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APPLICATION OF THE DISCRETE CLASSICAL CASE TO A $1-2$ TYPE RELATION

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Abstract. In this paper, we present a simple approach in order to build up recursively the connection coefficients between a sequence of polynomials $\{Q_n\}_{n\geq 0}$ and an orthogonal polynomials sequence $\{P_n\}_{n\geq 0}$ when

$$P_n(x) = Q_n(x) + r_n Q_{n-1}(x), \quad n \geq 0.$$

This yields the relation between the parameters of the corresponding recurrence relations. Some special cases are developed. More specifically, assuming that $\{P_n\}_{n\geq 0}$ is a discrete classical orthogonal polynomials sequence.

Keywords: orthogonal polynomials; darboux transformations; differential equations; discrete classical polynomials.

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1. Introduction

Let $\mathscr{P}$ be the linear space of polynomials in one variable with complex coefficients and let $\mathscr{P}'$ be its algebraic dual. We denote by $\langle u, f\rangle$ the action of u in $\mathscr{P}'$ on f in $\mathscr{P}$ and by $(u)_n := \langle u, x^n\rangle$, $n \geq 0$, the moments of u with respect to the monomial sequence $\{x^n\}_{n\geq 0}$. When $(u)_0 = 1$, the linear functional u is said to be normalized. The linear functional u is called regular

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(quasi-definite) if the leading principal submatrices $\mathscr{H}_n$ of the Hankel matrix $\mathscr{H}=\left(u_{i+j}\right)_{i,j\geq 0}$ related to the moments $u_n=\langle u,x^n\rangle$, $n\geq 0$, are nonsingular, for each $n\geq 0$ [8].

Let $\{P_n\}_{n\geq 0}$ be a monic orthogonal polynomials sequence, in short *MOPS*, with respect to the linear functional $u\in\mathscr{P}'$, *i.e.*

$$\langle u,P_n(x)P_m(x)\rangle = k_n\delta_{n,m},\qquad n,m\geq 0,$$

where $k_n\neq 0$, $n\geq 0$.

In this way, $\{P_n\}_{n\geq 0}$ satisfies the following three-term recurrence relation

$$\begin{cases} P_{n+1}(x)=(x-\beta_n)P_n(x)-\gamma_nP_{n-1}(x), & n\geq 1,\\ P_0(x)=1,\quad P_1(x)=x-\beta_0, & \end{cases} \tag{1}$$

where $\gamma_n\neq 0$, for each $n\geq 1$.

Definition 1. [8] The sequence $\{P\}_{n\geq 0}$ is called symmetric if it fulfils

$$P_n(-x)=(-1)^nP_n(x),\qquad n\geq 0.$$

Theorem 2. [8] *For each MOPS $\{P_n\}_{n\geq 0}$, the following statements are equivalent*

(i) $\{P_n\}_{n\geq 0}$ *is symmetric.*

(ii) $\{P_n\}_{n\geq 0}$ *satisfies the recurrence relation*

$$\begin{cases} P_{n+1}(x)=xP_n(x)-\gamma_nP_{n-1}(x), & n\geq 1,\\ P_0(x)=1,\quad P_1(x)=x, & \end{cases}$$

where $\gamma_n\neq 0$, for each $n\geq 1$.

Definition 3. An *OPS* $\{P_n\}_{n\geq 0}$ is said to be a sequence of discrete classical orthogonal polynomials if the *PS* $\{Q_n\}_{n\geq 0}$ defined by $(n+1)Q_n(x)=\Delta P_{n+1}(x)$ is also an *OPS* where Δ denotes the difference operator defined by

$$\Delta(f)(x)=f(x+1)-f(x).$$

Let u and v be two regular linear functionals and let $\{P_n\}_{n\geq 0}$ and $\{Q_n\}_{n\geq 0}$ be the corresponding sequences of monic orthogonal polynomials. Assume that there exist non-negative integer numbers M and N, and sequences of complex numbers $\{r_{i,n}\}_{n\geq 0}$ and $\{s_{k,n}\}_{n\geq 0}$ such that the structure relation

$$Q_n(x)+\sum_{i=1}^{M} r_{i,n}Q_{n-i}(x)=P_n(x)+\sum_{i=1}^{N} s_{i,n}P_{n-i}(x)$$

holds for $n\geq 0$. Further, assume that $r_{M,M+N}\neq 0$ and $s_{N,M+N}\neq 0$, $\det\left[\alpha_{ij}\right]_{i,j=1}^{M+N}\neq 0$, where the entries α_{ij} of the matrix are defined on the basis of $\{r_{i,n}\}_{n\geq 0}$ and $\{s_{k,n}\}_{n\geq 0}$. Then there exist two polynomials Φ and Ψ with $\deg\Phi=M$ and $\deg\Psi=N$ such that

$$\Phi(z)u=\Psi(z)v.$$

These polynomials Φ and Ψ can be constructed in an explicit way [15]. On the other hand, the converse result is also analyzed. A characterization theorem for the sequence $\{Q_n\}_{n\geq 0}$ to be orthogonal assuming $\{P_n\}_{n\geq 0}$ is orthogonal is obtained when $M=0$ and $N=1$, $M=1$ and $N=1$, $M=0$ and $N=2$, $M=1$ and $N=2$, $M=0$ and $N=3$, $M=0$ and $N=k$ [2, 3, 4, 5, 6]. In this contribution, the main purpose is to analyze the existence of a *MOPS* $\{Q_n\}_{n\geq 0}$ satisfying the connection relation of two consecutive elements

$$P_n(x)=Q_n(x)+r_nQ_{n-1}(x),\quad n\geq 0, \tag{2}$$

with the initial conditions $Q_0(x)=P_0(x)=1$ and $Q_{-1}(x)=P_{-1}(x)=0$, and where $\{r_n\}_{n\geq 0}$ is a sequence of complex numbers with the initial condition $r_0=0$ and $r_n\neq 0$ for all $n\geq 1$. This paper is organized as follows. In section 2, we develop some basic results and lemmas. Section 3, we discuss some particular cases. Finally, we illustrate this study with some examples of discrete classical orthogonal polynomials.

2. Characterization of Orthogonality

Let $\{P_n\}_{n\geq 0}$ be a *MOPS*, so it satisfies the three-term recurrence relation (1). Necessary and sufficient conditions for the orthogonality of $\{Q_n\}_{n\geq 0}$ are given by the following proposition.

Proposition 4. *Let $\{P_n\}_{n\geq 0}$ be a MOPS satisfies (1), and let $\{Q_n\}_{n\geq 0}$ be a sequence of polynomials given by the structure relation (2). Then, $\{Q_n\}_{n\geq 0}$ is a MOPS if and only if the following conditions hold*

(i) $r_{n-1}-\beta_{n-1}+\dfrac{\gamma_n}{r_n}=\mu,\quad n\geq 2,$ *where* $\mu=-\beta_0+\dfrac{\gamma_1}{r_1}.$

(ii) $\widetilde{\gamma}_n := \gamma_n + r_n(\beta_n - \beta_{n-1} - r_n + r_{n-1}) \neq 0,\ n \geq 1.$

Proof. Let

$$Q_n(x) = P_n(x) + \sum_{i=1}^{n} \alpha_{n,i} P_{n-i}(x), \quad n \geq 0 \tag{3}$$

where $\alpha_{n,i} = (-1)^i \prod_{j=0}^{i-1} r_{n-j}$, $1 \leq i \leq n$, $n \geq 1$.

Multiplying the left and the right hand sides the above expression by x and applying (1) to $xP_n(x)$, we get

$$\begin{aligned} xQ_n(x) &= P_{n+1}(x) + (\beta_n + \alpha_{n,1}) P_n(x) + (\gamma_n + \alpha_{n,1}\beta_{n-1} + \alpha_{n,2}) P_{n-1}(x) \\ &\quad + \sum_{i=2}^{n} [\alpha_{n,i-1}\gamma_{n-i+1} + \alpha_{n,i}\beta_{n-i} + \alpha_{n,i+1}] P_{n-i}(x). \end{aligned}$$

Hence, it follows

$$xQ_n(x) = P_{n+1}(x) + \widetilde{\beta}_n P_n(x) + \widetilde{\gamma}_n P_{n-1}(x) + \sum_{i=2}^{n} \widetilde{\alpha}_{n,i} P_{n-i}(x) \tag{4}$$

where

$$\begin{aligned} \widetilde{\beta}_n &: = \beta_n + \alpha_{n,1}, \\ \widetilde{\gamma}_n &: = \gamma_n + \alpha_{n,1}\beta_{n-1} + \alpha_{n,2} \end{aligned} \tag{5}$$

and

$$\widetilde{\alpha}_{n,i} := \alpha_{n,i-1}\gamma_{n-i+1} + \alpha_{n,i}\beta_{n-i} + \alpha_{n,i+1}, \ \ 2 \leq i \leq n, \text{ with } \alpha_{n,n+1} = 0. \tag{6}$$

Replacing (2) in (4), we obtain

$$\begin{aligned} xQ_n(x) &= Q_{n+1}(x) + \left(\widetilde{\beta}_n + r_{n+1}\right) Q_n(x) + \left(\widetilde{\gamma}_n + r_n\widetilde{\beta}_n\right) Q_{n-1}(x) \\ &\quad + (\widetilde{\alpha}_{n,2} + r_{n-1}\widetilde{\gamma}_n) Q_{n-2}(x) + \sum_{i=2}^{n-1} (\widetilde{\alpha}_{n,i+1} + r_{n-i}\widetilde{\alpha}_{n,i}) Q_{n-i-1}(x). \end{aligned}$$

So, $\{Q_n\}_{n\geq 0}$ will be orthogonal if and only if

$$\widetilde{\gamma}_n = \widetilde{\gamma}_n + r_n\widetilde{\beta}_n \neq 0, \quad n \geq 1, \tag{7}$$

$$\widetilde{\alpha}_{n,2} + r_{n-1}\widetilde{\gamma}_n = 0, \quad n \geq 1, \tag{8}$$

(9) $$\widetilde{\alpha}_{n,i+1}+r_{n-i}\widetilde{\alpha}_{n,i}=0,\quad 2\leq i\leq n.$$

From (8) and (9)

(10) $$\widetilde{\alpha}_{n,i}=(-1)^{i+1}\,\widetilde{\gamma}_n\prod_{j=1}^{i-1}r_{n-j},\ 2\leq i\leq n,$$

and from (6) and (8)

$$\alpha_{n,3}+\alpha_{n,2i}\beta_{n-2}+\alpha_{n,1}\gamma_{n-1}=-r_{n-1}\widetilde{\gamma}_n.$$

Moreover, from (6) and (10), for $2\leq i\leq n$,

$$\alpha_{n,i+1}+\alpha_{n,i}\beta_{n-i}+\alpha_{n,i-1}\gamma_{n-i+1}=(-1)^{i+1}\,\widetilde{\gamma}_n\prod_{j=1}^{i-1}r_{n-j}=-\frac{\widetilde{\gamma}_n}{r_n}\alpha_{n,i},$$

dividing in this last expression by $\alpha_{n,i}$, we get

$$r_{n-i}-\beta_{n-i}+\frac{\gamma_{n-i+1}}{r_{n-i+1}}=\frac{\widetilde{\gamma}_n}{r_n},\quad 2\leq i\leq n,$$

using (5), for $2\leq i\leq n$,

(11) $$r_{n-i}-\beta_{n-i}+\frac{\gamma_{n-i+1}}{r_{n-i+1}}=r_{n-1}-\beta_{n-1}+\frac{\gamma_n}{r_n}.$$

Taking into account (11), for n and $n-1$ instead of i, we get

$$-\beta_0+\frac{\gamma_1}{r_1}=r_1-\beta_1+\frac{\gamma_2}{r_2},$$

for r_1 is fix, we deduce r_2.

By iterating the process, we have

$$r_{n-2}-\beta_{n-2}+\frac{\gamma_{n-1}}{r_{n-1}}=r_{n-1}-\beta_{n-1}+\frac{\gamma_n}{r_n},\quad n\geq 2.$$

This last expression implies

(12) $$r_{n-1}-\beta_{n-1}+\frac{\gamma_n}{r_n}=\mu,\quad n\geq 2$$

where $\mu=-\beta_0+\dfrac{\gamma_1}{r_1}$.

Again, from (5) and (7)

(13) $$\widetilde{\gamma}_n=\gamma_n+r_n\left(\beta_n-\beta_{n-1}-r_n+r_{n-1}\right),\quad n\geq 1.$$

Dividing the left and the right hand sides in the above expression by r_n, leads to

$$\frac{\widetilde{\gamma}_n}{r_n} = \frac{\gamma_n}{r_n} + \beta_n - \beta_{n-1} - r_n + r_{n-1}, \quad n \geq 1,$$

using (12), we obtain

$$\frac{\widetilde{\gamma}_n}{r_n} = \mu + \beta_n - r_n = \frac{\gamma_{n+1}}{r_{n+1}} \neq 0.$$

This accomplish the proof. □

Let $\{t_n\}_{n\geq 1}$ be a sequence of complex numbers,
such that $t_1 = 1$ and $r_n = \dfrac{t_n}{t_{n+1}}$.
The relation (12) becomes

$$\frac{t_{n-1}}{t_n} - \beta_{n-1} + \frac{t_{n+1}}{t_n}\gamma_n = \mu, \quad n \geq 2,$$

hence

$$t_{n-1} - \beta_{n-1} t_n + \gamma_n t_{n+1} = \mu t_n, \quad n \geq 2. \tag{14}$$

From the relation (14) and taking into account

$$t_1(\mu) = 1, t_2(\mu) = \gamma_1^{-1}(\mu + \beta_0),$$

we can deduce that t_n is a polynomial of degree $n-1$.

If we denote by k_n its leading coefficient as well as we denote

$$t_n(\mu) = k_n \widetilde{t}_{n-1}(\mu),$$

then the relation (14) reads

$$\mu \widetilde{t}_{n-1}(\mu) = \widetilde{t}_n(\mu) - \beta_{n-1}\widetilde{t}_{n-1}(\mu) + \gamma_{n-1}\widetilde{t}_{n-2}(\mu), \quad n \geq 2,$$

with $\widetilde{t}_0(\mu) = 1, \widetilde{t}_1(\mu) = \mu + \beta_0$.

In other words,

$$\widetilde{t}_n(\mu) = (-1)^n P_n(-\mu),$$

and, as a consequence, we can deduce the explicit expression of t_n.

The next result gives the solution of the above linear difference equation.

Proposition 5. *Let*

$$\begin{aligned}\gamma_{2n+1}t_{2n+2} &= R_{2n+1}(\mu), \quad n \geq 0,\\ \gamma_{2n}t_{2n+1} &= S_{2n}(\mu), \qquad n \geq 1,\end{aligned}$$

then

$$\begin{cases} R_{2n+1}(\mu) = (\mu+\beta_{2n})\,\gamma_{2n}^{-1}S_{2n}(\mu) - \gamma_{2n-1}^{-1}R_{2n-1}(\mu), \quad n \geq 1,\\ R_1(\mu) = \mu+\beta_0 \end{cases}$$

and

$$\begin{cases} S_{2n}(\mu) = (\mu+\beta_{2n-1})\,\gamma_{2n-1}^{-1}R_{2n-1}(\mu) - \gamma_{2n-2}^{-1}S_{2n-2}(\mu), \quad n \geq 2,\\ S_0 = 1, \quad S_2(\mu) = (\mu+\beta_1)\,\gamma_1^{-1}R_1(\mu) - 1. \end{cases}$$

Proof. From the relation (14), we have

$$\gamma_n t_{n+1} = (\mu+\beta_{n-1})t_n - t_{n-1}, \quad n \geq 2,$$

then

$$\gamma_1 t_2 = \mu + \beta_0 = R_1(\mu)$$

with R_1 is an odd polynomial of degree 1.

On the other hand

$$\gamma_2 t_3 = (\mu+\beta_1)t_2 - t_1 = (\mu+\beta_1)\gamma_1^{-1}R_1(\mu) - 1 = S_2(\mu)$$

with S_2 is an even polynomial of degree 2.

Moreover

$$\gamma_3 t_4 = (\mu+\beta_2)t_3 - t_2 = (\mu+\beta_2)\gamma_2^{-1}S_2(\mu) - \gamma_1^{-1}R_1(\mu) = R_3(\mu)$$

with R_3 is an odd polynomial of degree 3.

And

$$\gamma_4 t_5 = (\mu+\beta_3)t_4 - t_3 = (\mu+\beta_3)\gamma_3^{-1}R_3(\mu) - \gamma_2^{-1}S_2(\mu) = S_4(\mu)$$

with S_4 is an even polynomial of degree 4.

Thus

$$\begin{aligned} R_{2n+1}(\mu) &= (\mu+\beta_{2n})\,\gamma_{2n}^{-1}S_{2n}(\mu) - \gamma_{2n-1}^{-1}R_{2n-1}(\mu), \quad n \geq 1,\\ S_{2n}(\mu) &= (\mu+\beta_{2n-1})\,\gamma_{2n-1}^{-1}R_{2n-1}(\mu) - \gamma_{2n-2}^{-1}S_{2n-2}(\mu), \quad n \geq 2 \end{aligned}$$

with initial conditions $R_1(\mu) = \mu + \beta_0$, $S_0(\mu) = 1$, $S_2(\mu) = (\mu + \beta_1)\gamma_1^{-1} R_1(\mu) - 1$. □

In the following result, we assume that the sequence of polynomials $\{Q_n\}_{n\geq 0}$ is a *MOPS*, i.e.

$$Q_{n+1}(x) = \left(x - \widetilde{\beta}_n\right) Q_n(x) - \widetilde{\gamma}_n Q_{n-1}(x), \qquad n \geq 0, \tag{15}$$

with the initial conditions $Q_0(x) = 1$, $Q_{-1}(x) = 0$ and the conditions $\widetilde{\gamma}_n \neq 0$, for each $n \geq 1$.

Lemma 6. *Let $\{Q_n\}_{n\geq 0}$ be a MOPS satisfying the decomposition (2), then*

$$\begin{cases} \widetilde{\beta}_n = \beta_n - r_n + r_{n+1}, & n \geq 0, \\ \widetilde{\gamma}_n = \gamma_n + r_n(\beta_n - \beta_{n-1} - r_n + r_{n-1}), & n \geq 1, \\ \widetilde{\gamma}_n = \dfrac{r_n}{r_{n+1}} \gamma_{n+1}, & n \geq 1. \end{cases} \tag{16}$$

3. Some Special Cases

We will consider the following two special cases.

3.1. $\{P_n\}_{n\geq 0}$ is symmetric

Let us suppose that $\{P_n\}_{n\geq 0}$ is a symmetric orthogonal polynomials sequence. From Proposition 5 and Theorem 2, we have

$$\begin{cases} R_{2n+1}(\mu) = \mu \gamma_{2n}^{-1} S_{2n}(\mu) - \gamma_{2n-1}^{-1} R_{2n-1}(\mu), & n \geq 1, \\ R_1(\mu) = \mu \end{cases}$$

and

$$\begin{cases} S_{2n}(\mu) = \mu \gamma_{2n-1}^{-1} R_{2n-1}(\mu) - \gamma_{2n-2}^{-1} S_{2n-2}(\mu), & n \geq 2, \\ S_0 = 1, \quad S_2(\mu) = \mu \gamma_1^{-1} R_1(\mu) - 1. \end{cases}$$

Moreover, from Lemma 6, we obtain

$$\begin{cases} \widetilde{\beta}_n = r_{n+1} - r_n, & n \geq 0, \\ \widetilde{\gamma}_n = \gamma_n + r_n(r_{n-1} - r_n), & n \geq 1, \\ r_n \widetilde{\gamma}_{n-1} = r_{n-1} \gamma_n, & n \geq 2. \end{cases} \tag{17}$$

Then

$$\frac{\widetilde{\gamma}_n}{r_n} + r_n = \frac{\widetilde{\gamma}_1}{r_1} + r_1, \quad n \geq 2, \text{ and } \frac{\gamma_1}{r_1} + r_1 \neq 0. \tag{18}$$

equivalently,

$$\frac{\gamma_{n+1}}{r_{n+1}} + r_n = \frac{\gamma_1}{r_1}, \quad n \geq 1.$$

By summation in (17), we get

$$\sum_{i=0}^{n}\widetilde{\beta}_i = r_{n+1}, \quad n \geq 0.$$

We deal with two particular cases.

(i) If $\widetilde{\beta}_n = \widetilde{\beta}$, $n \geq 0$, then

$$r_n = n\widetilde{\beta}, \;\; n \geq 1.$$

The relation (18) becomes

$$\frac{\widetilde{\gamma}_n}{n} = \widetilde{\gamma}_1 + (1-n)\,\widetilde{\beta}^2, \quad n \geq 1, \text{ and } \widetilde{\gamma}_1 = \gamma_1 - \widetilde{\beta}^2$$

with $\gamma_1 \neq \widetilde{\beta}^2$.

Then, the sequence $\{P_n\}_{n\geq 0}$ has the recurence coefficients

$$\frac{\gamma_{n+1}}{n+1} = \gamma_1 - n\widetilde{\beta}^2, \quad n \geq 1.$$

(ii) If $\widetilde{\beta}_n = 1+n$, $n \geq 0$, then

$$r_n = \frac{n(1+n)}{2}$$

and

$$\frac{\widetilde{\gamma}_n}{n(1+n)} = \frac{\widetilde{\gamma}_1}{2} - \frac{n(1+n)}{4} + \frac{1}{4}, \quad n \geq 1, \text{ and } \widetilde{\gamma}_1 = \gamma_1 - 1$$

with $\gamma_1 \neq 1$.

Then, the sequence $\{P_n\}_{n\geq 0}$ has the recurence coefficients

$$\frac{\gamma_{n+1}}{(n+1)(n+2)} = \frac{\gamma_1}{2} - \frac{n(1+n)}{4}, \quad n \geq 1.$$

3.2. The sequence $\{r_n\}_{n\geq 1}$ is constant

Let as consider $r_n = r$, $n \geq 1$, where $r \in \mathbb{R}\setminus\{0\}$. From Lemma 6, we get

$$\begin{cases} \widetilde{\beta}_n = \beta_n, \quad n \geq 1, \;\; \widetilde{\beta}_0 = \beta_0 + r, \\ \widetilde{\gamma}_n = \gamma_n + r(\beta_n - \beta_{n-1}), \quad n \geq 2, \\ \widetilde{\gamma}_1 = \gamma_1 + r(\beta_1 - \beta_0 - r), \\ \widetilde{\gamma}_n = \gamma_{n+1}, \quad n \geq 1. \end{cases} \tag{19}$$

Using the above system, we can deduce

$$\begin{aligned} \gamma_{n+1} &= \gamma_1 + r(\beta_n - \beta_1), \;\; n \geq 2, \\ \gamma_2 &= \gamma_1 + r(\beta_1 - \beta_0 - r). \end{aligned}$$

Then, we have two cases.

1st Case: Let consider $\beta_n=\beta$, $n\geq 0$, $\gamma_n=\gamma$, $n\geq 1$. From (19), we obtain

$$\begin{cases} \widetilde{\beta}_n=\beta, \quad n\geq 1,\ \widetilde{\beta}_0=\beta+r,\\ \widetilde{\gamma}_1=\gamma-r^2,\\ \widetilde{\gamma}_n=\gamma, \quad n\geq 1, \end{cases}$$

thus, $r=0$

so, $P_n\equiv Q_n,\ n\geq 0$.

2nd Case: If $\beta_n=\beta$, $n\geq 1$, and $\beta_0\neq\beta$, $\gamma_n=\gamma$, $n\geq 1$. From (19), we get

$$\begin{cases} \widetilde{\beta}_n=\beta, \quad n\geq 1,\ \widetilde{\beta}_0=\beta_0+r,\\ \widetilde{\gamma}_1=\gamma+r(\beta-\beta_0-r),\\ \widetilde{\gamma}_n=\gamma, \quad n\geq 1, \end{cases}$$

thus, $r=\beta-\beta_0$.

In this case $\{Q_n\}_{n\geq 0}$ and $\{P_n\}_{n\geq 0}$ have the same recurrence relation with initial conditions $Q_1(x)=P_1(x)-r$ and $Q_2(x)=P_2(x)-r(x-\beta_0-r)$.

4. Examples

In this section we illustrate some examples of classical discrete orthogonal polynomials. The three referred families of monic discrete orthogonal polynomials: Charlier $C_n^{(a)}$, Meixner $M_n^{(\nu,\mu)}$, Krawtchouk $K_n^{(p)}(.;N)$ and Hahn $h_n^{(\alpha,\beta)}(.;N)$ have the following hypergeometric representations [9]

$$\begin{aligned}
C_n^{(a)}(x) &= (-a)^n\,{}_2F_0\left(\begin{array}{cc} -n, & -x \\ - & \end{array};-\frac{1}{a}\right), \quad a>0,\\
M_n^{(\nu,\mu)}(x) &= (\nu)_n\left(\frac{\mu}{\mu-1}\right)^n {}_2F_1\left(\begin{array}{cc} -n, & -x \\ \nu & \end{array};1-\frac{1}{\mu}\right), \quad \nu>0,\ \mu\in\,]0,1[,\\
K_n^{(p)}(x;N) &= \frac{(-p)^n\,N!}{(N-n)!}\,{}_2F_1\left(\begin{array}{cc} -n, & -x \\ -N & \end{array};\frac{1}{p}\right),\ p\in\,]0,1[,\ N\in\mathbb{Z}^+,\ n\leq N,\\
h_n^{(\alpha,\,\beta)}(x;N) &= \frac{(-N+\alpha+1)_n}{(\alpha+\beta+n+1)_n}\,{}_3F_2\left(\begin{array}{ccc} -n & \alpha+\beta+n+1 & -x \\ -N & \alpha+1 & \end{array};1\right),\ \alpha>-1,\ \beta>-1.
\end{aligned}$$

where ${}_pF_q$ is the hypergeometric series defined by

$$ {}_pF_q\left(\begin{array}{cccc} a_1, & a_2, & \ldots, & a_p \\ b_1, & b_2, & \ldots, & b_q \end{array};z\right) := \sum_{k=0}^{\infty}\frac{(a_1)_k\ (a_2)_k\ \ldots(a_p)_k}{(b_1)_k\ (b_2)_k\ \ldots(b_q)_k}\frac{z^k}{k!} $$

with the Pochhammer symbol $(a)_n$ being defined by

$$ (a)_n = \begin{cases} 1, & \text{if } n=0, \\ a(a+1)(a+2)\ldots(a+n-1), & \text{if } n\geq 1. \end{cases} $$

And their recurrence relations can be written as follows [9]

$$ C_{n+1}^{(a)}(x) = (x-n-a)\,C_n^{(a)}(x) - an\,C_{n-1}^{(a)}(x), \quad n\geq 0, \tag{20} $$

with the initial conditions $C_0^{(a)}(x)=1$, $C_{-1}^{(a)}(x)=0$ and $a>0$.

$$ M_{n+1}^{(\nu,\mu)}(x) = \left(x - \frac{n+(n+\nu)\,\mu}{1-\mu}\right)M_n^{(\nu,\mu)}(x) - \frac{n(n+\nu-1)\mu}{(1-\mu)^2}\,M_{n-1}^{(\nu,\mu)}(x), \quad n\geq 0, \tag{21} $$

with the initial conditions $M_0^{(\nu,\mu)}(x)=1$, $M_{-1}^{(\nu,\mu)}(x)=0$ and $\nu>0$, $\mu\in\,]0,1[$.

$$ \begin{aligned} K_{n+1}^{(p)}(x;N) &= (x-n(1-p)-p(N-n))\,K_n^{(p)}(x;N) \\ &\quad -np(1-p)(N-n+1)\,K_{n-1}^{(p)}(x;N), \quad n\geq 0, \end{aligned} \tag{22} $$

with the initial conditions $K_0^{(p)}(x;N)=1$, $K_{-1}^{(p)}(x;N)=0$ and $p\in\,]0,1[$, $n\leq N$, $N\in\mathbb{Z}^+$.

$$ \begin{aligned} h_{n+1}^{(\alpha,\,\beta)}(x;N) &= \left(x - \frac{(\alpha+1)\,N\,(\alpha+\beta)+n\,(2N-\alpha+\beta)\,(\alpha+\beta+2N)}{4\,(\alpha+\beta+2n)\,(\alpha+\beta+2n+2)}\right)h_n^{(\alpha,\,\beta)}(x;N) \\ &\quad - \frac{n\,(N-n-1)\,(\alpha+\beta+n)\,(\alpha+n)\,(\beta+n)\,(\alpha+\beta+N+n+1)}{(\alpha+\beta+2n-1)\,(\alpha+\beta+2n)^2\,(\alpha+\beta+2n+1)}\,h_{n-1}^{(\alpha,\,\beta)}(x;N), \quad n\geq 0, \end{aligned} \tag{23} $$

with initial conditions $h_0^{(\alpha,\,\beta)}(x;N)=1$, $h_{-1}^{(\alpha,\,\beta)}(x;N)=0$ and $\alpha>-1$, $\beta>-1$.

4.1. *MOPS* of Charlier

From (16) and (20), the recurrence coefficients of $\{Q_n\}_{n\geq 0}$ are getting as follows

$$ \begin{aligned} \widetilde{\beta}_n &= r_{n+1} - r_n + n + a, \quad n\geq 0, \\ \widetilde{\gamma}_n &= na - \left(\widetilde{\beta}_{n-1} - n - a\right)\sum_{k=0}^{n-1}\left(\widetilde{\beta}_k - k - a\right), \quad n\geq 1. \end{aligned} $$

As such, the Δ operator acts as a lowering operator on this family of polynomials. The explicit expression is [14]

$$\frac{\Delta C_{n+1}^{(a)}(x)}{n+1} = C_n^{(a)}(x), \quad n \geq 0,$$

if we get $Q_n(x) = \dfrac{\Delta C_{n+1}^{(a)}(x)}{n+1}$, $n \geq 0$, then $Q_n \equiv C_n^{(a)}$.

Thus, there is no a sequence $\{r_n\}_{n\geq 0}$ such the decomposition (2) holds.

4.2. ***MOPS*** **of Meixner**

From (16) and (21), the recurrence coefficients of $\{Q_n\}_{n\geq 0}$ are obtained as follows

$$\begin{aligned}
\widetilde{\beta}_n &= r_{n+1} - r_n + \frac{n + (n+\nu)\mu}{1-\mu}, \quad n \geq 0, \\
\widetilde{\gamma}_n &= \frac{n(n+\nu-1)\mu}{(1-\mu)^2} - \left(\widetilde{\beta}_{n-1} - \frac{n + (n+\nu)\mu}{1-\mu}\right) \sum_{k=0}^{n-1} \left(\widetilde{\beta}_k - \frac{k+(k+\nu)\mu}{1-\mu}\right), \quad n \geq 1.
\end{aligned}$$

It is well known that the Meixner polynomials satisfy the following relation [14]

$$\frac{\Delta M_{n+1}^{(\nu,\mu)}(x)}{n+1} = M_n^{(\nu+1,\mu)}(x), \quad n \geq 0,$$

if we get $Q_n(x) = \dfrac{\Delta M_{n+1}^{(\nu,\mu)}(x)}{n+1}$, $n \geq 0$, then $Q_n \equiv M_n^{(\nu+1,\mu)}$.

And as

$$\begin{aligned}
\widetilde{\gamma}_n &= \frac{n}{n+1}\gamma_{n+1} \\
&= \frac{n}{n+1}\frac{(n+1)(n+\nu)\mu}{(1-\mu)^2} \\
&= \frac{n(n+\nu)\mu}{(1-\mu)^2}, \quad n \geq 1,
\end{aligned}$$

then, there is a sequence $\{r_n\}_{n\geq 0}$ such the decomposition (2) holds.

4.3. ***MOPS*** **of Krawtchouk**

From (16) and (22), the recurrence coefficients of $\{Q_n\}_{n\geq 0}$ are determined as follows

$$\widetilde{\beta}_n = r_{n+1} - r_n + n(1-p) + p(N-n), \quad n \geq 0,$$

$$\begin{aligned}
\widetilde{\gamma}_n &= np(1-p)(N-n+1) \\
&\quad - \left[\widetilde{\beta}_{n-1} - n(1-p) - p(N-n)\right] \sum_{k=0}^{n-1} \left[\widetilde{\beta}_k - k(1-p) - p(N-k)\right], \quad n \geq 1.
\end{aligned}$$

Moreover, the Krawtchouk polynomials satisfy the following relation [14]

$$\frac{\Delta K_{n+1}^{(p)}(x;N)}{n+1} = K_n^{(p)}(x;N-1), \quad n \geq 0,$$

if we get $Q_n(x) = \dfrac{\Delta K_{n+1}^{(p)}(x;N)}{n+1}$, $n \geq 0$, then $Q_n \equiv K_n^{(p)}(.;N-1)$.
And as

$$\begin{aligned}\widetilde{\gamma}_n &= \frac{n}{n+1}\gamma_{n+1} \\ &= \frac{n}{n+1}p(n+1)(1-p)(N-n) \\ &= np(1-p)(N-n), \quad n \geq 1,\end{aligned}$$

then, there is a sequence $\{r_n\}_{n\geq 0}$ such the decomposition (2) holds.

4.4. *MOPS* of Hahn

From (16) and (23), the recurrence coefficients of $\{Q_n\}_{n\geq 0}$ are getting as follows

$$\widetilde{\beta}_n = r_{n+1} - r_n + \frac{(\alpha+1)N(\alpha+\beta) + n(2N-\alpha+\beta)(\alpha+\beta+2N)}{4(\alpha+\beta+2n)(\alpha+\beta+2n+2)}, \quad n \geq 0,$$

$$\begin{aligned}\widetilde{\gamma}_n = \; & \frac{n(N-n-1)(\alpha+\beta+n)(\alpha+n)(\beta+n)(\alpha+\beta+N+n+1)}{(\alpha+\beta+2n-1)(\alpha+\beta+2n)^2(\alpha+\beta+2n+1)} \\ & -\left[\widetilde{\beta}_{n-1} - \frac{(\alpha+1)N(\alpha+\beta) + n(2N-\alpha+\beta)(\alpha+\beta+2N)}{4(\alpha+\beta+2n)(\alpha+\beta+2n+2)}\right] \\ & \times\sum_{k=0}^{n-1}\left[\widetilde{\beta}_k - \frac{(\alpha+1)N(\alpha+\beta) + k(2N-\alpha+\beta)(\alpha+\beta+2N)}{4(\alpha+\beta+2k)(\alpha+\beta+2k+2)}\right], \quad n \geq 1.\end{aligned}$$

Moreover, the Hahn polynomials satisfy the following relation [14]

$$\frac{\Delta h_{n+1}^{(\alpha,\,\beta)}(x;N)}{n+1} = h_n^{(\alpha+1,\,\beta+1)}(x;N-1), \quad n \geq 0,$$

if we get $Q_n(x) = \dfrac{\Delta h_n^{(\alpha,\,\beta)}(x;N)}{n+1}$, $n \geq 0$, then $Q_n \equiv h_n^{(\alpha+1,\,\beta+1)}(.;N-1)$.

And as

$$\begin{aligned}
\widetilde{\gamma}_n &\neq \frac{n}{n+1}\gamma_{n+1} \\
&= \frac{n}{n+1}\frac{(n+1)(N-n-2)(\alpha+\beta+n+1)(\alpha+n+1)(\beta+n+1)(\alpha+\beta+N+n+2)}{(\alpha+\beta+2n+1)(\alpha+\beta+2n+2)^2(\alpha+\beta+2n+3)} \\
&= \frac{n(N-n-2)(\alpha+\beta+n+1)(\alpha+n+1)(\beta+n+1)(\alpha+\beta+N+n+2)}{(\alpha+\beta+2n+1)(\alpha+\beta+2n+2)^2(\alpha+\beta+2n+3)}, \quad n \geq 1.
\end{aligned}$$

Thus, there is no a sequence $\{r_n\}_{n\geq 0}$ such the decomposition (2) holds.

Conflict of Interests

The author(s) declare that there is no conflict of interests.

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