result is achieved by applying several expansions of Banach's contraction principle to the entire cone metric space, as well as providing an illustration of the primary result.

Keywords: fractional derivative; fractional differentail equations; existence of solution; cone metric space; Banach contraction principle.

2010 AMS Subject Classification: 26A33, 39A05, 54E50, 45N05, 47G20, 34K05, 47H10.

1. Introduction

Fractional derivative is as old as calculus. L'Hospital in 1695 asked what does it mean $\frac{d^n\Upsilon}{d\vartheta^n}$ if $n = 1/2$. Since then, many researchers tried to put a definition of a fractional derivative. Most of them used an integral form for the fractional derivative. The goal of this research is to investigate the existence and uniqueness of solutions to fractional differential equations with nonlocal conditions in the form of cone metric space two of which are the most popular ones.

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Received January 23, 2022

i) Riemann–Liouville definition [1]. For $\alpha \in [n-1,n)$ the α - derivative of Υ is

$$D_a^{\alpha}\Upsilon(\vartheta) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n\vartheta}{d\xi^n}\int_a^{\alpha}\frac{\Upsilon(\vartheta)}{(\xi-\vartheta)^{\alpha-n+1}}d\vartheta$$

ii) Caputo definition [1]. For $\alpha \in [n-1,n)$ the α - derivative of Υ is

$$D_a^{\alpha}\Upsilon(\vartheta) = \frac{1}{\Gamma(n-\alpha)}\int_a^{\alpha}\frac{\Upsilon^{(n)}(\vartheta)}{(\xi-\vartheta)^{\alpha-n+1}}d\vartheta$$

iii) Caputo Integral. For $\alpha \in [n-1,n)$ the α - integral of Υ is

$$D_a^{\alpha}\Upsilon(\vartheta) = \frac{1}{\Gamma(\alpha)}\int_0^{\xi}(\xi-\vartheta)^{\alpha-1}d\vartheta.$$

Here we introduce the existence and uniqueness of solution of fractional differential equation with nonlocal conditions in cone metric space of the form:

$$D^{\alpha}\vartheta(\xi) = A(\xi)\vartheta(\xi) + \Upsilon(\xi,\vartheta(\xi),\vartheta(\xi-1)), \qquad \xi \in J = [0,b] \tag{1.1}$$

$$\vartheta(\xi-1) = \psi(\xi) \qquad\qquad \xi \in [0,1] \tag{1.2}$$

$$\vartheta(0) + \iota(\vartheta) = \vartheta_0, \tag{1.3}$$

where $A(\xi)$ is a bounded linear operator on a Banach space X with domain $D(A(\xi))$, the unknown $\vartheta(\cdot)$ takes values in the Banach space $X.Let$ $\Upsilon : J\times X\times X \to X$, $\iota : C(J,X) \to X$ are appropriate continuous functions and ϑ_0 is given element of X. $\psi(\xi)$ is a continuous function for $[0,1]$. We note that, if $\xi \in [0,1]$, the problem is reduced to fractional differential equation

$$D^{\alpha}\vartheta(\xi) = A(\xi)\vartheta(\xi) + \Upsilon(\xi,\vartheta(\xi),\psi(\xi)), \qquad \xi \in J = [0,b]$$

With initial condition $\vartheta(0)+\iota(\vartheta) = \vartheta_0$. Here, it is essential to obtain the solutions of (1.1)-(1.3) for [0,b]. Fractional differential equations (FDEs) are used to simulate a wide range of physical events, and they may be solved using a variety of methods [1, 3, 10].

2. DEFINITIONS AND PRELIMINARIES

Let us review the cone metric space ideas and we refer the reader to [2, 4, 5, 8] for the more details.

Definition 2.1. Let E be a real Banach space and P is a subset of E. Then P is called a cone if and only if,

1. P is closed, nonempty and $P \neq 0$.

2. $a, b \in \mathbb{R}, a, b \geq 0, \vartheta, \zeta \in P \Rightarrow a\vartheta + b\zeta \in P.$

3. $\vartheta \in P$ and $-\vartheta \in P \Rightarrow \vartheta = 0$.

Let $\leq$ be the partial ordering relation defined on E as $\vartheta \leq \zeta$ if and only if $\zeta - \vartheta \in P$. We shall write $\vartheta < \zeta$ to indicate that $\vartheta \leq \zeta$ but $\vartheta \neq \zeta$, while $\vartheta \ll \zeta$ will stand for $\zeta - \vartheta \in intP$. If there is a number $K > 0$ such that $0 \leq \vartheta \leq \zeta$ implies $\|\vartheta\| \leq k\|\zeta\|$, for every $\vartheta, \zeta \in E$ then cone P is called normal. Such least positive number satisfying above is called the normal constant of P. We always assume E is a real Banach space in the following fashion, P is cone in E with int $P \neq \phi$, and $\leq$ is partial ordering with respect to P.

Definition 2.2. Let X a nonempty set. Suppose that the mapping $d : X \times X \to E$ satisfies:

(d_1) $0 \leq d(\vartheta, \zeta)$ for all $\vartheta, \zeta \in \vartheta$ and $d(\vartheta, \zeta) = 0$ if and only if $\vartheta = \zeta$;

(d_2) $d(\vartheta, \zeta) = d(\zeta, \vartheta)$, for all $\vartheta, \zeta \in X$;

(d_3) $d(\vartheta, \zeta) \leq d(\vartheta, \chi) + d(\chi, \zeta)$, for all $\vartheta, \zeta \in X$.

Then d is called a cone metric on X and (X, d) is called a cone metric space. The concept of cone metric space is broader than the concept of metric space.

Definition 2.3. Let X be an ordered space. A function $\Phi : X \to X$ is said to a comparison function if every $\vartheta, \zeta \in X, \vartheta \leq \zeta$, implies that $\Phi(\vartheta) \leq \Phi(\zeta), \Phi(\vartheta) \leq \vartheta$ and $\lim_{n\to\infty} \|\Phi^n(\vartheta)\| = 0$, for every $\vartheta \in X$.

3. Fractional Order Differential Equation of Finite Delay in Cone Metric Space

Let X is a Banach space with norm $\|.\|$. $B = C(J,X)$ be the Banach space of all continuous function from J into X endowed with supremum norm

$$\|\vartheta\|_\infty = sup\{\|\vartheta(\xi)\| : \xi \in [0,b]\}$$

Let $P = (\vartheta,\zeta) : \vartheta,\zeta \geq 0 \subset E = \mathbb{R}^2$, and define

$$d(\Upsilon,\iota) = (\|\Upsilon-\iota\|_\infty, \alpha\|\Upsilon-\iota\|_\infty);$$

for every $\Upsilon,\iota \in B$, then it is easily seen that (B,d) is a cone metric space.

Definition 3.1. The function $\vartheta \in B$ satisfies the integral equation

Case I: If $\xi \in [0,1]$ then

$$\vartheta(\xi) = \vartheta_0 - \iota(\vartheta) + \frac{1}{\Gamma(\alpha)}\int_0^\xi (\xi-\omega)^{\alpha-1}A(\omega)\Upsilon(\omega,\vartheta(\omega),\vartheta(\omega-1))d\omega, \tag{3.1}$$

Case II: If $\xi \in [1,b]$ then

$$\begin{aligned}\vartheta(\xi) = \vartheta_0 - \iota(\vartheta) &+ \frac{1}{\Gamma(\alpha)}\int_0^1 (\xi-\omega)^{\alpha-1}A(\omega)\Upsilon(\omega,\vartheta(\omega),\vartheta(\omega-1))d\omega \\ &+ \frac{1}{\Gamma(\alpha)}\int_1^\xi (\xi-\omega)^{\alpha-1}A(\omega)\Upsilon(\omega,\vartheta(\omega),\vartheta(\omega-1))d\omega, \end{aligned} \tag{3.2}$$

is called the mild solution of the equation $(1.1)-(1.3)$.

We need the following theorem for further discussion:

Lemma 3.1. [9] Let (X,d) be a complete cone metric space, where P is a normal cone with normal constant K. Let $\Upsilon : X \to X$ be a function such that there exists a comparison function $\Phi : P \to P$ such that

$$d(\Upsilon(\vartheta),\Upsilon(\zeta)) \leq \Phi(d(\vartheta,\zeta)),$$

for every $\vartheta,\zeta \in X$. Then Υ has unique fixed point.

We list the following hypothesis for our convenience:

(H_1) A(ξ) is a bounded linear operator on X for each $\xi \in J$, the function $\xi \to A(\xi)$ is continuous in the uniform operator topology and hence there exists a constant K such that

$$K = \sup_{\xi \in J} \|A(\xi)\|.$$

(H_2) Let $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ be a comparison function

(i) There exist continuous function $p_1, p_2 : J \to \mathbb{R}^+$ such that

Case I: If $\xi \in [0,1]$ then

$$(\|\Upsilon(\xi,\vartheta(\xi),\psi(\xi)) - \Upsilon(\xi,\zeta(\xi),\psi(\xi))\|, \lambda\|\Upsilon(\xi,\vartheta(\xi),\psi(\xi)) - \Upsilon(\xi,\zeta(\xi),\psi(\xi))\|)$$
$$\leq p_1(\xi)\Phi(d(\vartheta,\zeta)),$$

Case II: If $\xi \in [1,b]$ then

$$(\|\Upsilon(\xi,\vartheta(\xi),\vartheta(\xi-1)) - \Upsilon(\xi,\zeta(\xi),\zeta(\xi-1))\|, \lambda\|\Upsilon(\xi,\vartheta(\xi),\vartheta(\xi-1)) - \Upsilon(\xi,\zeta(\xi),\zeta(\xi-1))\|)$$
$$\leq p_2(\xi)\Phi(d(\vartheta,\zeta)),$$

for every $\xi \in J$ and $\vartheta, \zeta \in X$

(ii) There exists a positive constant G such that

$$(\|\iota(\vartheta) - \iota(\zeta)\|, \lambda\|\iota(\vartheta) - \iota(\zeta)\|) \leq G\Phi(d(\vartheta,\zeta)),$$

for every $\vartheta, \zeta \in X$

$$(H_3) \qquad \sup_{\xi \in J}\left\{G + \frac{1}{\Gamma(\alpha)}\int_0^{\xi}(\xi-\omega)^{\alpha-1}K\big[p_1(\omega)+p_2(\omega)\big]d\omega\right\} = 1.$$

Our main results are given in the following theorem:

Theorem 3.2. If hypotheses $(H_1) - (H_3)$ hold, then the differential equation $(1.1) - (1.2)$ has a unique solution ϑ on J

Proof: The operartor $F : B \to B$ is defined by

Case I: If $\xi \in [0,1]$ then

$$F\vartheta(\xi) = \vartheta_0 - \iota(\vartheta) + \frac{1}{\Gamma(\alpha)}\int_0^{\xi}(\xi-\omega)^{\alpha-1}(A(\omega)\Upsilon(\omega,\vartheta(\omega),\vartheta(\omega-1))d\omega, \tag{3.3}$$

Case II: If $\xi \in [1,b]$ then

$$\begin{aligned} F\vartheta(t) = \vartheta_0 - \iota(\vartheta) &+ \frac{1}{\Gamma(\alpha)}\int_0^{1}(\xi-\omega)^{\alpha-1}(A(\omega)\Upsilon(\omega,\vartheta(\omega),\vartheta(\omega-1))ds \\ &+ \frac{1}{\Gamma(\alpha)}\int_1^{\xi}(\xi-\omega)^{\alpha-1}A(\omega)\Upsilon(\omega,\vartheta(\omega),\vartheta(\omega-1))d\omega, \end{aligned} \tag{3.4}$$

By using the hypothesis $(H_1)-(H_3)$, We have

Case I: If $\xi \in [0,1]$ then

$$\begin{aligned} &(\|F\vartheta(\xi)-F\zeta(\xi)\|, \lambda\|F\vartheta(\xi)-F\zeta(\xi)\|) \\ &\leq \Big(\|\iota(\vartheta)-\iota(\zeta)\| + \frac{1}{\Gamma(\alpha)}\int_0^{\xi}(\xi-\omega)^{\alpha-1}\|A(\omega)\|\|\Upsilon(\omega,\vartheta(\omega),\psi(\omega))-\Upsilon(\omega,\zeta(\omega),\psi(\omega))\|d\omega, \\ &\quad \lambda\|\iota(\vartheta)-\iota(\zeta)\| + \lambda\frac{1}{\Gamma(\alpha)}\int_0^{\xi}(\xi-\omega)^{\alpha-1}\|A(\omega)\|\|\Upsilon(\omega,\vartheta(\omega),\psi(\omega))-\Upsilon(\omega,\zeta(\omega),\psi(\omega))\|d\omega\Big) \\ &\leq (\|\iota(\vartheta)-\iota(\zeta)\|, \lambda\|\iota(\vartheta)-\iota(\zeta)\|) + \frac{1}{\Gamma(\alpha)}\int_0^{\xi}(\xi-\omega)^{\alpha-1}K(\|\Upsilon(\omega,\vartheta(\omega),\psi(\omega))-f(\omega,\zeta(\omega),\psi(\omega))\|, \\ &\quad \lambda\|f(\omega,\vartheta(\omega),\psi(\omega))-\Upsilon(\omega,\zeta(\omega),\psi(\omega))\|d\omega) \\ &\leq G\Phi(\|\vartheta-\zeta\|,\lambda\|\vartheta-\zeta\|) + \frac{1}{\Gamma(\alpha)}\int_0^{\xi}(\xi-\omega)^{\alpha-1}Kp_1(\omega)\Phi(\|\vartheta(\omega)-\zeta(\omega)\|,\lambda\|\vartheta(\omega)-\zeta(\omega)\|)d\omega \\ &\leq G\Phi(\|\vartheta-\zeta\|_\infty,\lambda\|\vartheta-\zeta\|_\infty) + \Phi(\|\vartheta-\zeta\|_\infty,\lambda\|\vartheta-\zeta\|_\infty)\frac{1}{\Gamma(\alpha)}\int_0^{\xi}(\xi-\omega)^{\alpha-1}Kp_1(\omega)\ \ d\omega \\ &\leq G\Phi(d(\vartheta,\zeta)) + \Phi(d(\vartheta,\zeta))\frac{1}{\Gamma(\alpha)}\int_0^{\xi}(\xi-\omega)^{\alpha-1}Kp_1(\omega)d\omega \\ &\leq \Phi(d(\vartheta,\zeta))\left\{G + \frac{1}{\Gamma(\alpha)}\int_0^{\xi}(\xi-\omega)^{\alpha-1}Kp_1(\omega)d\omega\right\} \\ &\leq \Phi(d(\vartheta,\zeta))\left\{G + \frac{1}{\Gamma(\alpha)}\int_0^{\xi}(\xi-\omega)^{\alpha-1}K\big[p_1(\omega)+p_2(\omega)\big]d\omega\right\} \\ &\leq \Phi(d(\vartheta,\zeta)) \end{aligned} \tag{3.5}$$

Case II: for $\xi \in [1,b]$

$$(\|F\vartheta(\xi)-F\zeta(\xi)\|,\lambda\|F\vartheta(\xi)-F\zeta(\xi)\|)$$

$$\leq \Big(\|\iota(\vartheta)-\iota(\zeta)\|+\frac{1}{\Gamma(\alpha)}\int_0^1(\xi-\omega)^{\alpha-1}\|A(\omega)\|\Big[\|\Upsilon(\omega,\vartheta(\omega),\psi(\omega))-\Upsilon(\omega,\zeta(\omega),\psi(\omega))\|d\omega\Big]$$

$$+\frac{1}{\Gamma(\alpha)}\int_1^\xi(\xi-\omega)^{\alpha-1}\|A(\omega)\|\Big[\|\Upsilon(\omega,\vartheta(\omega),\vartheta(\omega-1))-\Upsilon(\omega,\zeta(\omega),\zeta(\omega-1))\|\Big]d\omega,$$

$$\lambda\|\iota(\vartheta)-\iota(\zeta)\|+\lambda\frac{1}{\Gamma(\alpha)}\int_0^1(\xi-\omega)^{\alpha-1}\|A(\omega)\|\Big[\|\Upsilon(\omega,\vartheta(\omega),\psi(\omega))-\Upsilon(\omega,\zeta(\omega),\psi(\omega))\|d\omega\Big]$$

$$+\lambda\frac{1}{\Gamma(\alpha)}\int_1^\xi(\xi-\omega)^{\alpha-1}\|A(\omega)\|\Big[\|\Upsilon(\omega,\vartheta(\omega),\vartheta(\omega-1))-\Upsilon(\omega,\zeta(\omega),\zeta(\omega-1))\|\Big]d\omega\Big)$$

$$\leq G\Phi(d(\vartheta,\zeta))+\frac{1}{\Gamma(\alpha)}\int_0^1(\xi-\omega)^{\alpha-1}Kp_1(\omega)\Phi(d(\vartheta,\zeta))d\omega+\frac{1}{\Gamma(\alpha)}\int_1^\xi(\xi-\omega)^{\alpha-1}Kp_2(\omega)\Phi(d(\vartheta,\zeta))d\omega$$

$$\leq G\Phi(d(\vartheta,\zeta))+\frac{1}{\Gamma(\alpha)}\int_0^1(\xi-\omega)^{\alpha-1}K\big[p_1(\omega)+p_2(\omega)\big]\Phi(d(\vartheta,\zeta))d\omega$$

$$+\frac{1}{\Gamma(\alpha)}\int_1^\xi(\xi-\omega)^{\alpha-1}K\big[p_1(\omega)+p_2(\omega)\big]d\omega\Phi(d(\vartheta,\zeta))$$

$$\leq \Phi(d(\vartheta,\zeta))\Big\{G+\frac{1}{\Gamma(\alpha)}\int_0^1(\xi-\omega)^{\alpha-1}K\big[p_1(\omega)+p_2(\omega)\big]d\omega+\frac{1}{\Gamma(\alpha)}\int_1^\xi(\xi-\omega)^{\alpha-1}K\big[p_1(\omega)+p_2(\omega)\big]d\omega\Big\}$$

$$\leq \Phi(d(\vartheta,\zeta))\Big\{G+\frac{1}{\Gamma(\alpha)}\int_0^\xi(\xi-\omega)^{\alpha-1}K\big[p_1(\omega)+p_2(\omega)\big]d\omega\Big\}$$

$$\leq \Phi(d(\vartheta,\zeta)) \tag{3.6}$$

for every $\vartheta,\zeta\in B$. This implies that $d(\Upsilon\vartheta,\Upsilon\zeta)\leq\Phi(d(\vartheta,\zeta))$, for every $\vartheta,\zeta\in B$. Now an application of Lemma 3.1, the operator has a unique point in B. This means that the equation (1.1)-(1.2) has unique solution.

4. Application

In this section, we give an example to illustrate the usefulness of our result discussed in previous section. Let us consider the following fractional differential equation:

$$D^{\alpha}\vartheta(\xi)=\frac{35}{36}e^{-\xi}\vartheta(\xi)+\Upsilon(\xi,\vartheta(\xi),\vartheta(\xi-1)),\ \xi\in J=[0,2],\ \vartheta\in X \tag{4.1}$$

$$\vartheta(0)+\frac{\vartheta}{8+\vartheta}=\vartheta_0, \tag{4.2}$$

where,

$$\Upsilon(\xi,\vartheta(\xi),\vartheta(\xi-1))=\frac{\xi e^{-\xi}\vartheta(\xi)}{(9+e^{\xi})(1+\vartheta(\xi))},\qquad for\ \xi\in[0,1]$$

$$\Upsilon(\xi,\vartheta(\xi),\vartheta(\xi-1))=\frac{2\xi e^{-(\xi-1)}\vartheta(\xi-1)}{(9+e^{\xi-1})(1+\vartheta(\xi-1))},\quad for\ \xi\in[1,2]$$

Therefore, we have

$$A(\xi)=\tfrac{35}{36}e^{-\xi},\ \ \xi\in J$$

$$\Upsilon(\xi,\vartheta(\xi),\psi(\xi))=\tfrac{\xi e^{-\xi}\vartheta(\xi)}{(9+e^{\xi})(1+\vartheta(\xi))},\ \ (\xi,\vartheta)\in J\times X$$

$$\Upsilon(\xi,\vartheta(\xi),\vartheta(\xi-1))=\tfrac{2\xi e^{-(\xi-1)}\vartheta(\xi-1)}{(9+e^{\xi-1})(1+\vartheta(\xi-1))},\ \ (\xi,\vartheta)\in J\times X$$

$$\iota(\vartheta)=\frac{\vartheta}{8+\vartheta},\ \ \vartheta\in X$$

Now for $\vartheta,\zeta\in C(J,X)$ and $\xi\in J$, we have

Case I: for $\xi\in[0,1]$

$$(\|\Upsilon(\xi,\vartheta(\xi),\vartheta(\xi-1))-\Upsilon(\xi,\zeta(\xi),\zeta(\xi-1))\|,\lambda\|\Upsilon(\xi,\vartheta(\xi),\vartheta(\xi-1))-\Upsilon(\xi,\zeta(\xi),\zeta(\xi-1))\|)$$

$$=\tfrac{\xi e^{-\xi}}{9+e^{\xi}}(\|\tfrac{\vartheta(\xi)}{1+\vartheta(\xi)}-\tfrac{\zeta(\xi)}{1+\zeta(\xi)}\|,\lambda\|\tfrac{\vartheta(\xi)}{1+\vartheta(\xi)}-\tfrac{\zeta(\xi)}{1+\zeta(\xi)}\|)$$

$$= \frac{\xi e^{-\xi}}{9+e^{\xi}}(\|\frac{\vartheta(\xi)-\zeta(\xi)}{(1+\vartheta(\xi))(1+\zeta(\xi))}\|, \lambda\|\frac{\vartheta(\xi)-\zeta(\xi)}{(1+\vartheta(\xi))(1+\zeta(\xi))}\|)$$

$$\leq \frac{\xi e^{-\xi}}{9+e^{\xi}}(\|\vartheta(\xi)-\zeta(\xi)\|, \lambda\|\vartheta(\xi)-\zeta(\xi)\|)$$

$$\leq \frac{\xi e^{-\xi}}{9+e^{\xi}}(\|\vartheta-\zeta\|_{\infty}, \lambda\|\vartheta-\zeta\|_{\infty})$$

$$\leq \frac{\xi e^{-\xi}}{9+e^{\xi}} d(\vartheta,\zeta)$$

$$\leq \frac{\xi}{10}\Phi(d(\vartheta,\zeta)),$$

where $p_1(\xi) = \frac{\xi}{10}$, which is continuous function of J into $\mathbb{R}^+$ and a comparison function $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ such that $\Phi(d(\vartheta,\zeta)) = d(\vartheta,\zeta)$.

Case II: for $\xi \in [1,2]$

$$(\|\Upsilon(\xi,\vartheta(\xi),\vartheta(\xi-1)) - \Upsilon(\xi,\zeta(\xi),\zeta(\xi-1))\|, \lambda\|\Upsilon(\xi,\vartheta(\xi),\vartheta(\xi-1)) - \Upsilon(\xi,\zeta(\xi),\zeta(\xi-1))\|)$$

$$= \frac{2\xi e^{-(\xi-1)}}{9+e^{\xi-1}}(\|\frac{\vartheta(\xi-1)}{1+\vartheta(\xi-1)} - \frac{\zeta(\xi-1)}{1+\zeta(\xi-1)}\|, \lambda\|\frac{\vartheta(\xi-1)}{1+\vartheta(\xi-1)} - \frac{\zeta(\xi-1)}{1+\zeta(\xi-1)}\|)$$

$$= \frac{2\xi e^{-(\xi-1)}}{9+e^{\xi-1}}(\|\frac{\vartheta(\xi-1)-\zeta(\xi-1)}{(1+\vartheta(\xi-1))(1+\zeta(\xi-1))}\|, \lambda\|\frac{\vartheta(\xi-1)-\zeta(\xi-1)}{(1+\vartheta(\xi-1))(1+\zeta(\xi-1))}\|)$$

$$\leq \frac{2\xi e^{-(\xi-1)}}{9+e^{\xi-1}}(\|\vartheta(\xi-1)-\zeta(\xi-1)\|, \lambda\|\vartheta(\xi-1)-\zeta(\xi-1)\|)$$

$$\leq \frac{2\xi e^{-(\xi-1)}}{9+e^{\xi-1}}(\|\vartheta-\zeta\|_{\infty}, \lambda\|\vartheta-\zeta\|_{\infty})$$

$$\leq \frac{2\xi e^{-(\xi-1)}}{9+e^{\xi-1}} d(\vartheta,\zeta)$$

$$\leq \frac{\xi}{5}\Phi(d(\vartheta,\zeta)),$$

where $p_2(\xi) = \frac{\xi}{5}$, which is continuous function of J into $\mathbb{R}^+$ and a comparison function $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ such that $\Phi(d(\vartheta,\zeta)) = d(\vartheta,\zeta)$.

Similarly, we can have

$(\|\iota(\vartheta)-\iota(\zeta)\|,\lambda\|\iota(\vartheta)-\iota(\zeta)\|)$

$$\leq 8\left(\frac{\|\vartheta-\zeta\|}{(8+\|\vartheta\|)(8+\|\zeta\|)},\lambda\frac{\|\vartheta-\zeta\|}{(8+\|\vartheta\|)(8+\|\zeta\|)}\right)$$

$$\leq \frac{8}{64}(\|\vartheta-\zeta\|,\lambda\|\vartheta-\zeta\|)$$

$$\leq \frac{1}{8}(\|\vartheta-\zeta\|_{\infty},\lambda\|\vartheta-\zeta\|_{\infty})$$

$$\leq \frac{1}{8}\Phi(d(\vartheta,\zeta)),$$

where $G=\frac{1}{8}$, and the comparison function Φ defined as above. Hence the condition (H_1) holds with $K=\frac{35}{36}$.

Moreover,

$$\sup_{\xi\in J}\left\{G+\frac{1}{\Gamma(\alpha)}\int_0^{\xi}(\xi-\omega)^{\alpha-1}K\left[p_1(\omega)+p_2(\omega)\right]d\omega\right\}=\sup_{\xi\in J}\left\{\frac{1}{8}+\frac{35}{36}\frac{1}{\Gamma(\alpha)}\int_0^{\xi}(\xi-\omega)^{\alpha-1}\left(\frac{\omega}{10}+\frac{\omega}{5}\right)d\omega\right\}$$

$$=\sup_{\xi\in J}\left\{\frac{1}{8}+\frac{35}{36}\frac{1}{\Gamma(\alpha)}\int_0^{\xi}(\xi-\omega)^{\alpha-1}\left(\frac{3\omega}{10}\right)d\omega\right\}$$

$$=\sup_{\xi\in J}\left\{\frac{1}{8}+\frac{35}{36}\frac{1}{\Gamma(\alpha)}\left[\frac{3\omega^{\alpha+1}}{10\alpha(\alpha+1)}\right]_0^{\xi}\right\}$$

$$=\left\{\frac{1}{8}+\frac{35}{36}\frac{1}{\Gamma(\alpha)}\left[\frac{3(2^{\alpha+1})}{10\alpha(\alpha+1)}\right]\right\}$$

$$=\left\{\frac{1}{8}+\frac{35}{36}[\Delta]\right\}$$

$$=\left[\frac{1}{8}+\frac{35}{36}\times\frac{9}{10}\right]=\left[\frac{1}{8}+\frac{7}{8}\right]=1$$

where $\Delta=\frac{1}{\Gamma(\alpha)}\left[\frac{3(2^{\alpha+1})}{10\alpha(\alpha+1)}\right]$ choose α so that $\frac{1}{\Gamma(\alpha)}\left[\frac{3(2^{\alpha+1})}{10\alpha(\alpha+1)}\right]=\frac{9}{10}$, Since all the conditions of Theorem 3.1 are satisfied, the problem (4.1)-(4.2) has a unique solution ϑ on J.

5. Conclusion

In this study, we investigated the existence of finite delay fractional order differential equations in cone metric spaces and shown that the solution is unique. We demonstrated this result in entire cone metric space by employing various expansions of the Banach contraction principle. In addition, the above outcome was put into practice.

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