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COMPLEX DYNAMICS OF A DISCRETE-TIME MODEL WITH PREY REFUGE AND HOLLING TYPE-II FUNCTIONAL RESPONSE

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Abstract. The chaotic dynamics of a discrete-time predator-prey model with prey refuge and Holling type-II functional response are investigated. We investigate the system's existence and local stability. Using bifurcation theory, it is demonstrated that the system experiences period-doubling bifurcation and Neimark-Sacker bifurcation. Furthermore, numerical simulations are carried out to demonstrate the compatibility with analytical conclusions as well as the system's complexity.

Keywords: predator-prey; stability; period-doubling bifurcation; Neimark-Sacker bifurcation.

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1. Introduction

The predator-prey model has received a lot of attention in recent years because of its natural capacity to represent the existence and extinction of populations caused by the interaction between prey and its predator. For estimating population size, both discrete-time and continuous-time models are utilised. Though the majority of the dynamic behaviours of population models are based on continuous models driven by differential equations, discrete time models are more suited than continuous ones when the population size is rarely small or the population has

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nonoverlapping generations. It has been discovered that the dynamic behaviours of discrete systems are more complicated and have more rich dynamics than continuous systems.

The functional responses are functions of the density of prey. It represents the amount of prey devoured by each predator. It is a critical component of all predator-prey interactions in population dynamics. In 1965, Holling [1] established three categories of functional responses. Later, several researches, such as Crowley-Martin [2] and Beddington-DeAngelis [3, 4], offered many types of functional responses. Many researchers investigated systems based on predator-prey interactions, including many forms of functional responses [5, 6, 7, 8].

Nature can provide a certain level of protection to a limited number of prey populations by establishing refuges. The notion of a prey refuge has attracted the curiosity of theoretical ecologists. A prey refuge can be generally described as any approach that reduces predation risk, such as geographical or temporal refuges, prey aggregations, or decreased prey activity [9, 10, 11]. The presence of refuges can have substantial consequences on the coexistence of predators and prey, and numerous studies demonstrate that the most essential roles of prey refuges are avoiding prey extinction and dampening predator–prey oscillations [12, 14, 13].

In this work, we discuss the following continuous-time prey-predator interaction proposed in [15].

$$\begin{cases} \frac{dx}{dt} = \alpha x(1-\frac{x}{K}) - \frac{\beta(1-m)xy}{1+a(1-m)x}, \\ \frac{dy}{dt} = -\gamma y + \frac{c\beta(1-m)xy}{1+a(1-m)x}, \end{cases} \tag{1.1}$$

where $x(t)$ is the population of prey, $y(t)$ is the population of predator at any time t, α is the growth rate of prey, K is the carrying capacity of prey, β is the x removal by y, $m \in [0,1)$ represents the proportion of the prey which can take refuge to avoid predation, a represents x when y is half, c is the conversion of x biomass into y biomass, and γ is the reduction of y due to other factors.

We use the forward Euler technique on system (1.1) to produce the discrete-time predator-prey system shown below.

$$\begin{cases} x_{n+1} = x_n + h\left(\alpha x_n(1-\frac{x_n}{K}) - \frac{\beta(1-m)x_n y_n}{1+a(1-m)x_n}\right), \\ y_{n+1} = y_n + h\left(-\gamma y_n + \frac{c\beta(1-m)x_n y_n}{1+a(1-m)x_n}\right), \end{cases} \tag{1.2}$$

where $h > 0$ is the step length of the Euler scheme.

The paper is organized as follows: In section 2, the existence and stability of fixed points of system (1.2) are discussed. In section 3, we discuss local bifurcation analysis at fixed points of system (1.2) by using center manifold theorem and bifurcation theory. Some numerical examples are offered in section 4 to validate our theoretical conclusions. Some final thoughts are included in the section 5.

2. EXISTENCE AND STABILITY OF FIXED POINTS

In this section we studied the existence and stability conditions for the fixed points of system (1.2). By simple algebraic computations, we obtain that system (1.2) has three fixed points.

$$E_0(0,0),\ E_1(K,0),\ E_2\left(\frac{\gamma}{(1-m)(c\beta-\gamma a)}, \frac{\alpha c\gamma(1+a(1-m)K)(P_0-1)}{K(1-m)^2(c\beta-\gamma a)^2}\right),$$

where $P_0 = \frac{c\beta(1-m)K}{\gamma(1+a(1-m)K)}$. The first fixed point E_0 represents that both prey and predator populations goes to extinction. The second fixed point E_1 represents that in the absence of predators, the prey population approaches its carrying capacity K. Note that the first two fixed points E_0 and E_1 exist for all positive parametric values but the third fixed point exists only if $c\beta - \gamma a > 0$. For biologically purposes the fixed point E_2 is meaningful to discuss because E_2 is the only positive fixed point if $c\beta - \gamma a > 0$. Our main objective is to study local stability and bifurcation at the fixed point E_2.

The local stability of fixed points of the system (1.2) depends on the eigenvalues of the Jacobian matrix evaluated at the fixed points of the system (1.2). The Jacobian matrix of the system (1.2) evaluated at the point $(\bar{x},\bar{y})$ is

$$J(\bar{x},\bar{y}) = \begin{bmatrix} 1+\frac{\alpha h(K-2\bar{x})}{K}+\frac{\beta h(-1+m)\bar{y}}{(-1+a(-1+m)\bar{x})^2} & -\frac{\beta h(-1+m)\bar{x}}{-1+a(-1+m)\bar{x}} \\ -\frac{\beta ch(-1+m)\bar{y}}{(-1+a(-1+m)\bar{x})^2} & 1-\gamma h+\frac{\beta ch(-1+m)\bar{x}}{-1+a(-1+m)\bar{x}} \end{bmatrix} \tag{2.1}$$

To investigate the stability of fixed points of system (1.2), we use the following results.

Lemma 2.1. [16] *Let $F(w) = w^2 + Aw + B$ be the characteristic equation of the eigenvalues associated to the Jacobian matrix evaluated at a fixed point $(\bar{x},\bar{y})$ and w_1, w_2 are two roots of $F(w) = 0$, then $(\bar{x},\bar{y})$ is* *(i) sink and therefore locally asymptotically stable if* $|w_{1,2}| < 1$,

(ii) source and therefore unstable if $|w_{1,2}| > 1$,

(iii) saddle point if $|w_1| < 1$ *and* $|w_2| > 1$ *(or* $|w_1| > 1$ *and* $|w_2| < 1$*)*,

(iv) non-hyperbolic if either $|w_1| = 1$ *or* $|w_2| = 1$.

Lemma 2.2. [16] *Let* $F(w) = w^2 + Aw + B$. *Assume that* $F(1) > 0$. *If* w_1, w_2 *are two roots of* $F(w) = 0$, *then*

(i) $|w_{1,2}| < 1$ *iff* $F(-1) > 0$ *and* $B < 1$,

(ii) $|w_1| < 1$ *and* $|w_2| > 1$ *(or* $|w_1| > 1$ *and* $|w_2| < 1$*) iff* $F(-1) < 0$,

(iii) $|w_1| > 1$ *and* $|w_2| > 1$ *iff* $F(-1) > 0$ *and* $B > 1$,

(iv) $w_1 = -1$ *and* $|w_2| \neq 1$ *iff* $F(-1) = 0$ *and* $A \neq 0, 2$,

(v) w_1 *and* w_2 *are complex and* $|w_{1,2}| = 1$ *iff* $A^2 - 4B < 0$ *and* $B = 1$.

The Jacobian matrix evaluated at E_2 is

$$J(E_2) = \begin{bmatrix} 1 + \frac{\alpha\gamma h(\beta(c+acK(-1+m))+a\gamma(1+a(K-Km)))}{\beta cK(\beta c-a\gamma)(-1+m)} & -\frac{\gamma h}{c} \\ \frac{\alpha h(\gamma+\beta cK(-1+m)+a\gamma(K-Km))}{\beta K(-1+m)} & 1 \end{bmatrix}. \tag{2.2}$$

The characteristic polynomial of $J(E_2)$ is

$$F(w) = w^2 + Aw + B,$$

where

$$A = -2 + S_1 h,\ B = 1 - S_1 h + S_2 h^2$$

$$S_1 = -\frac{\alpha\gamma(\beta(c + acK(-1+m)) + a\gamma(1 + aK(1-m)))}{\beta cK(\beta c - a\gamma)(-1+m)},\ S_2 = \frac{\alpha\gamma(\gamma + \beta cK(-1+m) + a\gamma K(1-m))}{\beta cK(-1+m)}.$$

By simple computations, we obtain

$$F(0) = 1 - S_1 h + S_2 h^2,$$

$$F(1) = S_2 h^2 > 0,$$

$$F(-1) = 4 - 2S_1 h + S_2 h^2.$$

$$\begin{aligned} S_2 =& \frac{\alpha\gamma(\gamma + \beta cK(-1+m) + a\gamma K(1-m))}{\beta cK(-1+m)} \\ =& \alpha\gamma\left(1 - \frac{\gamma(1 + a(1-m)K)}{\beta cK(1-m)}\right) \\ =& \alpha\gamma(1 - \frac{1}{P_0}) \end{aligned}$$

It is clear that $F(1) > 0$ if and only if $P_0 > 1$.

Using lemma (2.2), we obtain the local dynamics of the fixed point E_2.

Proposition 2.3. *Assume that* $c\beta - \gamma a > 0$ *and* $P_0 = \frac{c\beta(1-m)K}{\gamma(1+a(1-m)K)} > 1$. *The fixed point* E_2 *of the system* (1.2) *is*

(i) a sink and therefore it is locally asymptotically stable if one of the following conditions holds

(a) $S_1 > 0,\ S_1^2 - 4S_2 < 0$ *and* $0 < h < \frac{S_1}{S_2}$,

(b) $S_1 > 0,\ S_1^2 - 4S_2 \geq 0$ *and* $0 < h < \frac{S_1 - \sqrt{S_1^2 - 4S_2}}{S_2}$,

(ii) a source and therefore it is unstable if one of the following conditions holds

(a) $S_1 \leq 0$

(b) $S_1 > 0,\ S_1^2 - 4S_2 \leq 0$ *and* $h > \frac{S_1}{S_2}$,

(c) $S_1 > 0,\ S_1^2 - 4S_2 > 0$ *and* $h > \frac{S_1 + \sqrt{S_1^2 - 4S_2}}{S_2}$,

(iii) a saddle point if the following condition holds

$S_1 > 0,\ S_1^2 - 4S_2 > 0$ *and* $\frac{S_1-\sqrt{S_1^2-4S_2}}{S_2} < h < \frac{S_1+\sqrt{S_1^2-4S_2}}{S_2}$,

(iv) non-hyperbolic point if one of the following conditions holds
(a) $S_1 > 0,\ S_1^2 - 4S_2 > 0$ *and* $h = \frac{S_1 \pm \sqrt{S_1^2-4S_2}}{S_2}$
(b) $S_1 > 0,\ S_1^2 - 4S_2 < 0$ *and* $h = \frac{S_1}{S_2}$

It is clear that if $S_1 > 0,\ S_1^2 - 4S_2 < 0$ and $h = \frac{S_1}{S_2}$, then eigenvalues of $J(E_2)$ are complex with unit modulus. Therefore, system (1.2) experiences Neimark-Sacker bifurcation at fixed point E_2 when parameters vary in a small neighbourhood of Ω_1.

$$\Omega_1 = \left\{ a,c,h,K,\alpha,\beta,\gamma \in \mathbb{R}^+, m \in [0,1) \middle| S_1 > 0,\ S_1^2 - 4S_2 < 0,\ h = \frac{S_1}{S_2} \right\}.$$

Moreover, if $S_1 > 0,\ S_1^2 - 4S_2 > 0$ and $h = \frac{S_1 \pm \sqrt{S_1^2-4S_2}}{S_2}$, then one of the eigenvalues of $J(E_2)$ is -1 and other eigenvalue λ satisfies $|\lambda| \neq 1$. Therefore a period-doubling bifurcation can occur if parameters vary in a small neighbourhood of Ω_2 or Ω_3.

$$\Omega_2 = \left\{ a,c,h,K,\alpha,\beta,\gamma \in \mathbb{R}^+, m \in [0,1) \middle| S_1 > 0,\ S_1^2 - 4S_2 > 0,\ h = \frac{S_1 + \sqrt{S_1^2 - 4S_2}}{S_2} \right\}.$$

$$\Omega_3 = \left\{ a,c,h,K,\alpha,\beta,\gamma \in \mathbb{R}^+, m \in [0,1) \middle| S_1 > 0,\ S_1^2 - 4S_2 > 0,\ h = \frac{S_1 - \sqrt{S_1^2 - 4S_2}}{S_2} \right\}.$$

3. Local Bifurcation Analysis

In this section, different bifurcation types are discussed at fixed points of the system (1.2). For detailed bifurcation theory, we refer the readers to [17, 18]. In recent years, bifurcation analysis has been extensively studied by many researchers. For instance we refer the readers to [19, 20, 21, 22, 23].

3.1. Period-Doubling Bifurcation at E_2:

In this section, we discuss period-doubling bifurcation at fixed point $E_2\left(\frac{\gamma}{(1-m)(c\beta-\gamma a)}, \frac{\alpha c\gamma(1+a(1-m)K)(P_0-1)}{K(1-m)^2(c\beta-\gamma a)^2}\right)$ for the domain Ω_3. Similar arguments can be used for the domain Ω_2.

Consider the domain

$$\Omega_3 = \left\{ a,c,h_1,K,\alpha,\beta,\gamma \in \mathbb{R}^+, m \in [0,1) \middle| S_1 > 0,\ S_1^2 - 4S_2 > 0,\ h_1 = \frac{S_1 - \sqrt{S_1^2 - 4S_2}}{S_2} \right\}.$$

Assuming that $(a,c,h_1,K,\alpha,\beta,\gamma,m) \in \Omega_3$, and δ be small perturbation in h_1, we consider the following perturbation of the system (1.2):

$$\begin{cases} x_{n+1} = x_n + (h+\delta)\left(\alpha x_n(1-\frac{x_n}{K}) - \frac{\beta(1-m)x_ny_n}{1+a(1-m)x_n}\right), \\ y_{n+1} = y_n + (h+\delta)\left(-\gamma y_n + \frac{c\beta(1-m)x_ny_n}{1+a(1-m)x_n}\right), \end{cases} \tag{3.1}$$

where $\delta, |\delta| \ll 1$, is a small perturbation parameter. We define $a_n = x_n - \frac{\gamma}{(1-m)(c\beta-\gamma a)}$, $b_n = y_n - \frac{\alpha c\gamma(1+a(1-m)K)(P_0-1)}{K(1-m)^2(c\beta-\gamma a)^2}$, to translate fixed point E_2 to origin. Under this translation map the system (3.1) becomes

$$\begin{bmatrix} a_{n+1} \\ b_{n+1} \end{bmatrix} = \begin{bmatrix} 1-S_1h_1 & -\frac{\gamma h_1}{c} \\ \frac{ch_1S_2}{\gamma} & 1 \end{bmatrix} \begin{bmatrix} a_n \\ b_n \end{bmatrix} + \begin{bmatrix} F(a_n,b_n,\delta) \\ G(a_n,b_n,\delta) \end{bmatrix}, \tag{3.2}$$

where

$$\begin{aligned} F(a_n,b_n,\delta) &= -\frac{\gamma}{c}\delta b_n + S_1\delta a_n + \frac{(\beta c - a\gamma)^2 h_1(-1+m)}{\beta c^2} a_nb_n + \frac{(\beta c - a\gamma)^2(-1+m)}{\beta c^2}\delta a_nb_n \\ &\quad - V_1a_n^2 - V_2\delta a_n^2 - V_3a_n^2b_n + V_4a_n^3 + O((|a_n|+|b_n|+|\delta|)^4), \\ G(a_n,b_n,\delta) &= \frac{cS_2}{\gamma}\delta a_n - V_5a_nb_n - V_6\delta a_nb_n + V_7a_n^2 + V_8\delta a_n^2 + V_9a_n^2b_n - V_{10}a_n^3 + O((|a_n|+|b_n|+|\delta|)^4). \end{aligned}$$

where

$$\begin{aligned} V_1 &= \frac{h_1\alpha\left(c^2(1+aK(-1+m))\beta^2 + ac(1-2aK(-1+m))\beta\gamma + a^2(-1+aK(-1+m))\gamma^2\right)}{c^2K\beta^2}, \\ V_2 &= \frac{\alpha\left(c^2(1+aK(-1+m))\beta^2 + ac(1-2aK(-1+m))\beta\gamma + a^2(-1+aK(-1+m))\gamma^2\right)}{c^2K\beta^2}, \\ V_3 &= \frac{ah_1(-1+m)^2(-c\beta+a\gamma)^3}{c^3\beta^2}, \\ V_4 &= \frac{a^2h_1(-1+m)\alpha(c\beta-a\gamma)^2(-cK(-1+m)\beta + (-1+aK(-1+m))\gamma)}{c^3K\beta^3}, \\ V_5 &= \frac{h_1(-1+m)(c\beta-a\gamma)^2}{c\beta},\ V_6 = \frac{(-1+m)(c\beta-a\gamma)^2}{c\beta}, \end{aligned}$$

$$V_7 = \frac{ah_1\alpha(c\beta - a\gamma)(cK(-1+m)\beta + \gamma + a(K-Km)\gamma)}{cK\beta^2},$$

$$V_8 = \frac{a\alpha(c\beta - a\gamma)(cK(-1+m)\beta + \gamma + a(K-Km)\gamma)}{cK\beta^2},\ V_9 = \frac{ah_1(-1+m)^2(-c\beta + a\gamma)^3}{c^2\beta^2},$$

$$V_{10} = \frac{a^2h_1(-1+m)\alpha(c\beta - a\gamma)^2(-cK(-1+m)\beta + (-1+aK(-1+m))\gamma)}{c^2K\beta^3}.$$

For $h_1 = \frac{S_1 - \sqrt{S_1^2 - 4S_2}}{S_2}$, the eigenvalues of $J(P_2)$ are $\lambda_1 = -1$ and $\lambda_2 = 3 - S_1h_1$.

Let

$$T = \begin{bmatrix} \frac{2\gamma}{c(-S_1+\sqrt{S_1^2-4S_2})} & -\frac{-S_1^2\gamma + S_1\gamma\sqrt{S_1^2-4S_2} + 2S_2\gamma}{S_2c(-S_1+\sqrt{S_1^2-4S_2})} \\ 1 & 1 \end{bmatrix}.$$

Under the following transformation

$$\begin{bmatrix} a_n \\ b_n \end{bmatrix} = T \begin{bmatrix} e_n \\ f_n \end{bmatrix}, \tag{3.3}$$

the system (3.2) becomes

$$\begin{bmatrix} e_{n+1} \\ f_{n+1} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} e_n \\ f_n \end{bmatrix} + \begin{bmatrix} F(e_n, f_n, \delta) \\ G(e_n, f_n, \delta) \end{bmatrix}, \tag{3.4}$$

where

$$\lambda_2 = \frac{2S_1^4 - 2S_1^3\sqrt{S_1^2-4S_2} - 11S_1^2S_2 + 7S_1S_2\sqrt{S_1^2-4S_2} + 12S_2^2}{S_2(-S_1^2 + S_1\sqrt{S_1^2-4S_2} + 4S_2)},$$

$$\begin{aligned} F(e_n, f_n, \delta) &= M_1e_n^2 + M_2e_n^3 + M_3e_nf_n + M_4e_n^2f_n + M_5f_n^2 + M_6e_nf_n^2 + M_7f_n^3 + M_8\delta e_n + M_9\delta e_n^2 \\ &\quad + M_{10}\delta f_n + M_{11}\delta e_nf_n + O((|e_n| + |f_n| + |\delta|)^4), \\ G(e_n, f_n, \delta) &= N_1e_n^2 + N_2e_n^3 + N_3e_nf_n + N_4e_n^2f_n + N_5f_n^2 + N_6e_nf_n^2 + N_7f_n^3 + N_8\delta e_n + N_9\delta e_n^2 \\ &\quad + N_{10}\delta f_n + N_{11}\delta e_nf_n + O((|e_n| + |f_n| + |\delta|)^4), \end{aligned}$$

$$M_1 = -(2(2\alpha(S_1-\sqrt{S_1^2-4S_2})S_2\gamma(a\beta c\gamma(1-2aK(-1+m))+a^2\gamma^2(-1+aK(-1+m))$$
$$+\beta^2c^2(1+aK(-1+m)))+(S_1-\sqrt{S_1^2-4S_2})^2S_2\beta c(\beta c-a\gamma)^2K(-1+m)$$
$$+(S_1^2-S_1\sqrt{S_1^2-4S_2}-2S_2)\gamma(\beta c-a\gamma)((-S_1+\sqrt{S_1^2-4S_2})\beta c(\beta c-a\gamma)K(-1+m)$$
$$-2a\alpha\gamma(\gamma+\beta cK(-1+m)+a\gamma(K-Km)))))$$
$$/((-S_1+\sqrt{S_1^2-4S_2})S_2(-S_1^2+S_1\sqrt{S_1^2-4S_2}+4S_2)\beta^2c^3K),$$

$$M_2 = -(8a\gamma(\beta c-a\gamma)^2(a\gamma(\alpha(S_1S_2-\sqrt{S_1^2-4S_2}S_2-S_1^2\gamma+S_1\sqrt{S_1^2-4S_2}\gamma+2S_2\gamma)(\gamma+\beta cK(-1+m))$$
$$\beta c(S_1S_2(\sqrt{S_1^2-4S_2}-3\gamma)+S_1^3\gamma-S_1^2(S_2+\sqrt{S_1^2-4S_2}\gamma)+S_2(2S_2+\sqrt{S_1^2-4S_2}\gamma))K(-1+m))$$
$$-a^2\alpha\gamma^2(S_1S_2-\sqrt{S_1^2-4S_2}S_2-S_1^2\gamma+S_1\sqrt{S_1^2-4S_2}\gamma+2S_2\gamma)K(-1+m)$$
$$+\beta^2c^2(-S_1S_2(\sqrt{S_1^2-4S_2}-3\gamma)-S_1^3\gamma+S_1^2(S_2\sqrt{S_1^2-4S_2}\gamma)-S_2(2S_2+\sqrt{S_1^2-4S_2}\gamma))$$
$$K(-1+m))(-1+m))/((S_1\sqrt{S_1^2-4S_2})^2S_2(-S_1^2+S_1\sqrt{S_1^2-4S_2}+4S_2)\beta^3c^5K),$$

$$M_3 = -((4\alpha(S_1-\sqrt{S_1^2-4S_2})(S_1^2-S_1\sqrt{S_1^2-4S_2}-2S_2)S_2\gamma(a\beta c\gamma(1-2aK(-1+m))$$
$$+a^2\gamma^2(-1+aK(-1+m))+\beta^2c^2(1+aK(-1+m)))+2(S_1^2-S_1\sqrt{S_1^2-4S_2}-2S_2)^2$$
$$S_2\beta c(\beta c-a\gamma)^2K(-1+m+2(S_1-\sqrt{S_1^2-4S_2})^2S_2^2\beta c(\beta c-a\gamma)^2K(-1+m)$$
$$+(S_1^2-S_1\sqrt{S_1^2-4S_2}-2S_2)\gamma(\beta c-a\gamma)(2(-S_1+\sqrt{S_1^2-4S_2})S_2\beta c(\beta c-a\gamma)K(-1+m)$$
$$-(-S_1+\sqrt{S_1^2-4S_2})(-S_1^2+S_1\sqrt{S_1^2-4S_2}+2S_2)\beta c(\beta c-a\gamma)K(-1+m)$$
$$-4a\alpha(S_1^2-S_1\sqrt{S_1^2-4S_2}-2S_2)\gamma(\gamma+\beta cK(-1+m)+a\gamma(K-Km)))))$$
$$/((-S_1+\sqrt{S_1^2-4S_2})S_2^2(-S_1^2+S_1\sqrt{S_1^2-4S_2}+4S_2)\beta^2c^3K),$$

$$M_4 = -(4a(S_1-\sqrt{S_1^2-4S_2})\gamma(3a\alpha(S_1-\sqrt{S_1^2-4S_2})(S_1^2S_1\sqrt{S_1^2-4S_2}-2S_2)S_2\gamma(\beta c-a\gamma)^2$$
$$(\gamma(-1+aK(-1+m))-\beta cK(-1+m))(1-m)-(S_1^2-S_1\sqrt{S_1^2-4S_2}-2S_2)\gamma(\beta c-a\gamma)^2,$$
$$(3a\alpha(S_1^2-S_1\sqrt{S_1^2-4S_2}-2S_2)\gamma(\gamma(-1+aK(-1+m))-\beta cK(-1+m))+(-S_1+\sqrt{S_1^2-4S_2})S_2\beta c$$
$$(\beta c-a\gamma)K(-1+m)-(-S_1+\sqrt{S_1^2-4S_2})(-S_1^2+S_1\sqrt{S_1^2-4S_2}+2S_2)\beta c(\beta c-a\gamma)K(-1+m))(1-m)$$
$$+(S_1\sqrt{S_1^2-4S_2})^2S_2^2\beta c(\beta c-a\gamma)^3K(-1+m)^2+(S_1-\sqrt{S_1^2-4S_2})^2S_2(-S_1^2+S_1\sqrt{S_1^2-4S_2}+2S_2)$$
$$\beta c(-\beta c+a\gamma)^3K(-1+m)^2))/((-S_1+\sqrt{S_1^2-4S_2})^3(S_1^2-S_1\sqrt{S_1^2-4S_2}-4S_2)S_2^2\beta^3c^5K),$$

$$M_5 = ((-S_1^2+S_1\sqrt{S_1^2-4S_2}+2S_2)(\alpha(S_1-\sqrt{S_1^2-4S_2})(S_1^2S_1\sqrt{S_1^2-4S_2}-2S_2)S_2\gamma(a\beta c\gamma(1-2aK(-1+m))$$
$$+a^2\gamma^2(-1+aK(-1+m))+\beta^2c^2(1+aK(-1+m)))+(S_1-\sqrt{S_1^2-4S_2})^2S_2^2\beta c(\beta c-a\gamma)^2K(-1+m)$$
$$-(-S_1+\sqrt{S_1^2-4S_2})S_2(-S_1^2+S_1\sqrt{S_1^2-4S_2}+2S_2)\beta c\gamma(\beta c-a\gamma)^2K(-1+m)$$
$$-a\alpha(S_1^2-S_1\sqrt{S_1^2-4S_2}-2S_2)^2\gamma^2(\beta c-a\gamma)(\gamma+\beta cK(-1+m)+a\gamma(K-Km))))$$
$$/((-S_1\sqrt{S_1^2-4S_2})S_2^3(-S_1^2+S_1\sqrt{S_1^2-4S_2}+4S_2)\beta^2c^3K),$$

$$M_6 = (a(S_1-\sqrt{S_1^2-4S_2})(-S_1^2+S_1\sqrt{S_1^2-4S_2}+2S_2)\gamma(\beta c-a\gamma)^2(6a\alpha(S_1-\sqrt{S_1^2-4S_2})$$
$$(S_1^2-S_1\sqrt{S_1^2-4S_2}-2S_2)S_2\gamma(\gamma(-1+aK(-1+m))-\beta cK(-1+m))-6a\alpha(S_1^2-S_1\sqrt{S_1^2-4S_2}-2S_2)^2$$
$$\gamma^2(\gamma(-1+aK(-1+m))-\beta cK(-1+m))-2(S_1^2-S_1\sqrt{S_1^2-4S_2}-2S_2)^2S_2\beta c(\beta c-a\gamma)K(-1+m)$$
$$-4(S_1-\sqrt{S_1^2-4S_2})^2S_2^2\beta c(\beta c-a\gamma)K(-1+m)-(-S_1+\sqrt{S_1^2-4S_2})(S_1^2-S_1\sqrt{S_1^2-4S_2}-2S_2)^2$$
$$\beta c\gamma(\beta c-a\gamma)K(-1+m)+4(-S_1+\sqrt{S_1^2-4S_2})S_2(S_1^2+S_1\sqrt{S_1^2-4S_2}+2S_2)$$
$$\beta c\gamma(\beta c-a\gamma)K(-1+m))(1-m))/((S_1+\sqrt{S_1^2-4S_2})^3(S_1^2-S_1\sqrt{S_1^2-4S_2}-4S_2)S_2^3\beta^3c^5K),$$

$$M_7 = (a(S_1^2-S_1\sqrt{S_1^2-4S_2}-2S_2)^2\gamma(\beta c-a\gamma)^2(S_1S_2-\sqrt{S_1^2-4S_2}S_2-S_1^2\gamma+S_1\sqrt{S_1^2-4S_2}\gamma+2S_2\gamma)$$
$$(a\gamma(\alpha(-S_1^2+S_1\sqrt{S_1^2-4S_2}+2S_2)(\gamma+\beta cK(-1+m))-(-S_1\sqrt{S_1^2-4S_2})S_2\beta cK(-1+m))$$
$$+(-S_1+\sqrt{S_1^2-4S_2})S_2\beta^2c^2K(-1+m)a^2\alpha(-S_1^2+S_1\sqrt{S_1^2-4S_2}+2S_2)\gamma^2K(-1+m))(-1+m))$$
$$/((S_1\sqrt{S_1^2-4S_2})^2S_2^4(-S_1^2+S_1\sqrt{S_1^2-4S_2}+4S_2)\beta^3c^5K),$$

$$M_8 = \frac{2S_2(-3S_1^2+3S_1\sqrt{S_1^2-4S_2}+4S_2)}{(-S_1+\sqrt{S_1^2-4S_2})(-S_1^2+S_1\sqrt{S_1^2-4S_2}+4S_2)},$$

$$M_9 = (4S_2(-\frac{2\alpha\gamma(a\beta c\gamma(1-2aK(-1+m))+a^2\gamma^2(-1+aK(-1+m))+\beta^2c^2(1+aK(-1+m)))}{K}$$
$$+(-S_1+\sqrt{S_1^2-4S_2})\beta c(\beta c-a\gamma)^2(-1+m)-\frac{1}{(S_1-\sqrt{S_1^2-4S_2})S_2K}(S_1^2-S_1\sqrt{S_1^2-4S_2}-2S_2)\gamma$$
$$(\beta c-a\gamma)((-S_1+\sqrt{S_1^2-4S_2})\beta c(\beta c-a\gamma)K(-1+m)-2a\alpha\gamma(\gamma+\beta cK(-1+m)+a\gamma(K-Km)))))$$
$$/((-S_1+\sqrt{S_1^2-4S_2})(-S_1^2+S_1\sqrt{S_1^2-4S_2}+4S_2)\beta^2c^3),$$

$$M_{10} = -\frac{4S_1(S_1^3 - S_1^2\sqrt{S_1^2-4S_2} - 3S_1S_2 + S_2\sqrt{S_1^2-4S_2})}{(-S_1+\sqrt{S_1^2-4S_2})(-S_1^2+4S_2+S_1\sqrt{S_1^2-4S_2})},$$

$$M_{11} = (S_2(\frac{1}{S_2K}4\alpha(-S_1^2+S_1\sqrt{S_1^2-4S_2}+2S_2)\gamma(a\beta c\gamma(1-2ak(-1+m))+a^2\gamma^2(-1+aK(-1+m))$$
$$+\beta^2c^2(1+aK(-1+m)))-(2(S_1^2-S_1\sqrt{S_1^2-4S_2}-2S_2)^2\gamma(-\beta c+a\gamma)(a\gamma(-2\alpha(\gamma+\beta cK(-1+m))$$
$$+S_1\beta cK(-1+m))-S_1\beta^2c^2K(-1+m)+2a^2\alpha\gamma^2K(-1+m)))$$
$$/((-S_1+\sqrt{S_1^2-4S_2})S_2^2K)+2(-S_1+\sqrt{S_1^2-4S_2})\beta c(\beta c-a\gamma)^2(-1+m)$$
$$-\frac{(-S_1+\sqrt{S_1^2-4S_2})(-S_1^2+S_1\sqrt{S_1^2-4S_2}+2S_2)\beta c(\beta c-a\gamma)^2(-1+m)}{S_2}))$$
$$/((-S_1+\sqrt{S_1^2-4S_2})(S_1^2+S_1\sqrt{S_1^2-4S_2}+4S_2)\beta^2c^3),$$

$$N_1 = -((4(\alpha\gamma(-S_1(a\beta c\gamma(1-2aK(-1+m))+a^2\gamma^2(-1+aK(-1+m))+\beta^2c^2(1+aK(-1+m)))$$
$$+a^2\gamma^2(\sqrt{S_1^2-4S_2}+2\gamma)(-1+aK(-1+m))-a\beta c\gamma(\sqrt{S_1^2-4S_2}+2\gamma)(-1+2aK(-1+m))$$
$$+\beta^2c^2(\sqrt{S_1^2-4S_2}+a(\sqrt{S_1^2-4S_2}+2\gamma)K(-1+m)))+\beta c(\beta c-a\gamma)^2(-S_1^2+2S_2-\sqrt{S_1^2-4S_2}\gamma$$
$$+S_1(\sqrt{S_1^2-4S_2}+\gamma))K(-1+m)))/((S_1+\sqrt{S_1^2-4S_2})(-S_1^2+S_1\sqrt{S_1^2-4S_2}+4S_2)\beta^2c^3K)),$$

$$N_2 = (8a\gamma(\beta c-a\gamma)^2(a\gamma(-\alpha(-S_1+\sqrt{S_1^2-4S_2}+2\gamma)(\gamma+\beta cK(-1+m))+\beta c(-S_1^2+2S_2-\sqrt{S_1^2-4S_2}\gamma$$
$$+S_1(\sqrt{S_1^2-4S_2}+\gamma))K(-1+m))+a^2\alpha\gamma^2(-S_1+\sqrt{S_1^2-4S_2}+2\gamma)K(-1+m)+$$
$$\beta^2c^2(S_1^2-2S_2+\sqrt{S_1^2-4S_2}\gamma-S_1(\sqrt{S_1^2-4S_2}+\gamma))K(-1+m))(-1+m))$$
$$/((S_1-\sqrt{S_1^2-4S_2})^2(-S_1^2+S_1\sqrt{S_1^2-4S_2}+4S_2)\beta^3c^5K),$$

$$N_3 = (2(2\alpha(S_1-\sqrt{S_1^2-4S_2})(S_1^2-S_1\sqrt{S_1^2-4S_2}-2S_2)\gamma(a\beta c\gamma(1-2aK(-1+m))$$
$$+a^2\gamma^2(-1+aK(-1+m))+\beta^2c^2(1+aK(-1+m)))(S_1^2-S_1\sqrt{S_1^2-4S_2}-2S_2)^2\beta c(\beta c-a\gamma)^2K(-1+m)$$
$$+(S_1\sqrt{S_1^2-4S_2})^2S_2\beta c(\beta c-a\gamma)^2K(-1+m)+\gamma(\beta c-a\gamma)(2(-S_1+\sqrt{S_1^2-4S_2})S_2\beta c(\beta c-a\gamma)K(-1+m)$$
$$-(-S_1+\sqrt{S_1^2-4S_2})(S_1^2+S_1\sqrt{S_1^2-4S_2}+2S_2)\beta c(\beta c-a\gamma)K(-1+m)-4a\alpha(S_1^2S_1\sqrt{S_1^2-4S_2}-2S_2)\gamma$$
$$(\gamma+\beta cK(-1+m)+a\gamma(K-Km)))))/((S_1+\sqrt{S_1^2-4S_2})S_2(-S_1^2+S_1\sqrt{S_1^2-4S_2}+4S_2)\beta^2c^3K),$$

$$N_4 = (8a\gamma(\beta c - a\gamma)^2(a\gamma(-3\alpha(-S_1^3 + S_1^2(\sqrt{S_1^2 - 4S_2} + \gamma) - S_2(\sqrt{S_1^2 - 4S_2} + 2\gamma) + S_1(3S_2 - \sqrt{S_1^2 - 4S_2}\gamma))$$
$$(\gamma + \beta cK(-1+m)) + \beta c(-2S_1^4 + 2S_1^3(\sqrt{S_1^2 - 4S_2} + \gamma) - S_1S_2(3\sqrt{S_1^2 - 4S_2} + 5\gamma)$$
$$+ S_1^2(7S_2 - 2\sqrt{S_1^2 - 4S_2}\gamma) + S_2(-2S_2 + \sqrt{S_1^2 - 4S_2}\gamma))K(-1+m)) + 3a^2\alpha\gamma^2(-S_1^3 + S_1^2(\sqrt{S_1^2 - 4S_2} + \gamma)$$
$$- S_2(\sqrt{S_1^2 - 4S_2} + 2\gamma) + S_1(3S_2 - \sqrt{S_1^2 - 4S_2}\gamma))K(-1+m) - \beta^2c^2(-2S_1^4 + 2S_1^3(\sqrt{S_1^2 - 4S_2} + \gamma)$$
$$- S_1S_2(3\sqrt{S_1^2 - 4S_2} + 5\gamma) + S_1^2(7S_2 - 2\sqrt{S_1^2 - 4S_2}\gamma) + S_2(-2S_2 + \sqrt{S_1^2 - 4S_2}\gamma))$$
$$K(-1+m))(-1+m))/((S_1 - \sqrt{S_1^2 - 4S_2})^2S_2(-S_1^2 + S_1\sqrt{S_1^2 - 4S_2} + 4S_2)\beta^3c^5K),$$

$$N_5 = ((S_1^2 - S_1\sqrt{S_1^2 - 4S_2} - 2S_2)(\alpha(S_1 - \sqrt{S_1^2 - 4S_2})(S_1^2 - S_1\sqrt{S_1^2 - 4S_2} - 2S_2)\gamma(a\beta c\gamma(1 - 2aK(-1+m))$$
$$+ a^2\gamma^2(-1 + aK(-1+m)) + \beta^2c^2(1 + aK(-1+m))) + 2(S_1^2 - S_1\sqrt{S_1^2 - 4S_2} - 2S_2)S_2\beta c(\beta c - a\gamma)^2$$
$$K(-1+m) + 2\gamma(\beta c - a\gamma)((-S_1 + \sqrt{S_1^2 - 4S_2})S_2\beta c(\beta c - a\gamma)K(-1+m) - a\alpha(S_1^2 - S_1\sqrt{S_1^2 - 4S_2} - 2S_2)$$
$$\gamma(\gamma + \beta cK(-1+m) + a\gamma(K - Km)))))/((-S_1 + \sqrt{S_1^2 - 4S_2})S_2^2(-S_1^2 + S_1\sqrt{S_1^2 - 4S_2} + 4S_2)\beta^2c^3K),$$

$$N_6 = (a(S_1 - \sqrt{S_1^2 - 4S_2})\gamma(6a\alpha(S_1 - \sqrt{S_1^2 - 4S_2})(S_1^2 - S_1\sqrt{S_1^2 - 4S_2} - 2S_2)^2\gamma(\beta c - a\gamma)^2$$
$$(\gamma(-1 + aK(-1+m)) - \beta cK(-1+m))(1-m) - 2(S_1^2 - S_1\sqrt{S_1^2 - 4S_2} - 2S_2)\gamma(\beta c - a\gamma)^2$$
$$(6a\alpha(S_1^2 - S_1\sqrt{S_1^2 - 4S_2} - 2S_2)\gamma(\gamma(-1 + aK(-1+m)) - \beta cK(-1+m)) + 4(S_1 + \sqrt{S_1^2 - 4S_2})S_2\beta c$$
$$(\beta c - a\gamma)K(-1+m) - (-S_1 + \sqrt{S_1^2 - 4S_2})(S_1^2 + S_1\sqrt{S_1^2 - 4S_2} + 2S_2)\beta c(\beta c - a\gamma)K(-1+m))(1-m)$$
$$+ (S_1\sqrt{S_1^2 - 4S_2})^2(S_1^2 - S_1\sqrt{S_1^2 - 4S_2} - 2S_2)^2\beta c(\beta c - a\gamma)^3K(-1+m)^2 + 4(S_1 - \sqrt{S_1^2 - 4S_2})^2$$
$$S_2(-S_1^2 + S_1\sqrt{S_1^2 - 4S_2} + 2S_2)\beta c(\beta c + a\gamma)^3K(-1+m)^2))$$
$$/((-S_1 + \sqrt{S_1^2 - 4S_2})^3(S_1^2 - S_1\sqrt{S_1^2 - 4S_2} - 4S_2)S_2^2\beta^3c^5K),$$

$$N_7 = (4a\gamma(\beta c - a\gamma)^2(S_1^5 - S_1^4(\sqrt{S_1^2 - 4S_2} + \gamma) - S_2^2(\sqrt{S_1^2 - 4S_2} + 2\gamma) + S_1^2S_2(3\sqrt{S_1^2 - 4S_2} + 4\gamma)$$
$$+ S_1S_2(5S_2 - 2\sqrt{S_1^2 - 4S_2}\gamma) + S_1^3(-5S_2 + \sqrt{S_1^2 - 4S_2}\gamma))(a\gamma(-\alpha(S_1^2 + S_1\sqrt{S_1^2 - 4S_2} + 2S_2)$$
$$(\gamma + \beta cK(-1+m)) + (-S_1 + \sqrt{S_1^2 - 4S_2})S_2\beta cK(-1+m)) - (-S_1 + \sqrt{S_1^2 - 4S_2})S_2\beta^2c^2K(-1+m)$$
$$+ a^2\alpha(S_1^2 + S_1\sqrt{S_1^2 - 4S_2} + 2S_2)\gamma^2K(-1+m))(-1+m))$$
$$/((S_1 - \sqrt{S_1^2 - 4S_2})^2S_2^3(-S_1^2 + S_1\sqrt{S_1^2 - 4S_2} + 4S_2)\beta^3c^5K),$$

$$N_8 = \frac{4S_1S_2}{S_1^2 - S_1\sqrt{S_1^2-4S_2} - 4S_2},$$

$$\begin{aligned}N_9 = {} & (8S_2(\alpha\gamma(-S_1(a\beta c\gamma(1-2aK(-1+m)) + a^2\gamma^2(-1+aK(-1+m)) + \beta^2c^2(1+aK(-1+m))) \\ & + a^2\gamma^2(\sqrt{S_1^2-4S_2}+2\gamma)(-1+aK(-1+m)) - a\beta c\gamma(\sqrt{S_1^2-4S_2}+2\gamma)(-1+2aK(-1+m)) \\ & + \beta^2c^2(\sqrt{S_1^2-4S_2} + a(\sqrt{S_1^2-4S_2}+2\gamma)K(-1+m))) + \beta c(\beta c - a\gamma)^2(-S_1^2+2S_2-\sqrt{S_1^2-4S_2}\gamma \\ & + S_1(\sqrt{S_1^2-4S_2}+\gamma))K(-1+m)))/((S_1-\sqrt{S_1^2-4S_2})^2(-S_1^2+S_1\sqrt{S_1^2-4S_2}+4S_2)\beta^2c^3K),\end{aligned}$$

$$N_{10} = -\frac{2(-S_1^4+S_1^3\sqrt{S_1^2-4S_2}+S_1^2S_2+S_1\sqrt{S_1^2-4S_2}S_2+4S_2^2)}{(-S_1+\sqrt{S_1^2-4S_2})(-S_1^2+S_1\sqrt{S_1^2-4S_2}+4S_2)},$$

$$\begin{aligned}N_{11} = {} & (S_2(-\frac{1}{S_2K}4\alpha(-S_1^2+S_1\sqrt{S_1^2-4S_2}+2S_2)\gamma(a\beta c\gamma(1-2aK(-1+m)) + a^2\gamma^2(-1+aK(-1+m)) \\ & + \beta^2c^2(1+aK(-1+m))) - 2(S_1+\sqrt{S_1^2-4S_2})\beta c(\beta c - a\gamma)^2(-1+m) \\ & + \frac{(-S_1+\sqrt{S_1^2-4S_2})(S_1^2+S_1\sqrt{S_1^2-4S_2}+2S_2)\beta c(\beta c-a\gamma)^2(-1+m)}{S_2} \\ & + 2\gamma(\beta c - a\gamma)(-2\beta c(\beta c - a\gamma)(-1+m) + \frac{(-S_1^2+S_1\sqrt{S_1^2-4S_2}+2S_2)\beta c(\beta c-a\gamma)(-1+m)}{S_2} \\ & - \frac{4a\alpha(S_1^2-S_1\sqrt{S_1^2-4S_2}-2S_2)\gamma(\gamma+\beta cK(-1+m)+a\gamma(K-Km))}{(S_1-\sqrt{S_1^2-4S_2})S_2K}))) \\ & /((-S_1+\sqrt{S_1^2-4S_2})(-S_1^2+S_1\sqrt{S_1^2-4S_2}+4S_2)\beta^2c^3).\end{aligned}$$

Next, we determine the center manifold $W^C(0,0,0)$ for (3.4), which can be represented as follows:

$$W^C(0,0,0) = \left\{(e_n, f_n, \delta) \in \mathbb{R}^3 \middle| f_n = c_1e_n^2 + c_2e_n\delta + c_3\delta^2 + O((|e_n|+|\delta|)^3)\right\},$$

where

$$c_1 = \frac{N_1}{1-\lambda_2},\ c_2 = -\frac{N_8}{1+\lambda_2},\ c_3 = 0.$$

Thus the system (3.4) restricted to the center manifold is given by

$$\tilde{F}: e_{n+1} = -e_n + M_1 e_n^2 + M_8 \delta e_n + (M_2 + \frac{M_3 N_1}{1-\lambda_2}) e_n^3 - \frac{M_{10} N_8}{1+\lambda_2} \delta^2 e_n$$

(3.5) $$+ (M_9 + \frac{M_{10} N_1}{1-\lambda_2} - \frac{M_3 N_8}{1+\lambda_2}) \delta e_n^2 + O((|e_n| + |\delta|)^4).$$

In order for map (3.5) to undergo period-doubling bifurcation, we require that the following two quantities l_1 and l_2 are non-zero, where

$$l_1 = \tilde{F}_\delta \tilde{F}_{e_n e_n} + 2\tilde{F}_{e_n \delta}\Big|_{(0,0)}, \; l_2 = \frac{1}{2}(\tilde{F}_{e_n e_n})^2 + \frac{1}{3}\tilde{F}_{e_n e_n e_n}$$

From simple computations, we obtain

$$l_1 = 2M_8, \; l_2 = 2(M_1^2 + M_2 + \frac{M_3 N_1}{1-\lambda_2}).$$

Due to the above analysis, we have the following result.

Theorem 3.1. *The system* (1.2) *experiences period-doubling bifurcation at the fixed point* $E_2\left(\frac{\gamma}{(1-m)(c\beta-\gamma a)}, \frac{\alpha c\gamma(1+a(1-m)K)(P_0-1)}{K(1-m)^2(c\beta-\gamma a)^2}\right)$ *if* $l_1, l_2 \neq 0$ *and* h *varies in a small neighbourhood of* $h_1 = \frac{S_1 - \sqrt{S_1^2 - 4S_2}}{S_2}$. *Moreover, if* $l_2 > 0$ *(respectively* $l_2 < 0$*), then the period-2 orbits that bifurcate from* $E_2\left(\frac{\gamma}{(1-m)(c\beta-\gamma a)}, \frac{\alpha c\gamma(1+a(1-m)K)(P_0-1)}{K(1-m)^2(c\beta-\gamma a)^2}\right)$ *are stable (respectively, unstable).*

3.2. Neimark-Sacker Bifurcation at $E_2\left(\frac{\gamma}{(1-m)(c\beta-\gamma a)}, \frac{\alpha c\gamma(1+a(1-m)K)(P_0-1)}{K(1-m)^2(c\beta-\gamma a)^2}\right)$:

In this section, we discuss Neimark-Sacker(NS) bifurcation at fixed point $E_2\left(\frac{\gamma}{(1-m)(c\beta-\gamma a)}, \frac{\alpha c\gamma(1+a(1-m)K)(P_0-1)}{K(1-m)^2(c\beta-\gamma a)^2}\right)$ for the domain Ω_1.

Consider the domain

$$\Omega_1 = \left\{ a, c, h_1, K, \alpha, \beta, \gamma \in \mathbb{R}^+, m \in [0,1) \middle| S_1 > 0, \; S_1^2 - 4S_2 < 0, \; h_2 = \frac{S_1}{S_2} \right\}.$$

Assuming that $(a, c, h_2, K, \alpha, \beta, \gamma, m) \in \Omega_1$, and δ be small perturbation in h_2, we consider the following perturbation of the system (1.2):

(3.6) $$\begin{cases} x_{n+1} = x_n + (h+\delta)\left(\alpha x_n(1 - \frac{x_n}{K}) - \frac{\beta(1-m)x_n y_n}{1+a(1-m)x_n}\right), \\ y_{n+1} = y_n + (h+\delta)\left(-\gamma y_n + \frac{c\beta(1-m)x_n y_n}{1+a(1-m)x_n}\right), \end{cases}$$

We define $a_n = x_n - \frac{\gamma}{(1-m)(c\beta-\gamma a)}$, $b_n = y_n - \frac{\alpha c\gamma(1+a(1-m)K)(P_0-1)}{K(1-m)^2(c\beta-\gamma a)^2}$, to translate fixed point $E_2\left(\frac{\gamma}{(1-m)(c\beta-\gamma a)}, \frac{\alpha c\gamma(1+a(1-m)K)(P_0-1)}{K(1-m)^2(c\beta-\gamma a)^2}\right)$ to origin. Under this translation map the system (3.6) becomes

$$\begin{bmatrix} a_{n+1} \\ b_{n+1} \end{bmatrix} = \begin{bmatrix} 1-\frac{S_1(S_1+S_2\delta)}{S_2} & -\frac{\gamma(S_1+S_2\delta)}{cS_2} \\ \frac{c(S_1+S_2\delta)}{\gamma} & 1 \end{bmatrix} \begin{bmatrix} a_n \\ b_n \end{bmatrix} + \begin{bmatrix} F(a_n,b_n) \\ G(a_n,b_n) \end{bmatrix}, \tag{3.7}$$

where

$$F(a_n,b_n) = W_1 a_n b_n - W_2 a_n^2 - W_3 a_n^2 b_n + W_4 a_n^3 + O((|a_n|+|b_n|)^4),$$

$$G(a_n,b_n) = -W_5 a_n b_n + W_6 a_n^2 + W_7 a_n^2 b_n - W_8 a_n^3 + O((|a_n|+|b_n|)^4).$$

where

$$W_1 = \frac{(\beta c - a\gamma)^2(-1+m)(S_1+S_2\delta)}{S_2\beta c^2},$$
$$W_2 = -\frac{\alpha(a\beta c\gamma(1-2aK(-1+m)) + a^2\gamma^2(-1+aK(-1+m))\beta^2c^2(1+aK(-1+m))(S_1+S_2\delta)}{S_2\beta^2c^2K},$$
$$W_3 = -\frac{a(-\beta c + a\gamma)^3(-1+m)^2(S_1+S_2\delta)}{S_2\beta^2c^3},$$
$$W_4 = \frac{a^2\alpha(\beta c - a\gamma)^2(\gamma(-1+aK(-1+m)) - \beta cK(-1+m))(-1+m)(S_1+S_2\delta)}{S_2\beta^3c^3K},$$
$$W_5 = -\frac{(\beta c - a\gamma)^2(-1+m)(S_1+S_2\delta)}{S_2\beta c},$$
$$W_6 = \frac{a\alpha(\beta c - a\gamma)(\gamma + \beta cK(-1+m) + a\gamma(K-Km))(S_1+S_2\delta)}{S_2\beta^2cK},$$
$$W_7 = \frac{a(-\beta c + a\gamma)^3(-1+m)^2(S_1+S_2\delta)}{S_2\beta^2c^2},$$
$$W_8 = -\frac{a^2\alpha(\beta c - a\gamma)^2(\gamma(-1+aK(-1+m)) - \beta cK(-1+m))(-1+m)(S_1+S_2\delta)}{S_2\beta^3c^2K}.$$

The characteristic equation of the linearized part of the system (3.7) at the fixed point $(0,0)$ is

$$\lambda^2 - p(\delta)\lambda + q(\delta) = 0, \tag{3.8}$$

where

$$p(\delta) = 2 - \frac{S_1^2}{S_2} - S_1\delta,$$
$$q(\delta) = 1 + S_1\delta + S_2\delta^2.$$

The roots of the equation (3.8) are complex with the property $|\lambda_{1,2}| = 1$, which are given by

$$\lambda_{1,2} = \frac{p(\delta) \pm i\sqrt{4q(\delta) - p^2(\delta)}}{2}.$$

By computations, we obtain

$$|\lambda_1| = |\lambda_2| = \sqrt{q(\delta)}$$

and

$$\left(\frac{d|\lambda_1|}{d\delta}\right)_{\delta=0} = \left(\frac{d|\lambda_2|}{d\delta}\right)_{\delta=0} = \frac{S_1}{2} > 0.$$

Moreover, it is required that $\lambda_1^i, \lambda_2^i \neq 1$ for $i = 1,2,3,4$ at $\delta = 0$ which is equivalent to $p(0) \neq \pm 2, 0, 1$. Since $S_1 > 0,\ S_1^2 - 4S_2 < 0$ and $p(0) = 2 - \frac{S_1^2}{S_2}$, therefore $p(0) \neq \pm 2$. We only need to require that $p(0) \neq 0, 1$, which leads to $S_1^2 \neq 2S_2, S_2$.

To transform the linear part of (3.7) into its canonical form at $\delta = 0$, we use the aforementioned transformation:

$$\begin{bmatrix} a_n \\ b_n \end{bmatrix} = \begin{bmatrix} -\frac{S_1\gamma}{cS_2} & 0 \\ \frac{S_1^2}{2S_2} & -\frac{S_1\sqrt{4S_2 - S_1^2}}{2S_2} \end{bmatrix} \begin{bmatrix} e_n \\ f_n \end{bmatrix}. \tag{3.9}$$

Under the transformation (3.9), the system (3.7) becomes

$$\begin{bmatrix} e_{n+1} \\ f_{n+1} \end{bmatrix} = \begin{bmatrix} \mu & -\nu \\ \nu & \mu \end{bmatrix} \begin{bmatrix} e_n \\ f_n \end{bmatrix} + \begin{bmatrix} F(e_n, f_n) \\ G(e_n, f_n) \end{bmatrix}, \tag{3.10}$$

where

$$\mu = 1 - \frac{S_1^2}{2S_2},\ \nu = \frac{S_1\sqrt{4S_2 - S_1^2}}{2S_2},$$
$$F(e_n, f_n) = M_1e_n^2 + M_2e_n^3 + M_3e_nf_n + M_4e_n^2f_n + O((|e_n| + |f_n|)^4),$$
$$G(e_n, f_n) = N_1e_n^2 + N_2e_n^3 + N_3e_nf_n + N_4e_n^2f_n + O((|e_n| + |f_n|)^4),$$

$$M_1 = \frac{1}{2S_2^2\beta^2c^3}S_1^2(\frac{1}{K}2(1-\frac{S_1^2}{S_2})\gamma(a\beta c\gamma(1-2aK(-1+m))+a^2\gamma^2(-1+aK(-1+m))$$
$$+\beta^2c^2(1+aK(-1+m)))+S_1\beta c(\beta c-a\gamma)^2(-1+m)),$$
$$M_2 = \frac{1}{2S_2^4\beta^3c^5K}aS_1^3\gamma(\beta c-a\gamma)^2(a\gamma(2S_1^2(\gamma+\beta cK(-1+m))-2S_2(\gamma+\beta cK(-1+m))S_1S_2$$
$$\beta cK(-1+m))-S_1S_2\beta^2c^2K(-1+m)-2a^2(S_1^2-S_2)\gamma^2K(-1+m))(-1+m),$$
$$M_3 = -\frac{S_1^2\sqrt{-S_1^2+4S_2}(\beta c-a\gamma)^2(-1+m)}{2S_2^2\beta c^2},$$
$$M_4 = -\frac{aS_1^3\sqrt{-S_1^2+4S_2}\gamma(-\beta c+a\gamma)^3(-1+m)^2}{2S_2^3\beta^2c^4},$$
$$N_1 = (S_1^3(-2S_1(S_1^2-S_2)\gamma(a\beta c\gamma(1-2aK(-1+m))+a^2\gamma^2(-1+aK(-1+m)$$
$$+\beta^2c^2(1+aK(-1+m))+S_1^2S_2\beta c(\beta c-a\gamma)^2K(-1+m)-2S_1S_2\beta c\gamma(\beta c-a\gamma)^2K(-1+m)$$
$$+4a(S_1^2-S_2)\gamma^2(\beta c-a\gamma)(\gamma+\beta cK(-1+m)+a\gamma(K-Km))))/(2S_2^3\sqrt{-S_1^4+4S_1^2S_2}\beta^2c^3K),$$

$$N_2 = -((aS_1^3(S_1-2\gamma)\gamma(\beta c-a\gamma)^2(a\gamma(-2S_1^2(\gamma+\beta cK(-1+m))+2S_2(\gamma+\beta cK$$
$$(-1+m))-S_1S_2\beta cK(-1+m)+S_1S_2\beta^2c^2K(-1+m)$$
$$+2a^2(S_1^2-S_2)\gamma^2K(-1+m))(-1+M))/(2S_2^4\sqrt{-S_1^2+4S_2}\beta^3c^5K)),$$
$$N_3 = -\frac{S_1^2(S_1-2\gamma)(\beta c-a\gamma)^2(-1+m)}{2S_2^2\beta c^2},$$
$$N_4 = -\frac{aS_1^3(S_1-2\gamma)\gamma(-\beta c+a\gamma)^3(-1+m)^2}{2S_2^3\beta^2c^4}.$$

In a system with NS bifurcation, the aforementioned value L determines the direction in which the invariant curve occurs.

$$L = \left(\left[-Re\left(\frac{(1-2\lambda_1)\lambda_2^2}{1-\lambda_1}\eta_{20}\eta_{11}\right)-\frac{1}{2}|\eta_{11}|^2-|\eta_{02}|^2+Re(\lambda_2\eta_{21})\right]\right)_{\delta=0},$$

where

$$\eta_{20} = \frac{1}{8}\left[F_{e_ne_n} - F_{f_nf_n} + 2G_{e_nf_n} + i(G_{e_ne_n} - G_{f_nf_n} - 2F_{e_nf_n})\right],$$
$$\eta_{11} = \frac{1}{4}\left[F_{e_ne_n} + F_{f_nf_n} + i(G_{e_ne_n} + G_{f_nf_n})\right],$$
$$\eta_{02} = \frac{1}{8}\left[F_{e_ne_n} - F_{f_nf_n} - 2G_{e_nf_n} + i(G_{e_ne_n} - G_{f_nf_n} + 2F_{e_nf_n})\right],$$
$$\eta_{21} = \frac{1}{16}\left[F_{e_ne_ne_n} + F_{e_nf_nf_n} + G_{e_ne_nf_n} + G_{f_nf_nf_n} + i(G_{e_ne_ne_n} + G_{e_nf_nf_n} - F_{e_ne_nf_n} - F_{f_nf_nf_n})\right].$$

We derive the aforementioned theorem for the presence and direction of NS bifurcation from the above computations.

Theorem 3.2. *Suppose that $S_1 > 0, S_1^2 - 4S_2 < 0$ and $S_1^2 \neq 2S_2, S_2$. If $L \neq 0$, then the system* (1.2) *undergoes NS bifurcation at the fixed point $E_2\left(\frac{\gamma}{(1-m)(c\beta-\gamma a)}, \frac{\alpha c\gamma(1+a(1-m)K)(P_0-1)}{K(1-m)^2(c\beta-\gamma a)^2}\right)$ when the parameter h varies within a neighbourhood of $h_2 = \frac{S_1}{S_2}$. In addition, an attracting invariant closed curve bifurcates from the fixed point if $L < 0$, and a repelling invariant closed curve bifurcates from the fixed point if $L > 0$.*

4. Numerical Examples

In this section, we will provide some numerical simulations to back up our theoretical analysis of the model's multiple qualitative characteristics. We consider the following set of parameter values for bifurcation analysis.

TABLE 1. Parameter values

Cases	Fixed parameters and initial conditions	varying parameter
Case (i)	$a = 0.002,\ c = 0.01,\ K = 50,\ \alpha = 2,\ \beta = 0.1,$ $\gamma = 0.02,\ m = 0.01,\ x_0 = 21,\ y_0 = 12.$	$2.5 \leq h \leq 2.99$
Case (ii)	$a = 0.002,\ c = 0.01,\ K = 200,\ \alpha = 2,\ \beta = 0.1,$ $\gamma = 0.02,\ m = 0.01,\ x_0 = 20,\ y_0 = 20.$	$3.5 \leq h \leq 4.5$

Example 4.1. *Period-Doubling bifurcation of the model* (1.2) *at E_2 with respect to bifurcation parameter h.*

We take parameters values as in case (i) *of table* (1)*. The positive fixed point of* (1.2) *for these parametric values is* $E_2(21.0438, 12.187)$*. The eigenvalues of* $J(E_2)$ *for* $h = 2.60959$ *are* $\lambda_1 = -1, \lambda_2 = 0.924279$*, indicating that the model* (1.2) *is experiencing period doubling bifurcation at* $E_2(21.0438, 12.187)$ *as the bifurcation parameter* h *crosses* $h = h_1 = 2.60959$*. Figures (1a, 1b) show bifurcation diagrams for both prey and predator populations, respectively, for* $h \in [2.5, 2.99]$*. These figures express that fixed point* $E_2(21.0438, 12.187)$ *is locally asymptotically stable for* $0 < h < 2.60959$*, but loses its stability at* $h = 2.60959$*, where the model* (1.2) *undergoes period-doubling bifurcation. Moreover, for these values it is obtained that* $l_1 = 1.77408$ *and* $l_2 = 96.3329$*.*

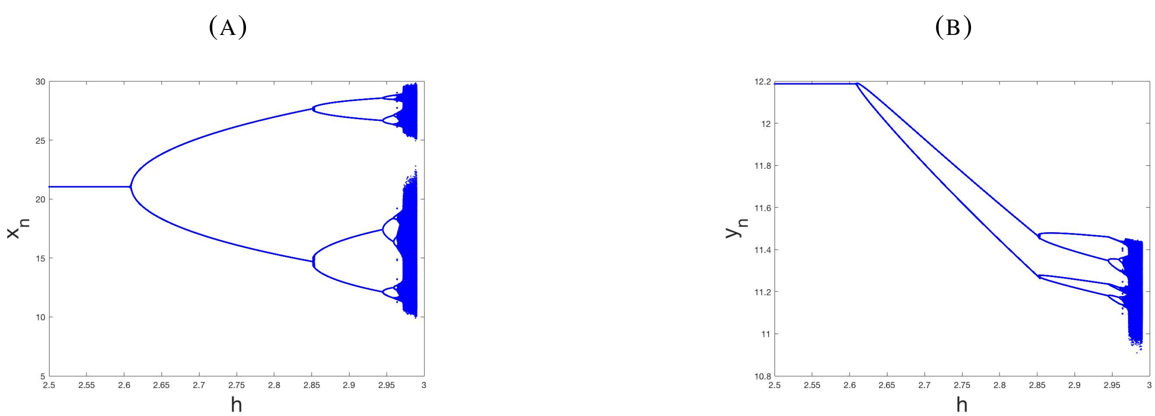

FIGURE 1. Bifurcation diagrams for case (i) set of values of table (1).

Example 4.2. *Neimark-Sacker bifurcation of the model* (1.2) *at* E_2 *with respect to bifurcation parameter* h*.*

We take parameters values as in case (ii) *of table* (1)*. The positive fixed point of* (1.2) *for these parametric values is* $E_2(21.0438, 18.8296)$*. The eigenvalues of* $J(E_2)$ *for* $h = 4.04124$ *are* $\lambda_1 = 0.719427 - 0.694568i, \lambda_2 = 0.719427 + 0.694568i$*, indicating that the model* (1.2) *is experiencing Neimark-Sacker bifurcation at* $E_2(21.0438, 18.8296)$ *as the bifurcation parameter* h *crosses* $h = h_2 = 4.04124$*. Figures (2a, 2b) show bifurcation diagrams for both prey and predator populations, respectively, for* $h \in [3.5, 4.5]$*.*

The fixed point E_2 *is locally asymptotically stable for these parametric values if and only if* $0 < h < 4.04124$*. Figures (2c,2d) show phase portraits of the model* (1.2) *for some values of h. These figures express that fixed point* $E_2(21.0438, 18.8296)$ *is locally asymptotically stable for* $0 < h < 4.04124$*, but loses its stability at* $h = 4.04124$*, where the model* (1.2) *undergoes Neimark-Sacker bifurcation. An invariant closed curve appears at* $h = 4.04124$ *and it increases its radius as h increases. Moreover, for these values it is obtained that* $L = -0.00350112$*.*

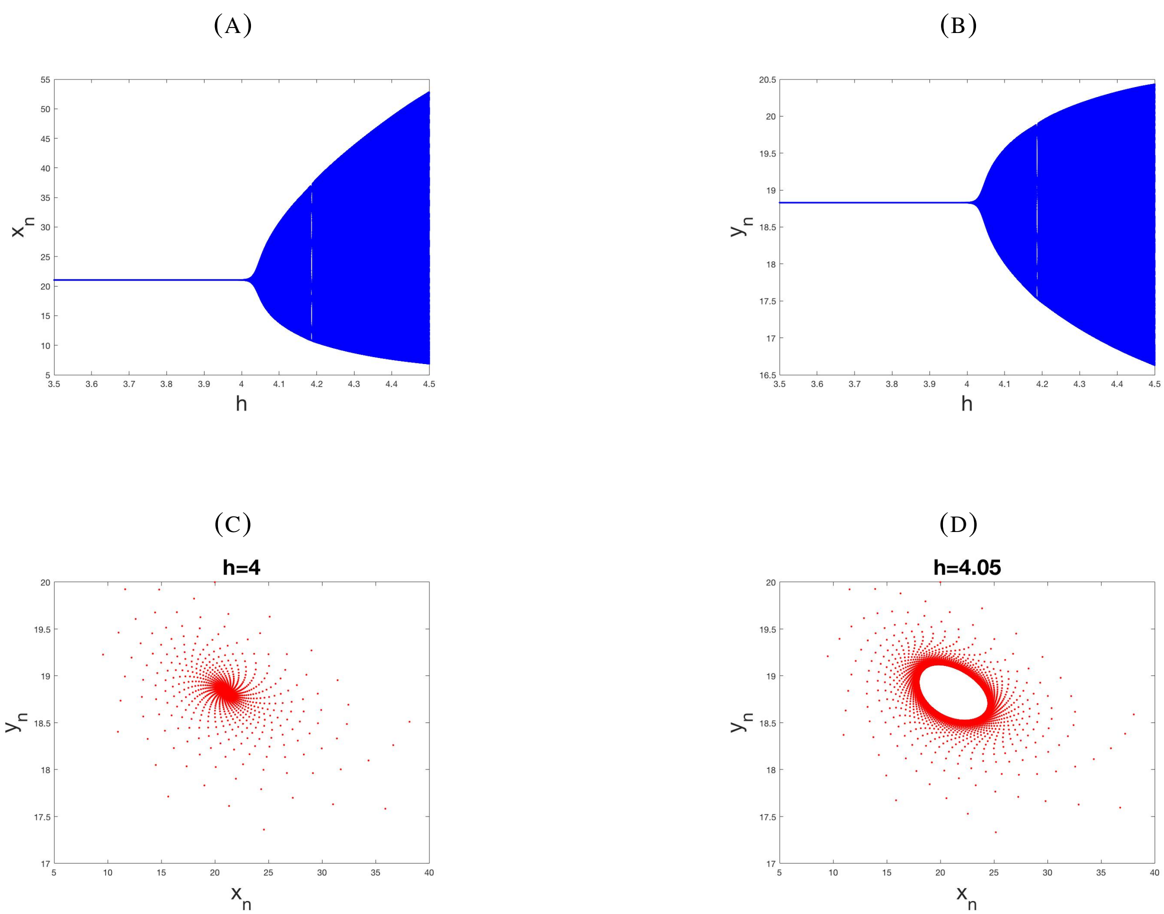

FIGURE 2. Bifurcation diagrams, phase portraits for some values of h for case (ii) set of values of table (1).

5. CONCLUSION

In this study, we explored the nonlinear dynamics of a discrete-time predator-prey model with Holling type-II functional response and prey refuge produced using the forward Euler discretization approach. Using bifurcation theory and the centre manifold theorem, it is demonstrated that the positive fixed point of the system has period-doubling bifurcation and Neimark-Sacker bifurcation. Based on the pictures, we can see that a small integral step size h can stabilise the dynamical system (1.2), but a big integral step size can destabilise the system, resulting in more complicated dynamical behaviours.

Conflict of Interests

The author(s) declare that there is no conflict of interests.

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