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ON THE DIVISOR GRAPH OF FINITE COMMUTATIVE RING

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Abstract. In this paper, we introduce a graphical structure of non empty finite commutative ring R called as divisor graph of R, denoted as $\mathbb{D}[R]$, is undirected simple graph with vertex set V= $R-\{0,1\}$ and for distinct vertices $a,b \in V, a \sim b$ if and only if either $a \mid b$ or $b \mid a$, i.e. $\exists\ c \in R$ such that $a = bc$ or $b = ac$. We will discuss structure and properties of divisor graph of ring Z_n. Moreover, we also determine diameter, girth, eulerian, planar, clique number of the $\mathbb{D}[Z_n], \forall$ n. The main objective of this paper is to study interplay of ring theoretic properties of R with graph theoretic properties of $\mathbb{D}[Z_n]$.

Keywords: graph; Eulerian graph; connected graph; girth; clique number; chromatic number.

2010 AMS Subject Classification: 05C25, 05C45, 05C69, 05C07, 05C12.

1. Introduction

The study of zero divisor graph was initiated by I. Beck[6] in 1988. He introduced graph to commutative ring with vertex set as set of all zero divisors. Then Anderson and Livingston[4] has changed vertex set which was defined by I.Beck[6] and studied the properties of zero divisor

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graph over the commutative ring. Later on many researcher studied properties of graph on various algebraic structures such as group, semi group, commutative ring, non commutative ring, field, vector space in [1, 2, 3, 5, 7, 8, 9]. Recently B. S. Reddy, R. S. Jain, N. Laxmikanth[3], studied Eulerians of zero divisor graph Z_n for natural number n. In the current era S. Akabari, M. Habibi[9], studied zero divisor graph on ideals of the ring. Later on Anshuman Das studied Non-Zero Component Union Graph of a Finite Dimensional Vector Space[1], Subspace Inclusion Graph of a Vector Space[2], R. A. Muneshwar and K. L. Bondar[7], introduced an open subset inclusion graph of a topological space and discussed some properties of this graph such as diameter, girth, connectivity, maximal independent sets, different variants of domination number, clique number and chromatic number, degree and connectivity. R. A. Muneshwar and K. L. Bondar[8], introduced an the intersection graph of a topological space and proved that if (X,τ) is the discrete topological space and $|X| \geq 3$ then this graph is a connected and also find its diameter and girth.

In this paper we introduced graph on ring of integer modulo n, denoted by $\mathbb{D}[Z_n]$ called divisor graph of Z_n and studied some properties of the graph $\mathbb{D}[Z_n]$. The main objective of this paper is to study interplay of ring theoretic properties of Z_n with graph theoretic properties of $\mathbb{D}[Z_n]$.

2. DEFINITION AND PRELIMINARIES

In this section we recall some notations and basic definitions of ring theory and graph theory. An ordered pair $G = (V,E)$ is called graph where V is set of vertices and E is set of edges, is the binary relation on V. If there is an edge between any two vertices u,v of V then they are said to be adjacent vertices. $H = (W,F)$ is subgraph of $G = (V,E)$ where $\phi \neq W \subseteq V$ and $F \subseteq E$. If V is finite, the graph G is said to be finite, otherwise graph is infinite. If all the vertices of G are pairwise adjacent, then G is said to be complete graph. A complete subgraph of a graph G is called a clique. A clique with maximum size is called clique number of graph G. It is written as $\omega(G)$,. The chromatic number of G, denoted as $\chi(G)$, is the minimum number of colours needed to label the vertices so that the adjacent vertices receive different colours.

A graph is said to be triangulated if for any vertex u in V, there exist v,w in V, such that (u,v,w) is a triangle. A path in graph G is a sequence of adjacent vertices and edges.For vertices x and y of G, let $d(x,y)$ be the lenth of a shortest path from vertex x to y. clearly $d(x,x) = 0$

and $d(x,y) = \infty$ if there is no path connecting x and y.The diameter of a graph G is defined as $diam(G) = Sup\{d(u,v) : u$ and v are vertices of $G\}$, is the largest distance between pairs of vertices of the graph, if it exists. Otherwise, $diam(G)$ is defined as ∞. The girth of a graph is the length of its shortest cycle, if it exists. Otherwise, it is defined as ∞. If a and b belong to a commutative ring R and a is non zero, we say that a divides b (or that a is a factor of b), write as $a \mid b$, if there exists an element c in R such that $b = ac$. If a does not divide b, we write $a \nmid b$. A zero divisor is a element r in a ring R, such that $r \cdot s = 0$ for some non zero s in R. The greatest integer function $[x]$ indicates an integral part of the real number x which is the nearest and smallest integer to x. For $n \geqslant 1$, the Euler's phi function $\phi(n)$ denote the number of positive integers not exceeding n that are relatively prime to n.

3. Divisor Graph of Ring

Definition 3.1 Let R be any commutative ring. We associate a simple undirected graph to ring R denoted by $\mathbb{G}[R]$ with vertex set $V = R$ and for non zero $r_1, r_2 \in V, r_1 \sim r_2$ if and only if either $r_1|r_2$ or $r_2|r_1$, i.e. $\exists\ r_3 \in R$ such that $r_1 = r_2r_3$ or $r_2 = r_1r_3$.

Note that if $r \neq 0 \in R$, then $r \sim 0$ and $r \sim 1$(if unity is exist). To avoid this triviality we will redefine vertex set.

Let divisor graph of ring with vertex set $V = \{r \in R | r \neq 0, r \neq 1\}$ and for distinct $r_1, r_2 \in V, r_1 \sim r_2$ or $(r_1, r_2) \in E$ if and only if either $r_1|r_2$ or $r_2|r_1$. We will denote this graph by $\mathbb{D}[R]$ and observe that $\mathbb{D}[R]$ is induced subgraph of $\mathbb{G}[R]$. We feel $\mathbb{D}[R]$ will better illustrate the structure of ring R.

Example 3.2 For n=2, the $\mathbb{D}[Z_n]$ is empty graph. Since vertex set of $\mathbb{D}[R]$ is $R - \{0,1\}$, hence $\mathbb{D}[Z_n]$ is empty graph.

Example 3.3 For n=3, $\mathbb{D}[Z_n]$ is single vertex graph.

Example 3.4 For n=4,5,6, $\mathbb{D}[Z_n]$ is as follows,

FIGURE 1. $\mathbb{D}[z_4]$

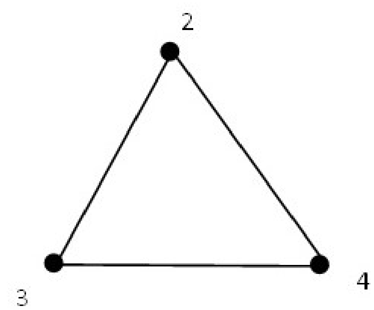

FIGURE 2. $\mathbb{D}[Z_5]$

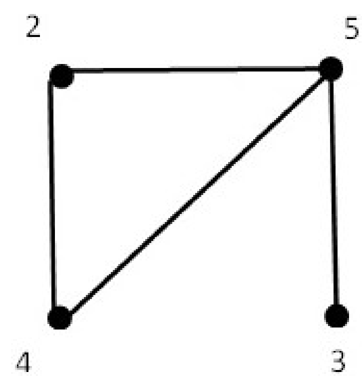

FIGURE 3. $\mathbb{D}[Z_6]$

After above discussion we will describe structure of $\mathbb{D}[Z_n]$.

Structure of $\mathbb{D}[Z_n]$

Let R be ring of integer modulo n, where $n = p_1^{k_1} p_2^{k_2} \cdots p_m^{k_m}$. A vertex set $V(\mathbb{D}[Z_n]), V = Z_n - \{0,1\} = \{2,3,\cdots,(n-1)\} = A \cup B$, where $A = U(R) - \{1\}$ and $B = Z(R) - \{0\}$. Since by definition 3.1, units are adjacent to each other, then we divide set of zero divisor into subsets V_i,

$B = \cup V_i$, where $V_i = \langle p_i \rangle, i = 1,2,3,\cdots,m$. Since not all vertices of V_i are adjacent to vertices of V_j. Define subsets of V_i by such that for $i \mid n, j|n$, $W_i = \{k(i)|k \in R, k \notin \langle j \rangle, i \nmid j, j \nmid i, i \neq j\}$ and $W_i \cap W_j = \phi$. Let W_i and W_j are subset of vertex set V, then $W_i \leftrightarrow W_j$ denote that each vertex of W_i is adjacent to every vertex of W_j and $W_i \not\leftrightarrow W_j$ denote that no vertex of W_i is adjacent to any vertex of W_j. A loop at subset A of V denote vertices of A are mutually adjacent.

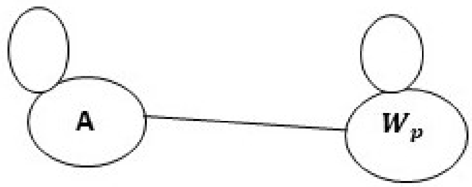

FIGURE 4. $\mathbb{D}[Z_{p^k}]$

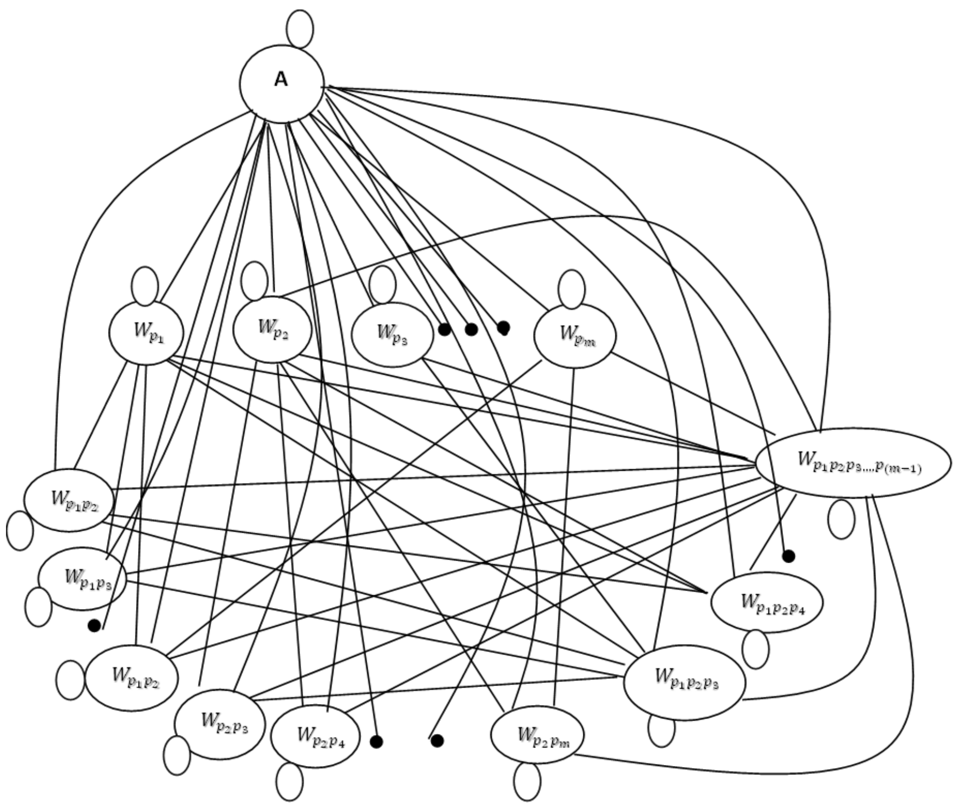

FIGURE 5. $\mathbb{D}[Z_{p_1p_2..p_m}]$

Theorem 3.5 If R is ring and I is subring of R then $\mathbb{D}[I]$ is subgraph of $\mathbb{D}[R]$.

Proof: The proof is follows from definition 3.1

Theorem 3.6 If $u \in U(R)$, where $U(R)$ is set of units of R then $u \sim y, \forall y \in R$.

Proof: Let u is unit element and y any other element in ring R then by definition of unit, we get $u^{-1} \in R, u \cdot u^{-1} = 1$. Multiplying this expression by y, we obtain $y \cdot u \cdot u^{-1} = y \cdot 1$, As R is commutative, finally we get expression $u \cdot k = y$, where $k = y \cdot u^{-1}$. Thus by divisibility relation $u \sim y$, $\forall y \in R$. Hence theorem is proved.

Corollary 3.7 If vertices u and v are associates in ring R, then $u \sim v$.

Proof: Let u and v are associate in R then there exists unit $w \in R$ such that $u = wv$. Thus $u \sim v$. Since w is unit in R therefore $\exists w^{-1}$ such that $w^{-1}u = w^{-1}wv$ i.e. $w^{-1}u = v$. Hence $v \sim u$.

Corollary 3.8 If $U(R)$ is set of unit element of R then $\mathbb{D}[U(R)]$ is complete subgraph of $\mathbb{D}[R]$

Proof: Let R be finite ring and $U(R)$ be its set of unit elements. Let $u, v \in U(R)$ are any two distinct elements then from theorem 3.6, $u \sim v$, $\forall\ u, v \in U(R)$. Thus $\mathbb{D}[U(R)]$ is complete subgraph of $\mathbb{D}[R]$.

Theorem 3.9 $\mathbb{D}[Z_n]$ is connected graph.

Proof: Let R is ring of integer modulo n and $r_1, r_2 \in V(\mathbb{D}[Z_n])$.

Claim: There exits a path connecting r_1 and r_2.

Since elements of Z_n are either unit or a zero divisor. If either $r_1 \in U(R)$ or $r_2 \in U(R)$ then theorem 3.6, shows that $r_1 \sim r_2$. If $r_1, r_2 \in Z^*(R)$ and $r_1 \nsim r_2$, then $\exists r_3 \in U(R)$ such that $r_1 \sim r_3 \sim r_2$. Hence we obtain path connecting r_1 and r_2. This show that $\mathbb{D}[Z_n]$ is connected graph.

Theorem 3.10 $\mathbb{D}[Z_n]$ is triangulated for $n \geq 5$ and not triangulated for $n = 1, 2, 3, 4, 6$.

Proof: At first we show that $\mathbb{D}[Z_n]$ is not triangulated for $n = 6$.

case I: For n=6, a vertex set V is $\{2, 3, 4, 5\}$ where $A = \{5\}, B = \{2, 3, 4\}$, then from figure 3, vertex $3 \sim 5$ only. i.e. 3 is not vertex of triangle. Hence $\mathbb{D}[Z_n]$ is not triangulated for $n = 6$

case II: For $n = 1, 2, 3, 4$ A vertex set is ϕ,$\{2\}$ and $\{2, 3\}$ respectively. Therefore $\mathbb{D}[Z_n]$ is not triangulated.

case III: For $n \geq 5$, number of units are $\phi(5) = 5 - 1 = 4$. Since all units adjacent to every elements of ring. i.e. every vertex is vertex of triangle and hence theorem is proved.

4. DIVISOR GRAPH OF Z_n

Theorem 4.1 $\mathbb{D}[R]$ is empty graph if and only if $R = Z_1, R = Z_2$.

Proof: The proof follows from definition 3.1.

Theorem 4.2 For ring of integer modulo n, $\mathbb{D}[Z_n]$ is complete graph iff $n = p^k, k \geq 1$.

Proof: Since $V(\mathbb{D}[Z_n]), V = \{2,3,....,(n-1)\} = A \cup B$ where $A = U(R) - \{1\}$ and $B = Z(R) - \{0\}$. Let $u, v \in V$ be arbitrary vertices and we claim that $u \sim v$. If either u or v in A then from theorem 3.6, vertex $u \sim v$. If both $u, v \in B$,then for some $r_1, r_2 \in R$ such that $u < v, u = r_1 p$ and $v = r_2 p$. If v is not multiple of u and $gcd(u,v) = p$ then $gcd(r_1, r_2) = 1$. If possible both r_1 and r_2 are in $Z(R)$ then $gcd(r_1, r_2) = p$ or $gcd(r_1, r_2) = q$. Which is contradiction to our supposition. Hence either r_1 or r_2 will be unit, if r_1 is unit, we obtain vertex $r_3 \in R$ such that $r_2 = r_3 r_1$, Multiply last expression by p. Finally we obtain $u = r_3 v$. Thus $\mathbb{D}[Z_{p^k}]$ is complete graph.

Conversely consider $\mathbb{D}[Z_n]$ is complete graph and assume that, $n \neq p^k$ i.e. n can be expressed as product of power of distinct primes. For simplicity let $n = pq, p \neq q$. Since $p, q \in V$ and $p \nmid q$, hence graph is disconnected. Which is contradiction. Thus $\mathbb{D}[Z_n]$ is complete graph if and only if $n = p^k$.

Corollary 4.3 If R is finite field then $\mathbb{D}[R]$ is complete graph. Converse need not be true.

Proof: The proof follows from theorem 4.2. For converse see figure 1.

Theorem 4.4 For $n = p^k$, zero divisor graph $\Gamma(Z_n)$ is complete subgraph of divisor graph $\mathbb{D}[Z_n]$.

Proof: Since $V = Z_n - \{0,1\} = A \cup B$ where $A = U(R) - \{1\}$ and $B = Z(R) - \{0\}$ i.e. $V(\Gamma[Z_n]) \subseteq V(\mathbb{D}[Z_n])$ and by theorem 4.2 , $\mathbb{D}[Z_{p^k}]$ is complete graph. Thus theorem is proved.

Theorem 4.5 The divisor graph $\mathbb{D}[Z_{p^k}]$ is Eulerian graph if p is odd prime.

Proof: Since by theorem 4.2, divisor graph $\mathbb{D}[Z_{p^k}]$ is complete graph K_{p^k-2}, then of every vertex has degree $p^k - 3$. If p is odd[even] then $p^k - 3$ is even[odd]. Hence by theorem 2.4[3], $\mathbb{D}[Z_{p^k}]$ is Eulerian if p is odd prime.

Theorem 4.6 If R is ring of integer modulo n and $n = p_1^{k_1} \cdot p_2^{k_2} \cdot p_3^{k_3} \cdots p_m^{k_m}$, then for $d_i | n$,

$$\begin{cases} u_i \sim v_i & \text{if } u_i, v_i \in W_{d_i}, \\ u_i \nsim v_j & \text{if } u_i \in W_{p_i^{k_i}} \text{ and } v_j \in W_{p_j^{k_j}}, i \neq j. \end{cases}$$

Proof: Let $R = Z_n$ and d_i be divisors of n $\forall i$

Case I: Let $W_{d_i} = \{s_i(d_i) | s_i \in R, s_i \notin < d_j >, d_i \nmid d_j, d_j \nmid d_i, i \neq j\}$ and $u_i, v_i \in W_{d_i}$ be any two vertices, then for some choice of $r_1, r_2 \in R, u_i = r_1 d_i, v_i = r_2 d_i$. If $u_i | v_i$ then from definition 3.1, $u_i \sim v_i$. If v_i is not multiple of u_i then $gcd(u_i, v_i) = d, gcd(r_1, r_2) = d'$ such that $d_i | d' | d$, Note that s_i is either multiple of d_i or a unit element. Choose $r'_1, r'_2 \in R$, such that $r_1 = r'_1 d', r_2 = r'_2 d', gcd(r'_1, r'_2) = d''$ and $d_i | d'' | d' | d$. Continue this fassion, after k steps we obtain $gcd(r_1^{(k)}, r_2^{(k)}) = 1$ and $1 | d_i | \cdots d'' | d' | d$, where either $r_1^{(k)}$ or $r_2^{(k)}$ is unit element. If $r_1^{(k)}$ is unit then by theorem 3.6, $r_2^{(k)} = k\dot{r}_1^{(k)}$ for some $k \in R$. Hence by substituting this values we get $v_i | u_i$. Thus $v_i \sim u_i$.

Case III: Let $u_i \in W_{p_i^{k_i}}, v_j \in W_{p_j^{k_j}}, i \neq j$, where $W_{p_i^{k_i}} = \{s_i(p_i^{k_i}) | s_i \in R, s_i \notin < p_j >, i \neq j\}$, $W_{p_j^{k_j}} = \{s_j(p_j^{k_j}) | s_j \in R, s_j \notin < p_i >, i \neq j\}$. Assume that, $u_i \sim v_j$ then either $u_i | v_j$ or $v_j | u_i$. i.e. $v_j \in W_{p_i^{k_i}}$ or $u_i \in W_{p_j^{k_j}}$. Which is contradiction to assumption. Hence $u_i \nsim v_j$.

Theorem 4.7 If $n = p_1^{k_1} \cdot p_2^{k_2} \cdot p_3^{k_3} \cdots p_m^{k_m}, k_i \geq 2$, then $\Gamma(Z_n)$ is not subgraph of $\mathbb{D}[Z_n]$.

Proof: Let $n = p_1^{k_1} \cdot p_2^{k_2} \cdot p_3^{k_3} \cdots p_m^{k_m}$ and $V = A \cup B$, where $A = U(R) - \{1\}$ and $B = Z(R) - \{0\}$. It is observed that B is vertex set of zero divisor graph. Then we obtain $u = p_1^{k_1} \cdot p_2^{k_2} \cdot p_3^{k_3} \cdots p_{m-1}^{k_{m-1}}$ and $v = p_m$ in set B, such that $u \cdot v = 0$. i.e. $u \sim v$ in $\Gamma(Z_n)$, but $u \nsim v$ in $\mathbb{D}[R]$, Since $u \nmid v$. Thus result is valid.

Theorem 4.8 If $n = p_1^{k_1} \cdot p_2^{k_2} \cdot p_3^{k_3} \cdots p_m^{k_m}, m > 1$ then $\mathbb{D}[Z_n]$ is not Eulerian graph if p_i is even prime for some i.

Proof: Let $R = Z_n$ and $n = p_1^{k_1} \cdot p_2^{k_2} \cdot p_3^{k_3} \cdots p_m^{k_m}$ and vertex set of $\mathbb{D}[Z_n], V = A \cup B$ where $A = U(R) - \{1\}, B = Z(R) - \{0\}$ and $|A| = \phi(n) - 1, |B| = n - \phi(n) + 1$. Suppose that p_1 is even prime number then then all multiples of p_1 and all units are adjacent to p_1. Therefore

$$\begin{aligned} deg(p_1) &= |A| + \left[\frac{n}{p_1}\right] - 2. \ (\because V = R - \{0, 1\}) \\ &= even - 1 + even - 2 \\ &= odd \end{aligned}$$

Hence by theorem 2.4[3], It shows that, $\mathbb{D}[R]$ is not Eulerian if p_i is even prime for some i.

5. Diameter and Girth of $\mathbb{D}[Z_n]$:

Theorem 5.1 If R is ring of integer modulo n then

$$diam(\mathbb{D}[R]) = \begin{cases} \infty & \text{if } n \leq 3. \\ 1 & \text{if } n > 3, n \doteq p^k, \\ 2 & \text{if } n > 3, n \neq p^k \end{cases}$$

Proof: In ring Z_n, The co-totient function $\phi(n)$ and $n - \phi(n)$ counts no. of units and no.of zero divisors respectively. Since $Diam(G) = Sup\{d(u,v) : u, v \text{ are vertices of graph }\}$. Let $n \leq 3$, a vertex V is either empty or a singleton set, $Diam(\mathbb{D}[Z_n]) = \infty$. If $n = p^k$ then by theorem 4.2, divisor graph $\mathbb{D}[Z_n]$ is complete graph then $Diam(\mathbb{D}[Z_n]) = 1$. Now for $n > 3, n \neq p^k$ divisor graph $\mathbb{D}[Z_n]$ is not a complete graph. i.e. we will get non adjacent vertices (say) r_1, r_2. Then by theorem 3.6,$\exists r_3 \in R$ such that $r_1 \sim r_3 \sim r_2$. This shows that $Diam(\mathbb{D}[Z_n]) = 2$.

Theorem 5.2 If R is ring of integer modulo n then

$$girth(\mathbb{D}[Z_n]) = \begin{cases} \infty & \text{if } n < 5 \\ 3 & \text{if } n \geq 5. \end{cases}$$

Proof: Let $n < 5$, $| V(\mathbb{D}[Z_n]) | < 3$ which is not enough to form cycle and hence $girth(\mathbb{D}[R]) = \infty$. Let $n \geq 5$ consider following cases.

Case I: If $n \geq 5, n = p^k$ then by theorem 4, $\mathbb{D}[Z_{p^k}]$ is complete graph of order $K_{p^k - 2}$. Thus $girth(\mathbb{D}[Z_n]) = 3$.

Case II: If $n \geq 5, n \neq p^k, n \neq 6$. Vertex set V contains more than two unit elements. These unit elements will form 3-cycle with any other elements of ring. For $n = 6$ see FIG. 3, Thus $girth(\mathbb{D}[Z_n]) = 3$.

6. Clique Number of Z_n :

In this section, we calculate clique number of divisor graph. A clique number is largest complete subgraph of graph.

Theorem 6.1 If $n = p^k, \omega(\mathbb{D}[Z_n]) = p^k - 2$, for some integer $k \geqslant 1$.

Proof: Since $\mathbb{D}[Z_{p^k}]$ is complete graph of order $p^k - 2$. This shows that $\omega(\mathbb{D}[Z_{p^k}]) = p^k - 2$.

Theorem 6.2 If $n = p_1 p_2, p_1 < p_2$ then $\omega(\mathbb{D}[Z_n]) = \phi(n) - 1 + p_2 - 1$.

Proof: Let R is ring of integer modulo n and $n = p_1 p_2$. Let vertex set $V = A \cup B$, where $A = U(R) - \{1\}$ and $B = Z(R) - \{0\}$. Let the subset B be divided as $B = W_{p_1} \cup W_{p_2}$, where $W_{p_1} = \{k_1 p_1 | k_1 \neq 0 \in R, k_1 \notin \langle p_2 \rangle\}$ and $W_{p_2} = \{k_2 p_2 | k_2 \neq 0 \in R, k_2 \notin \langle p_1 \rangle\}$. From figure 5, for $u, v \in W_{p_i}, i = 1,2$ then $u \sim v$. As $p_1 < p_2$, then multiples of p_1 are more than multiples of p_2. Hence by theorem 3.6, maximum clique is of order $\mid A \mid + \mid W_{p_1} \mid = \phi(n) - 1 + p_2 - 1$. Thus $\omega(\mathbb{D}[Z_n]) = \phi(n) - 1 + p_2 - 1$.

Theorem 6.3 For $n = p_1 \cdot p_2 \cdot p_3 \cdots p_m$, where $p_1 < p_j, j = 2,3 \cdots, m$ then

$$\omega(\mathbb{D}[Z_n]) = \phi(n) - 1 + (p_m - 1)[(p_{m-1} - 1)[\ldots[(p_2 - 1) + 1]\ldots] + 1]$$

Proof: Let R is ring of integer modulo n and $n = p_1 \cdot p_2 \cdot p_3 \cdots p_m$ Then let $A = U(R) - \{1\}$ and $Z(R) = \cup_i V_i$ such that $V_i = \langle p_i \rangle, i = 1,2,\ldots,m$, Since $p_1 p_j \nmid p_1 p_s, j \neq s, s = j = 2,3,\ldots,m$ and $p_1 | p_1 p_2 | p_1 p_2 p_3 | \cdots p_1 p_2 \cdots p_{(m-2)} | p_1 p_2 \cdots p_{(m-1)}$. We delete some vertices which makes problem to form complete subgraph. From figure 5 and theorem 4.6, $W_{p_1} \leftrightarrow W_{p_1 p_2} \leftrightarrow W_{p_1 p_2 p_3} \cdots \leftrightarrow W_{p_1 p_2 p_3 \cdots p_{p_(m-1)}}$. Therefore subset $W_{p_1} \cup W_{p_1 p_2} \cup W_{p_1 p_2 p_3} \cup \cdots \cup W_{p_1 p_2 \ldots p_{(m-1)}} \cup A$ forms maximum clique in $\mathbb{D}[Z_n]$.

Thus $\omega(\mathbb{D}[Z_n]) = |A| + |W_{p_1}| + |W_{p_1 p_2}| + \cdots + |W_{p_1 p_2 p_3 \cdots p_{(m-1)}}|$.

To determine formula consider following values of n.

Case I: Let $n = p_1 \cdot p_2 \cdot p_3$ then $W_{p_1} = \{k \cdot (p_1) | k \in R, k \notin \langle p_j \rangle, j \neq 2,3\}$ and $W_{p_1 p_2} = \{k \cdot (p_1 p_2) | k \notin \langle p_z \rangle z \neq 1,2\}$ Since $p_1 p_2 \nmid p_1 p_3$. Therefore set $W_{p_1} \cup W_{p_1 p_2}$, forms largest complete subgraph and

$$\begin{aligned}
|W_{p_1}| &= \left[\frac{n}{p_1}\right] - \left[\frac{n}{p_1 p_2}\right] - \left[\frac{n}{p_1 p_3}\right] + \left[\frac{n}{p_1 p_2 p_3}\right] \\
&= p_2 p_3 - p_3 - p_2 + 1 \\
&= p_3(p_2 - 1) - (p_2 - 1) \\
&= (p_2 - 1)(p_3 - 1) \\
|W_{p_1 p_2}| &= \left[\frac{n}{p_1 p_2}\right] - \left[\frac{n}{p_1 p_2 p_3}\right] \\
&= (p_3 - 1)
\end{aligned}$$

$$\begin{aligned}
\therefore \omega(\mathbb{D}[Z_n]) =& |A| + |W_{p_1} \cup W_{p_1p_2}| \\
=& |A| + |W_{p_1}| + |W_{p_1p_2}| \\
=& \phi(n) - 1 + \left[\frac{n}{p_1}\right] - \left[\frac{n}{p_1p_2}\right] - \left[\frac{n}{p_1p_3}\right] + \left[\frac{n}{p_1p_2p_3}\right] + \left[\frac{n}{p_1p_2}\right] - \left[\frac{n}{p_1p_2p_3}\right] \\
=& \phi(n) - 1 + (p_2 - 1)(p_3 - 1) + (p_3 - 1) \\
=& \phi(n) - 1 + (p_3 - 1)[(p_2 - 1) + 1]
\end{aligned}$$

Case II: Let $n = p_1 \cdot p_2 \cdot p_3 \cdot p_4$ then observe that $W_{p_1} \cup W_{p_1p_2} \cup W_{p_1p_2p_3}$ form largest complete subgraph.

$$\begin{aligned}
\omega(\mathbb{D}[Z_n]) &= |A| + |W_{p_1} \cup W_{p_1p_2} \cup W_{p_1p_2p_3}| \\
&= |A| + |W_{p_1}| + |W_{p_1p_2}| + |W_{p_1p_2p_3}| \\
&= \phi(n) - 1 + \left[\frac{n}{p_1}\right] - \left[\frac{n}{p_1p_2}\right] - \left[\frac{n}{p_1p_3}\right] - \left[\frac{n}{p_1p_4}\right] + \left[\frac{n}{p_1p_2p_3}\right] + \left[\frac{n}{p_1p_2p_4}\right] \\
&+ \left[\frac{n}{p_1p_3p_4}\right] - \left[\frac{n}{p_1p_2p_3p_4}\right] + \left[\frac{n}{p_1p_2}\right] - \left[\frac{n}{p_1p_2p_3}\right] - \left[\frac{n}{p_1p_2p_4}\right] + \left[\frac{n}{p_1p_2p_3p_4}\right] \\
&+ \left[\frac{n}{p_1p_2p_3}\right] - \left[\frac{n}{p_1p_2p_3p_4}\right] \\
&= \phi(n) - 1 + (p_2 - 1)(p_3 - 1)(p_4 - 1) + (p_3 - 1)(p_4 - 1) + (p_4 - 1) \\
&= \phi(n) - 1 + (p_4 - 1)[(p_2 - 1)(p_3 - 1) + (p_3 - 1) + 1] \\
&= \phi(n) - 1 + (p_4 - 1)[(p_3 - 1)[(p_2 - 1) + 1] + 1]
\end{aligned}$$

Therefore we may conclude that, if $n = p_1 \cdot p_2 \cdot p_3 \cdots p_m$ then

$$\omega(\mathbb{D}[Z_n]) = \phi(n) - 1 + (p_m - 1)[(p_{m-1} - 1)[\ldots[(p_2 - 1) + 1]\ldots] + 1]$$

Theorem 6.4 For $n = p_1^{k_1} \cdot p_2^{k_2} \cdot p_3^{k_3} \cdots p_m^{k_m}, k_1 \geq 1, p,$

$$\begin{aligned}
\omega(\mathbb{D}[Z_n]) &= \phi(n) - 1 + p_1^{k_1-1} p_2^{k_2-1} p_3^{k_3-1} .. p_m^{k_m-1} (p_2 - 1)(p_3 - 1) \cdots (p_m - 1) \\
&+ p_2^{k_2-1} p_3^{k_3} \cdots p_m^{k_m} - 1
\end{aligned}$$

Proof: Let $n = p_1^{k_1} \cdot p_2^{k_2} \cdot p_3^{k_3} \cdots p_m^{k_m}$. Let d_i be divisors of n and consider the partition of vertex set $V = A \cup_i T_{d_i}$, where, $T_{d_i} = \{s(d_i) | s \in R, s \notin < d_j >, i \neq j, d_i \nmid d_j, d_j \nmid d_i\}$. It is clear that if $k_i = 1 \forall i$ then $T_{d_i} = W_{d_i}$. Since ${p_i}^{k_i} \nmid p_i p_j, k_i > 1, i, j = 2, 3, \cdots, m, i \neq j$ but $p_1^{k_1} p_2 \mid p_1^{k_1} p_2 p_3 \cdots p_m$.

Hence from theorem 4.6, $T_{p_1} \cup T_{p_1^{k_1} p_2} \cup T_{p_1^{k_1} p_2 p_3} \cup \cdots \cup T_{p_1^{k_1} p_2 p_3 \cdots p_m}$ forms maximum clique in $\mathbb{D}[Z_n]$. To determine generalised formula consider the following cases.

Case I : if $n = p_1^{k_1} p_2, k_1 > 1$

$$\begin{aligned}
\omega(\mathbb{D}[Z_n]) &= |A| + |T_{p_1}| + |T_{p_1^{k_1} p_2}| \\
&= \phi(n) - 1 + \left[\frac{n}{p_1}\right] - \left[\frac{n}{p_1 p_2}\right] + 0 \\
&= \phi(n) - 1 + p_1^{k_1-1} p_2 - p_1^{k_1-1} \\
&= \phi(n) - 1 + p_1^{k_1-1}(p_2 - 1)
\end{aligned}$$

Case II: if $n = p_1^{k_1} \cdot p_2^{k_2}, k_1 > 1$, Since $p_1^{k_1} p_2 \mid p_1^{k_1} \cdot p_2^{k_2}$ and $p^k \nmid p_1 p_2$ then

$$\begin{aligned}
\omega(\mathbb{D}[Z_n]) &= |A| + |T_{p_1}| + |T_{p_1^{k_1} p_2}| \\
&= \phi(n) - 1 + \left[\frac{n}{p_1}\right] - \left[\frac{n}{p_1 p_2}\right] + \left[\frac{n}{p_1^{k_1} p_2}\right] - \left[\frac{n}{p_1^{k_1} p_2^{k_2}}\right] \\
&= \phi(n) - 1 + p_1^{k_1-1} p_2^{k_2} - p_1^{k_1-1} p_2^{k_2-1} + p_2^{k_2-1} - 1 \\
&= \phi(n) - 1 + \left[p_1^{k_1-1} p_2^{k_2-1}(p_2 - 1)\right] + p_2^{k_2-1} - 1
\end{aligned}$$

case III: if $n = p_1^{k_1} \cdot p_2^{k_2} \cdot p_3^{k_3}$ then

$$\begin{aligned}
\omega(\mathbb{D}[Z_n]) &= |A| + |T_{p_1}| + |T_{p_1^{k_1} p_2}| + |T_{p_1^{k_1} p_2 p_3}| \\
&= \phi(n) - 1 + p_1^{k_1-1} p_2^{k_2-1} p_3^{k_3-1}(p_2 - 1)(p_3 - 1) + p_2^{k_2-1} p_3^{k_3} - 1.
\end{aligned}$$

By observing above value of clique number we conclude the following
if $n = p_1^{k_1} \cdot p_2^{k_2} \cdot p_3^{k_3} \cdots p_m^{k_m}$ then

$$\begin{aligned}
\omega(\mathbb{D}[Z_n]) &= \phi(n) - 1 + p_1^{k_1-1} p_2^{k_2-1} p_3^{k_3-1} \cdots p_m^{k_m-1}(p_2 - 1)(p_3 - 1) \cdots (p_m - 1) \\
&+ p^{k_2-1} p_3^{k_3} \cdots p_m^{k_m} - 1.
\end{aligned}$$

7. Planarity and Chromatic Number of Z_n :

Theorem 7.1 $\mathbb{D}[Z_n]$ is planar graph if $n \leq 6$ and not planar graph if $n > 6$.

Proof: Let $n > 6$. Since vertex set is disjoint union of A and B where $A = U(R) - \{1\}$. For $n > 6, |A| \geq 4$. These unit vertices forms complete subgraph K_5 either with unit vertices or with zero divisors. Hence $\mathbb{D}[Z_n]$ is not planar graph if $n > 6$. For $n \leq 6$ see figure 1,2,3.

Theorem 7.2 For positive integer n, $\omega(\mathbb{D}[Z_n]) = \chi(\mathbb{D}[Z_n])$.

Proof: Let R be ring of integer modulo n. Since for any graph G, $\chi([G]) \geq \omega([G])$. We just need to prove $\chi([G]) \leq \omega([G])$. For this consider the following cases.

Case I: If $n = p^k$, then $\mathbb{D}[R]$ is complete graph. i.e. result is true.

Case II: If $n = p_1 \cdot p_2 \cdot p_3 \cdots p_m$. Then vertex set can be rewrite as $V = A \cup B$ where $A = U(R) - \{1\}$ and $B = Z(R) = \cup_i V_i$ such that $V_i = \langle p_i \rangle, i = 1, 2, \ldots, m$. We have, if $n = p_1 \cdot p_2 \cdot p_3 \cdots p_m$ then

$$\omega(\mathbb{D}[Z_n]) = |A| + |W_{p_1}| + |W_{p_1 p_2}| + |W_{p_1 p_2 p_3}| + |W_{p_1 p_2 \cdot\cdot p_{(m-1)}}|$$

Where $W_{d_i} = \{s(d_i) | s \in R, s \notin < d_j >, i \neq j, d_i \nmid d_j, d_j \nmid d_i\}$ and $W_i \nleftrightarrow W_j, i \nmid j, i \neq j$ and $W_i \leftrightarrow W_j, i \mid j$. To generalised result start with $n = p_1 p_2$,

Let $n = p_1 p_2$ then vertex set $V = A \cup V_1 \cup V_2$, where $V_1 = W_{p_1}, V_2 = W_{p_2}, |W_{p_1}| = (p_2 - 1)$ and $|W_{p_2}| = (p_1 - 1)$. We have $A \longleftrightarrow W_{p_1}, A \longleftrightarrow W_{p_2}, W_{p_1} \nleftrightarrow W_{p_2}$ Hence total no. distinct color will be $|A| + |W_{p_1}|$. Thus we conclude that, $\omega(\mathbb{D}[Z_n]) = \chi(\mathbb{D}[Z_n])$.

For $n = p_1 p_2 p_3$, We have,

$$\begin{aligned}
V &= A \cup V_1 \cup V_2 \cup V_3 \\
V_1 &= W_{p_1} \cup S_{p_{1,2}} \cup S_{p_{1,3}} \cup S_{p_{1,2,3}}, \\
V_2 &= W_{p_2} \cup S_{p_{1,2}} \cup S_{p_{1,3}} \cup S_{p_{1,2,3}} \\
V_3 &= W_{p_3} \cup S_{p_{1,3}} \cup S_{p_{2,3}} \cup S_{p_{1,2,3}}
\end{aligned}$$

and $W_{p_1} \nleftrightarrow W_{p_2}, W_{p_1} \nleftrightarrow W_{p_3}, W_{p_2} \nleftrightarrow W_{p_3}, W_{p_1 p_2} \nleftrightarrow W_{p_1 p_3}, W_{p_1 p_2} \nleftrightarrow W_{p_2 p_3}, W_{p_2 p_3} \nleftrightarrow W_{p_1 p_3}$

$$\begin{aligned}
|W_{p_1}| &= \left[\frac{n}{p_1}\right] - \left[\frac{n}{p_1 p_2}\right] - \left[\frac{n}{p_1 p_3}\right] + \left[\frac{n}{p_1 p_2 p_3}\right] \\
|W_{p_2}| &= \left[\frac{n}{p_2}\right] - \left[\frac{n}{p_1 p_2}\right] - \left[\frac{n}{p_2 p_3}\right] + \left[\frac{n}{p_1 p_2 p_3}\right]
\end{aligned}$$

$$\begin{aligned} |W_{p_1}| - |W_{p_2}| &= p_2p_3 - p_1p_3 - p_2 + p_1 \\ &= p_2(p_3-1) - p_1(p_3-1) = (p_3-1)(p_2-p_1) > 0 \end{aligned}$$

i.e.$| W_{p_1} |>| W_{p_2} |>| W_{p_3} |$ simillarly it is easy to show that $|W_{p_1p_2}| > |W_{p_2p_3}| > |W_{p_1p_3}|$. Hence same colour can be used for above non adjacent vertices. Therefore, for $n = p_1p_2p_3$, $\chi(\mathbb{D}[Z_n]) = |A| + |W_{p_1}| + |W_{p_1p_2}| = \omega(\mathbb{D}[Z_n])$. Thus we conclude that, result is true for $n = p_1p_2\cdots p_m$.

case III: Let $n = p_1^{k_1} \cdot p_2^{k_2} \cdot p_3^{k_3} \cdots p_m^{k_m}$, then $p_i{}^{k_i} \nmid p_ip_j, i \neq j = 1,2,3,\ldots,m$ but $p_1^{k_1} \cdot p_2 \mid p_1^{k_1} \cdot p_2^{k_2} \cdot p_3^{k_3} \cdots p_m^{k_m}$. Let the partition of vertex set $V = A \cup_i T_{d_i}$, where $T_{d_i} = \{s(d_i) | s \in R, s \notin < d_j > , i \neq j, d_i \nmid d_j, d_j \nmid d_i\}$. We have $p_1 | p_p^{k_1} p_2 | p_1^{k_1} p_2p_3 | \cdots | \cdots | p_1^{k_1} p_2 \cdots p_m$ and hence $T_{p_1} \cup T_{p_1^{k_1}p_2} \cup T_{p_1^{k_1}p_2p_3} \cdots \cup T_{p_1^{k_1}p_2\cdots p_m}$.

Consider $n = p_1^{k_1}p_2^{k_2}$, We have $V = T_{p_1} \cup T_{p_1p_2} \cup T_{p_2}$ such that $T_{p_1} \not\leftrightarrow T_{p_2}, T_{p_1} \leftrightarrow T_{p_1^{k_1}p_2}, T_{p_2} \leftrightarrow T_{p_1^{k_1}p_2}$, and $T_{p_1p_2} \leftrightarrow T_{p_1^{k_1}p_2}$,

$$\begin{aligned} |T_{p_1}| - |T_{p_2}| &= p_1^{k_1-1}p_2^{k_2} - p_1^{k_1}p_2^{k_2-1} \\ &= p_1^{k_1-1}p_2^{k_2-1}(p_2-p_1) \\ |T_{p_1p_2}| &= p_1^{k_1-1}p_2^{k_2-1} - 1 \\ |T_{p_1^{k_1}p_2}| &= p_2^{k_2-1} - 1 \\ \omega(\mathbb{D}[Z_n]) &= |A| + |T_{p_1}| + |T_{p_1^{k_1}p_2}| \end{aligned}$$

Thus for $n = p_1^{k_1} \cdot p_2^{k_2}$, we have shown that $\omega(\mathbb{D}[Z_n]) = \chi(\mathbb{D}[Z_n])$. In this manner we may show that $\omega(\mathbb{D}[Z_n]) = \chi(\mathbb{D}[Z_n]), \forall\, n \in N$.

8. Conclusion

In this paper we introduced a divisor graph $\mathbb{D}[R]$ of a commutative ring and studied relationship of $\mathbb{D}[R]$ and ring R, Also learn the basic properties such as subgraph, connectedness, completeness, Eulerian graph, girth, diameter, clique number, chromatic number, planarity of graph etc.

Conflict of Interests

The author(s) declare that there is no conflict of interests.

REFERENCES

[1] A. Das, Non-zero component graph of a finite dimensional vector space, Commun. Algebra, 44 (2016), 3918-3926.

[2] A. Das, Subspace inclusion graph of a vector space, Commun. Algebra, 44 (2016), 4724-4731.

[3] B. S. Reddy, R. S. Jain, N. Laxmikanth, Eulerian of zero divisor graph $\Gamma(Z_n)$, arXiv preprint arXiv:2001.01219 (2020).

[4] D. F. Anderson. and P. S. Livingston, The zero-divisor graph of a commutative ring, J. Algebra, 217 (1999), 434-447.

[5] D. B. West, Introduction to graph theory, Prentice Hall, (2001).

[6] I. Beck, Colouring of a commutative ring, J. Algebra, 116 (1988), 208–226.

[7] R. A. Muneshwar, K. L. Bondar, Open subset inclusion graph of a topological space, J. Discr. Math. Sci. Cryptography, 22 (2019), 1007-1018.

[8] R. A. Muneshwar, K. L. Bondar, Some significant properties of the intersection graph derived from topological space using intersection of open sets, Far East J. Math. Sci. 11 (2019), 29-48.

[9] S. Akbari, M. Habibi, A. Majidinya, R. Manaviyat, On the inclusion ideal graph of a ring, Electron. Notes Discr. Math. 45 (2014), 73–78.

[10] S. Vivek, B. Vikas, Linear algebra, Narosa Publishing House, (2001).