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ON QUASI-CAYLEY FUZZY GRAPHS

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Abstract. In this note, our aim is to initiate the notion of the Cayley fuzzy graphs on fuzzy quasi-subgroups. We discuss its basic properties along with few of its characterizations.

Keywords: quasi-Cayley graphs; complete quasi-Cayley graphs; strong quasi-Cayley graphs.

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1. Introduction

Cayley graph theory become a significant branch of algebraic graph theory from past few decades. Cayley graphs are applicable towards modeling of interconnection networks. In this regard numerous studies have been presented in the literature (see, e.g., Akers and Krishnamurthy [2]; Cooperman and Finkelstein [5]; Heydemann [8] etc). Variety of daily life problems in computer science, biology, coding theory, graphs and groups theory were solved by scientists by using Cayley graphs. The computer science community is currently considering Cayley graphs as architecture models for large-scale parallel processor machines. Any parallel processing or distributed system relies heavily on the design of interconnection networks. The

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network topology chosen has a considerable impact on the distributed system's performance. On the other hands, quasi-group is an interesting algebraic structure which plays a significant role where groups failed to deal with.

Some areas of the classical graph theoretic problems are still unclear in many circumstances. The traffic flow on the roads or vehicle capacity on the roads network are the examples of such parts which sometimes not be known exactly. In such situations, logically we use fuzzy sets and fuzzy logic to handle the vagueness. On the other hands, in social science, fuzzy graphs play a significant role to model the data with uncertainties. In comparison to the classical graphs, the fuzzy graphs models are more flexible and compatible. Fuzzy graphs are particularly useful when vertices or edges contain uncertainties.

The notion of fuzzy sets was first introduced by Zadeh [20]. Rosenfeld [16] initiated the theory of fuzzy groups at the base of fuzzy sets. Afterwards, he introduced the notion of fuzzy graphs which was based on the fuzzy relations [17]. Fuzzy graphs were further extended by Mordeson and Nair [13], they investigated many graph theoretic properties of fuzzy graphs. Bhattacharya [4] added many new results in the theory of fuzzy graphs. Sunita and Vijayakumar [18] presented the idea of fuzzy trees. Further to this, Mathew and Sunitha [10] further presented different categories of the arcs in fuzzy graphs and they also initiated the concepts of node and arc connectivity of fuzzy graphs (Mathew and Sunitha [11]). Different operations on the fuzzy graphs were suggested by Mordeson and Peng [12]. Subsequently, these operations were generalized towards interval-valued fuzzy graphs [3], Cayley fuzzy graphs [6] etc. Moreover, recently the first and third authors (with Babir Ali) introduced the generalization of fuzzy graphs termed bipolar picture fuzzy graphs [9].

Recently, the notion of Cayley fuzzy graphs on fuzzy groups are introduced in [19]. In this note, our aim is to initiate the notion of the Cayley fuzzy graphs on fuzzy quasi-subgroups. We also explore few of its basic characterizations.

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2. Preliminaries

A groupoid $(G,*)$ is said to be a quasigroup, if for arbitrary element $a,b \in G$ such that each of the equations $a*x=b,\ x*a=b$ has a unique solution in G [15]. We may define quasigroup as an algebra $Q=(G,\cdot,\backslash,/)$ with three binary operations $\cdot,\backslash,/$ satisfying

$$(x\cdot y)\backslash y = x,\ x\backslash(x\cdot y)=y$$

$$(x/y)\cdot y = x,\ x\cdot(x\backslash y)=y.$$

A sub-quasigroup is a subset S of a quasigroup $Q=(G,\cdot,\backslash,/)$ which is closed with respect to these three operations.

Definition 2.1. [20] A fuzzy set is a pair (ξ,X), where X is a nonempty set and $\xi : X \longrightarrow [0,1]$ is a membership function.

The set of all fuzzy subsets of a set X is the fuzzy power set of X and it will be denoted by $FP(X)$.

Definition 2.2. [21] Let $\xi_1,\xi_2 \in FP(X)$. If $\xi_1(x) \leq \xi_2(x), \forall x \in X$, then we say that ξ_1 is contained in ξ_2, and we express it as $\xi_1 \subseteq \xi_2$.

Definition 2.3. [21] Let $\xi \in FP(X)$. Then, the set $\{x|x\in X, \xi(x)>0\}$ is the support of ξ and is denoted by ξ^*.

Definition 2.4. [21] Let $\xi_1,\xi_2 \in FP(X)$. Then their union and intersection can be described as follows

$$(\xi_1 \cup \xi_2)(x) = \xi_1(x) \vee \xi_2(x)$$

$$(\xi_1 \cap \xi_2)(x) = \xi_1(x) \wedge \xi_2(x), \forall x \in X.$$

Definition 2.5. [21] Let $\xi \in FP(X)$. For $a \in [0,1]$. The we define ξ_a as follows

$$\xi_a = \{x|x\in X, \xi(x) \geq a\}$$

we call ξ_a $a-$level set of ξ.

Proposition 2.6. [21] *Let* $\xi_1,\xi_2 \in FP(X)$. *Then*

(1) $\xi_1 \subseteq \xi_2, a \in [0,1] \Rightarrow \xi_{1_a} \subseteq \xi_{2_a}$,

(2) $a, b \in [0,1], a \leq b \Rightarrow \xi_{1_b} \subseteq \xi_{1_a}$,

(3) $\xi_1 = \xi_2 \Leftrightarrow \xi_{1_a} = \xi_{2_a},\ for\ all\ a \in [0,1]$.

Definition 2.7. [21] Let $\xi_1 \in FP(X_1) and \xi_2 \in FP(X_2)$. Then $\xi_1 \times \xi_2 \in FP(X_1 \times X_2)$ can be defined as

$$\xi_1 \times \xi_2(x_1, x_2) = \xi_1(x_1) \wedge \xi_2(x_2), \text{ for all } (x_1, x_2) \in X_1 \times X_2$$

and is called the product of ξ_1 and ξ_2.

Definition 2.8. [20] A fuzzy relation on S is a fuzzy subset of $S \times S$. A fuzzy relation Υ on S is a fuzzy relation on the fuzzy subset ξ if $\Upsilon(x,y) \leq \xi(x) \wedge \xi(y)$ for all x, y in X.

Definition 2.9. [7] A fuzzy set ξ in a quasigroup $G = (G, \cdot, \backslash, /)$ is a fuzzy subquasigroup of G if

$$\xi(x * y) \geq min\{\xi(x), \xi(y)\}$$

for all $* \in \{\cdot, \backslash, /\}$ and $x, y \in G$.

Proposition 2.10. [7] *A fuzzy set ξ of a quasigroup $G = (G, \cdot, \backslash, /)$ is a fuzzy subquasigroup iff for every $a \in [0,1]$, ξ_a is either empty or a subquasigroup of G.*

Definition 2.11. [14] Let X be a non empty set. A fuzzy graph is a $V = (X, \tau, \rho)$ where τ is a fuzzy subset of X, ρ is a symmetric fuzzy relation on τ . i.e., $\tau : X \to [0,1]$ and $\rho : X \times X \to [0,1]$ such that $\rho(x,y) \leq \tau(x) \wedge \tau(y)$ for all $x, y \in X$.

3. Quasi-Cayley Fuzzy Graphs on Fuzzy Quasi-Cayley Groups

In this section, we introduce and discuss the notions of quasi-Cayley fuzzy graphs on quasigroups and quasi-Cayley fuzzy graphs of fuzzy quasi-subgroups of the quasigroup.
Quasi-Cayley fuzzy graph on a quasigroup can be defined as follows.

Definition 3.1. Let $Q = (G, \cdot, \backslash, /)$ be a quasigroup with (right) identity e, and let $\xi \in F(G)$ be a fuzzy quasi-subgroup. Suppose that $\sigma \subseteq \xi$ such that σ^*, the support of σ is a non-empty subset of G, $\sigma(e)=0$, $\sigma(e/x)=\sigma(e\backslash x)=\sigma(x)$ and $\sigma(x*y^{-1})\leq\xi(x)\wedge\xi(y)$, for all $x,y\in G$ and $*\in\{\cdot,\backslash,/)$. The fuzzy graph $X=(G,\xi,\nu)$ such that mapping ν is defined by

$$\nu(x,y)=\sigma(x*y^{-1}),\ for\ all\ (x,y)\in G^2$$

where $*\in\{\cdot,\backslash,/)$ is called the quasi-Cayley fuzzy graph of ξ (fuzzy quasi-subgroup) in G relative to σ. We will denote it by $CayF(Q,\xi,\sigma)$.

The fuzzy subset ξ with the above properties is referred to as a Cayley fuzzy subset of σ in G. Moreover, It is easy to observe that (G,ξ,σ) is a fuzzy graph, since for any $(x,y)\in G^2$, we have

$$\nu(x,y)=\sigma(x*y^{-1})\leq\xi(x)\wedge\xi(y).$$

Remark 3.2. Every Cayley fuzzy graph (on a fuzzy group) is a quasi-Cayley fuzzy graph (on a fuzzy quasigroup).

By a quasi-Cayley subset of a quasigroup Q we mean a subset A of Q such that whenever $a\in A$ then $a^{-1}\in A$ and A generates Q.

Lemma 3.3. *Let σ be a quasi-Cayley fuzzy subset of a fuzzy sub-quasigroup ξ in G. For any $a\in(0,1]$, if $\sigma_a\neq\phi$, then σ_a is a quasi-Cayley subset of ξ_a.*

Proof. Suppose that $\xi\in F(G)$ is a fuzzy sub-quasigroup of G and σ is a quasi-Cayley fuzzy subset of ξ in G. If $\sigma_a, a\in(0,1]$ is nonempty, then for any $x\in G$,

$$x\in\sigma_a\Rightarrow\sigma(x)\geq a\Rightarrow\xi(x)\geq a\Rightarrow x\in\xi_a,$$

therefore, $\sigma_a\subseteq\xi_a$. Also for any $x\in G$, we have:

$$x\in\sigma_a\Leftrightarrow\sigma(x)\geq a, x\neq e\Leftrightarrow\sigma(x^{-1})\geq a, x^{-1}\neq e\Leftrightarrow x^{-1}\in\sigma_a.$$

Hence, σ_a is a quasi-Cayley subset of ξ_a. □

For given quasi-Cayley fuzzy graph $CayF(G,\xi,\sigma)$, if $\sigma_a\neq\phi, a\in(0,1]$, the quasi-Cayley graph $Cay(G,\sigma_a)$ is called $a-$level Cayley graph of $CayF(G,\xi,\sigma)$.

Theorem 3.4. *Let $V=CayF(G,\xi,\sigma)$ be a quasi-Cayley fuzzy graph. If for $a\in(0,1], \sigma_a\neq\phi$, then $V_a=Cay(\xi_a,\sigma_a)$.*

Proof. Obviously, the vertex set of V_a is ξ_a. Denote the membership function of the set G^2 by ν, i.e, $\nu(x,y) = \sigma(xy^{-1}), (x,y) \in G^2$. Then $X_a = (\xi_a, \nu_a)$. For both elements $x,y \in \xi_a$, we have:

$$(x,y) \in \nu_a \Leftrightarrow \nu(x,y) \geq a$$

$$\Leftrightarrow \sigma(xy^{-1}) \geq a$$

$$\Leftrightarrow xy^{-1} \in \sigma_a,$$

the proof is complete. □

It is easy to see that for a quasi-Cayley fuzzy subset σ of $\xi \in F(G)$, σ^* is a quasi-Cayley subset of $\xi^* = G$, so similar to the above theorem, we can conclude the following proposition.

Proposition 3.5. *Let $V = CayF(G,\xi,\sigma)$ be a quasi-Cayley fuzzy graph. Then $V^* = Cay(\xi^*,\sigma^*) = Cay(G,\sigma^*)$.*

Conclusion. One can easily shift all the results described in [21] towards quasi-Cayley fuzzy graphs on fuzzy quasigroups.

Conflict of Interests

The author(s) declare that there is no conflict of interests.

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