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ON THE NONLINEAR CIRCLE PLUS OPERATOR RELATED TO THE LAPLACIAN

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Abstract. In this paper, we study the solution of nonlinear equation

$$\oplus^k u(x) = f(x, \triangle^{k-1}\Box^k L^k u(x)),$$

where the operator $\oplus^k$ is defined by

$$\oplus^k = \left[\left(\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2}\right)^4 - \left(\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right)^4\right]^k,$$

or the operator $\oplus^k$ can be express by $\oplus^k = \triangle^k\Box^k L^k$. The operator $\triangle^k$ is Laplacian operator, $\Box^k$ is ultrahyperbolic operator and L^k is operator defined by

$$L^k = \left[\left(\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2}\right)^2 + \left(\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right)^2\right]^k,$$

$p+q=n$ is the dimension of the n-dimension Euclidean space $\mathbb{R}^n$, $x=(x_1,x_2,\ldots,x_n)\in\mathbb{R}^n$, k is a positive integer, $u(x)$ is an unknown and f is a given function. It is found that the existence of the solution $u(x)$ of such equation depending on the condition of f and $\triangle^{k-1}\Box^k L^k u(x)$ and moreover such solution $u(x)$ related to the Laplacian depending on the conditions of p, q and k.

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1. Introduction

The operator $\oplus^k$ has been studied first by Kananthai, Suantai and Longani [5] and is defined by

$$\oplus^k = \left[\left(\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2}\right)^2 - \left(\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right)^2\right]^k \times \left[\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2} - \mathrm{i}\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right]^k$$

$$(1)\qquad \times\left[\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2} + \mathrm{i}\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right]^k = \left[\left(\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2}\right)^4 - \left(\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right)^4\right]^k,$$

where $p+q=n$ is the dimension of $\mathbb{R}^n$, $\mathrm{i}=\sqrt{-1}$ and k is a nonnegative integer. The diamond operator is denoted by

$$(2)\qquad \Diamond^k = \left(\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2}\right)^2 - \left(\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right)^2.$$

The operator L_1 and L_2 are defined by

$$(3)\qquad L_1 = \sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2} - \mathrm{i}\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}$$

and

$$(4)\qquad L_2 = \sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2} + \mathrm{i}\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}.$$

Thus equation (1) can be written as

$$\oplus^k = \Diamond^k L_1^k L_2^k.$$

Otherwise, the operator $\Diamond$ can also be expressed in the form $\Diamond = \Box\triangle = \triangle\Box$, where $\Box$ is the ultra-hyperbolic operator defined by

$$(5)\qquad \Box = \sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2} - \sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2},$$

where $p+q=n$ and $\triangle$ is the Laplacian defined by

$$(6)\qquad \triangle = \sum_{i=1}^{n}\frac{\partial^2}{\partial x_i^2}.$$

The linear equation $\Diamond^k u(x) = f(x)$, see [6], has been already studied and the convolution $u(x) = (-1)^k K_{2k,2k}(x) * f(x)$ has been obtained as a solution of such an equation where $K_{2k,2k}(x) = R^H_{2k}(x) * R^e_{2k}(x)$. The function $R^H_{2k}(x)$ and $R^e_{2k}(x)$ are defined by (9) and (11), respectively, with $\alpha = \beta = 2k$.

Kananthai, Suantai and Longani, see[4], has been studied the operator $\oplus^k$. They obtained

$$K(x) = [R^H_{2k}(u) * (-1)^k R^e_{2k}] * (-1)^k (-\mathrm{i})^{q/2} S_{2k}(w) * (-1)^k (\mathrm{i})^{q/2} T_{2k}(z)$$

is the elementary solution of such operator.

In this work, we study the nonlinear equation of the form

$$\oplus^k u(x) = f(x, \triangle^{k-1}\Box^k L^k u(x)). \tag{7}$$

with f defined and continuous for all $x \in \Omega \cup \partial\Omega$ where Ω is an open subset of $\mathbb{R}^n$ and $\partial\Omega$ denotes the boundary of Ω. We can find the solution $u(x)$ of (7) which is unique under the condition $|f(x, \triangle^{k-1}\Box^k L^k u(x))| \leq N$ where N is a constant for all $x \in \Omega$ and the boundary condition $\triangle^{k-1}\Box^k L^k u(x) = 0$ for $x \in \partial\Omega$.

2. Preliminaries

Definition 2.1. Let $x = (x_1, x_2, ..., x_n)$ be a point in the space $\mathbb{R}^n$ of the n-dimensional Euclidean space and write

$$v = x_1^2 + x_2^2 + ... + x_p^2 - x_{p+1}^2 - ... - x_{p+q}^2, \tag{8}$$

where $p + q = n$ is the dimension of $\mathbb{R}^n$.

Denote by $\Gamma_+ = \{x \in \mathbb{R}^n : x_1 > 0 \text{ and } u > 0\}$ the set of an interior of the forward cone and $\overline{\Gamma}_+$ denotes it closure and $\mathbb{R}^n$ is the n-dimensional Euclidean space.

For any complex number α, define

$$R^H_\alpha(v) = \begin{cases} \frac{v^{\frac{\alpha-n}{2}}}{K_n(\alpha)}, & \text{for } x \in \Gamma_+ \\ 0, & \text{for } x \notin \Gamma_+, \end{cases} \tag{9}$$

where the constant $K_n(\alpha)$ is given by the formula

$$K_n(\alpha) = \frac{\pi^{\frac{n-1}{2}}\Gamma(\frac{2+\alpha-n}{2})\Gamma(\frac{1-\alpha}{2})\Gamma(\alpha)}{\Gamma(\frac{2+\alpha-p}{2})\Gamma(\frac{p-\alpha}{2})}.$$

The function $R^H_\alpha(v)$ is called the hyperbolic kernel of Marcel Riesz and was introduced by *Y. Nozaki [4, p72]*. It is well known that $R^H_\alpha(v)$ is an ordinary function if $Re(\alpha) \geq n$ and is a distribution of α if $Re(\alpha) < n$. Let supp $R^H_\alpha(v)$ denote the support of $R^H_\alpha(v)$ and suppose supp $R^H_\alpha(v) \subset \bar{\Gamma}_+$, that is supp $R^H_\alpha(v)$ is compact.

Definition 2.2. Let $x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$ and write

$$|x| = x_1^2 + x_2^2 + ... + x_n^2. \tag{10}$$

For any complex number β, define

$$R^e_\beta(x) = 2^{-\beta}\pi^{\frac{-n}{2}}\Gamma(\frac{n-\beta}{2})\frac{|x|^{\frac{\beta-n}{2}}}{\Gamma(\frac{\beta}{2})}. \tag{11}$$

The function $R^e_\beta(x)$ is called the elliptic kernel of Marcel Riesz and is ordinary function for $Re(\beta) \geq n$ and is a distribution of β for $Re(\beta) < n$.

Definition 2.3. Let $x = (x_1, x_2, ..., x_n)$ be a point of $\mathbb{R}^n$ and write

$$z = x_1^2 + x_2^2 + ... + x_p^2 + i\left(x_{p+1}^2 + x_{p+2}^2 + ... + x_{p+q}^2\right) \tag{12}$$

and

$$w = x_1^2 + x_2^2 + ... + x_p^2 - i\left(x_{p+1}^2 + x_{p+2}^2 + ... + x_{p+q}^2\right), \tag{13}$$

For any complex number γ and ν, we define

$$T_\nu(z) = 2^{-\nu}\pi^{\frac{-n}{2}}\Gamma(\frac{n-\nu}{2})\frac{z^{\frac{\nu-n}{2}}}{\Gamma(\frac{\nu}{2})} \tag{14}$$

and

$$S_\gamma(w) = 2^{-\gamma}\pi^{\frac{-n}{2}}\Gamma(\frac{n-\gamma}{2})\frac{w^{\frac{\gamma-n}{2}}}{\Gamma(\frac{\gamma}{2})}. \tag{15}$$

The function $S_\gamma(w)$ and $T_\nu(z)$ is an ordinary function if $Re(\gamma) \geq n$ and $Re(\nu) \geq n$, is a distribution of γ for $Re(\gamma) < n$ and ν for $Re(\nu) < n$.

Lemma 2.1. Given the equation

$$\triangle^k u(x) = 0, \tag{16}$$

where $\triangle^k$ is the Laplacian operator iterated k-times defined by equation (6) we obtain $u(x) = ((-1)^{k-1} R^e_{2(k-1)}(x))^{(m)}$ as a solutions of *(16)* where $m = (n-4)/2, n \geq 4$ is non-negative integer and n is even and $R^e_{2(k-1)}(x)$ defined by equation *(11)* with m derivatives and $\beta = 2(k-1)$.

Proof. see [6, Lemma 2.2].

Lemma 2.2. Given the equation

$$\Box^k u(x) = 0, \tag{17}$$

where $\Box^k$ is the Ultra-hyperbolic operator iterated k-times defined by equation *(5)* we obtain $u(x) = (R^H_{2(k-1)}(v))^{(m)}$ as a solutions of *(17)* where $m = (n-4)/2, n \geq 4$ is non-negative integer and n is even and $R^H_{2(k-1)}(v)$ defined by equation *(9)* with m derivatives and $\alpha = 2(k-1)$.

Proof. see [6, Lemma 2.3].

Lemma 2.3. The function $T_{2k}(z) * S_{2k}(w)$ is an elementary solutions of the operator $L^k = L_1^k L_2^k = L_2^k L_1^k$, denoted by

$$L^k = \left[\left(\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2} \right)^2 - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right)^2 \right]^k, \tag{18}$$

where $T_{2k}(z)$ and $S_{2k}(w)$ are defined by equation *(14)* and *(15)*, respectively, with $\gamma = \nu = 2k$. The operator L_1^k and L_2^k are defined by equation *(3)* and *(4)*, respectively.

Proof. We need to show that $L_1^k[(-1)^k (\mathrm{i})^{\frac{q}{2}} T_{2k}(z)] = \delta$ and $L_2^k[(-1)^k (-\mathrm{i})^{\frac{q}{2}} S_{2k}(w)] = \delta$. At first we have to show that

$$L_1^k T_\nu(z) = (-1)^k T_{\nu-2k}(z), \;\; L_2^k S_\gamma(w) = (-1)^k S_{\gamma-2k}(w) \tag{19}$$

and also

$$T_{-2k}(z) = (-1)^k (-\mathrm{i})^{\frac{q}{2}} L_1^k \delta, \;\; S_{-2k}(w) = (-1)^k (\mathrm{i})^{\frac{q}{2}} L_2^k \delta. \tag{20}$$

Now for $k = 1$,

$$\begin{aligned} L_1 T_\nu(z) &= \left(\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2} - \mathrm{i} \sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right) T_\nu(z) \\ &= 2^{-\nu} \pi^{\frac{-n}{2}} \frac{\Gamma(\frac{n-\nu}{2})}{\Gamma(\frac{\nu}{2})} (\nu - n)(\nu - 2) z^{\frac{\nu-2-n}{2}} \\ &= (-1) 2^{-\nu-2} \frac{\Gamma(\frac{n-\nu-2}{2})}{\Gamma(\frac{\nu-2}{2})} z^{\frac{\nu-2-n}{2}} \\ &= -T_{\nu-2}(z). \end{aligned}$$

By repeating $k-$times in operating L_1 to $T_\nu(z)$, we obtain $L_1^k T_\nu(z) = (-1)^k T_{\nu-2k}(z)$. Similarly, $L_2^k S_\gamma(w) = (-1)^k S_{\gamma-2k}(w)$.

Now consider

$$z = x_1^2 + x_2^2 + ... + x_p^2 + \mathrm{i} \left(x_{p+1}^2 + x_{p+2}^2 + ... + x_{p+q}^2 \right), p + q = n$$

by changing the variable

$$x_1 = y_1, x_2 = y_2, ..., x_p = y_p,$$

$$x_{p+1} = \frac{y_{p+1}}{\sqrt{\mathrm{i}}} + x_{p+2} = \frac{y_{p+2}}{\sqrt{\mathrm{i}}}, ..., x_{p+q} = \frac{y_{p+q}}{\sqrt{\mathrm{i}}}.$$

Thus we have $z = y_1^2 + y_2^2 + ... + y_p^2 + y_{p+1}^2 + y_{p+2}^2 + ... + y_{p+q}^2$.

Denote $z = r^2 = y_1^2 + y_2^2 + ... + y_n^2$ and consider the generalized function $z^\lambda = r^{2\lambda}$ where λ is any complex number. Now $\left\langle z^\lambda, \varphi \right\rangle = \int_{R^n} z^\lambda \varphi(x) \mathrm{d}x$, where $\varphi \in \mathfrak{D}$ the space of infinitely differentiable functions with compact supports. Thus

$$\begin{aligned} \left\langle z^\lambda, \varphi \right\rangle &= \int_{R^n} r^{2\lambda} \varphi \frac{\partial(x_1, x_2, \ldots, x_n)}{\partial(y_1, y_2, \ldots, y_n)} \mathrm{d}y_1 \mathrm{d}y_2 \cdots \mathrm{d}y_n \\ &= \frac{1}{(\mathrm{i})^{q/2}} \int_{R^n} r^{2\lambda} \varphi \mathrm{d}y \\ &= \frac{1}{(\mathrm{i})^{q/2}} \left\langle r^{2\lambda}, \varphi \right\rangle. \end{aligned}$$

By Gelfand and Shilov [3, p.271], the function $r^{2\lambda}$ have simple poles at $\lambda = (-n/2) - k, k$ is nonnegative and for $k = 0$ we can find the residue of $r^{2\lambda}$ at $\lambda = -n/2$ and by [3, p.73], we obtain

$$\operatorname*{res}_{\lambda = -\frac{n}{2}} (r^{2\lambda}) = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})} \delta(x).$$

Thus

$$\operatorname*{res}_{\lambda=-\frac{n}{2}}(z^{\lambda}) = (-\mathrm{i})^{\frac{q}{2}} \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}\delta(x). \tag{21}$$

We next find the residues of z^{λ} at $\lambda = (-n/2) - k$. Now, by computing directly we have

$$L_1 z^{\lambda+1} = 2(\lambda+1)(2\lambda+n)z^{\lambda}.$$

By $k-$ fold iteration, we obtain

$$\begin{aligned} L_1^k z^{\lambda+k} = 4^k \quad & (\lambda+1)(\lambda+2)\cdots(\lambda+k)\left(\lambda+\tfrac{n}{2}\right)\times \\ & \left(\lambda+\tfrac{n}{2}+1\right)\cdots\left(\lambda+\tfrac{n}{2}+k-1\right) z^{\lambda} \end{aligned}$$

or

$$\begin{aligned} z^{\lambda} = & \frac{1}{4^k(\lambda+1)(\lambda+2)\cdots(\lambda+k)}\times \\ & \frac{1}{\left(\lambda+\frac{n}{2}\right)\left(\lambda+\frac{n}{2}+1\right)\cdots\left(\lambda+\frac{n}{2}+k-1\right)} L_1^k z^{\lambda+k}. \end{aligned}$$

Thus

$$\operatorname*{res}_{\lambda=-\frac{n}{2}-k}(z^{\lambda}) = \frac{1}{4^k k\left(\frac{n}{2}+k-1\right)\left(\frac{n}{2}+k-2\right)\cdots\left(\frac{n}{2}\right)} \operatorname*{res}_{\lambda=-\frac{n}{2}} L_1^k z^{\lambda+k}.$$

By (21) and the properties of Gamma functions, we obtain

$$\operatorname*{res}_{\lambda=-\frac{n}{2}-k}(z^{\lambda}) = (-\mathrm{i})^{q/2} \frac{2\pi^{\frac{n}{2}}}{4^k\Gamma(\frac{n}{2}+k)} L_1^k\delta(x). \tag{22}$$

Now we consider $T_{-2k}(z)$ we have

$$\begin{aligned} T_{-2k}(z) &= \lim_{\nu\to-2k} T(z) \\ &= \pi^{-\frac{n}{2}} \frac{\lim_{\nu\to-2k} z^{(\nu-n)/2}}{\lim_{\nu\to-2k}\Gamma(\frac{\nu}{2})} \lim_{\nu\to-2k} 2^{-\nu}\Gamma\left(\frac{n-\nu}{2}\right) \\ &= \pi^{-\frac{n}{2}} \frac{\lim_{\nu\to-2k}(\nu+2k) z^{(\nu-n)/2}}{\lim_{\nu\to-2k}\Gamma(\nu+2k)(\frac{\nu}{2})} \lim_{\nu\to-2k} 2^{2k}\Gamma\left(\frac{n+2k}{2}\right) \\ &= 4^k\pi^{-\frac{n}{2}} \frac{\operatorname*{res}_{\nu=-2k} z^{(\nu-n)/2}}{\operatorname*{res}_{\nu=-2k}\Gamma(\frac{\nu}{2})}\Gamma\left(\frac{n+2k}{2}\right). \end{aligned}$$

Since $\operatorname*{res}_{\lambda=-\frac{n}{2}-k} z^{\lambda} = \operatorname*{res}_{\nu=-2k} z^{(\nu-n)/2}$ and $\operatorname*{res}_{\nu=-2k}\Gamma(\frac{\nu}{2}) = \frac{2(-1)^k}{k!}$, by (22) and the properties of Gamma function we obtain

$$T_{-2k}(z) = (-1)^k(-\mathrm{i})^{\frac{q}{2}} L_1^k\delta(x).$$

Similarly

$$S_{-2k}(w) = (-1)^k(\mathrm{i})^{\frac{q}{2}} L_2^k\delta(x).$$

Thus we have

$$T_0(z) = (-\mathrm{i})^{\frac{q}{2}}\delta(x)\ , S_0(w) = (\mathrm{i})^{\frac{q}{2}}\delta(x). \tag{23}$$

Now, from (19) $L_1^k T_{2k}(z) = (-1)^k T_0(z)$ for $\nu = 2k$. Thus by (23) we obtain $L_1^k(-1)^k(\mathrm{i})^{\frac{q}{2}}T_{2k}(z) = \delta(x)$. It follows that $(-1)^k(\mathrm{i})^{\frac{q}{2}}T_{2k}(z)$ is an elementary solution of the operator L_1^k. Similarly
$(-1)^k(-\mathrm{i})^{\frac{q}{2}}S_{2k}(w)$ is also an elementary solution of the operator L_2^k. Thus we have

$$L^k(T_{2k}(z) * S_{2k}(w)) = L_2^k(-1)^k(\mathrm{i})^{\frac{q}{2}}T_{2k}(z) * L_1^k(-1)^k(-\mathrm{i})^{\frac{q}{2}}S_{2k}(w) = \delta.$$

Lemma 2.4. Given the equation

$$\triangle u(x) = f(x, u(x)), \tag{24}$$

where f is defined and has continuous first derivatives for all $x \in \Omega \cup \partial\Omega, \Omega$ is an open subset of $\mathbb{R}^n$ and $\partial\Omega$ denotes the boundary of Ω. Assume f is a bounded, that is $|f(x,u)| \leq N$ and the boundary condition $u(x) = 0$ for $x \in \partial\Omega$. Then we obtain $u(x)$ as a uniqe solution of *(24)*.

Proof. We can prove this lemma by the method of iterations and the Schauder's estimates, see [1, pp. 369-372].

3. Main results

Theorem 3.1. *Given the nonlinear equation*

$$\oplus^k u(x) = f(x, \triangle^{k-1}\Box^k L^k u(x)), \tag{25}$$

where $\oplus^k$ is the operator iterated k times, defined by (1), $\triangle^{k-1}$ is the Laplacian iterated $k-1$ times defined by (6) and $\Box^k$ is the ultrahyperbolic operator iterated k times defined by (5). Let f be defined and have continuous first derivatives for all $x \in \Omega \cup \partial\Omega, \Omega$ is an open subset of $\mathbb{R}^n$ and $\partial\Omega$ denotes the boundary of Ω and n is even with $n \geq 4$. Let f be a bounded function, that is

$$|f(x, \triangle^{k-1}\Box^k L^k u(x))| \leq N \tag{26}$$

and the boundary condition

$$\triangle^{k-1}\Box^k L^k u(x) = 0, \ \ for \ x \in \partial\Omega; \tag{27}$$

then we obtain

$$u(x) = (-1)^{k-1} R^e_{2(k-1)}(x) * R^H_{2k}(v) * S_{2k}(w) * T_{2k}(z) * W(x) \tag{28}$$

as a solution of (25) with the boundary condition

$$u(x) = S_{2k}(w) * T_{2k}(z) * R^H_{2k}(v) * (-1)^{k-2}(R^e_{2(k-2)}(x))^{(m)}$$

for $x \in \partial\Omega, m = (n-4)/2, \ k = 2, 3, 4, \ldots$ and v is given by (8), $W(x)$ is a continuous function for $x \in \Omega \cup \partial\Omega, R^e_{2(k-2)}(x)$ and $R^H_{2k}(v)$ are given by (11) and (9), respectively, with $\beta = 2(k-2)$and $\alpha = 2k$. Moreover, for $q = 0$ then (25) becomes

$$\triangle_p^{4k} u(x) = f(x, \triangle^{4k-1} u(x)), \tag{29}$$

with boundary condition

$$\triangle^{4k-1} u(x) = 0, \ \ for \ x \in \partial\Omega, \tag{30}$$

where $\triangle_p^{4k}$ is the Laplacian of p-dimension iterated $4k$-times. we have

$$u(x) = (-1)^{k-1} R^e_{2(k-1)}(x) * R^e_{6k}(x) * W(x) \tag{31}$$

as a solution of (29) where $|x| = x_1^2 + x_2^2 + ... + x_p^2$.

Proof. From equation (25), we have

$$\oplus^k u(x) = \triangle(\triangle^{k-1}\Box^k L^k u(x)) = f(x, \triangle^{k-1}\Box^k L^k u(x)). \tag{32}$$

Since $u(x)$ has continuous derivatives up to order $4k$ for $k=1,2,3,\dots$ we can assume

$$\triangle^{k-1}\Box^k L^k u(x) = W(x),\ \ for\ x\in\partial\Omega. \tag{33}$$

Thus, (32) can be written in the form

$$\oplus^k u(x) = \triangle W(x) = f(x, W(x)), \tag{34}$$

by (26)

$$|f(x, W(x))| \le N, \tag{35}$$

and by (27), $W(x)$=0 or

$$\triangle^{k-1}\Box^k L^k u(x) = 0,\ \ for\ x\in\partial\Omega. \tag{36}$$

Thus by Lemma 2.4 there exist a unique solution $W(x)$ of (34) which satisfies (35). Now consider (33), we have $\triangle^{k-1}(-1)^{k-1}R^e_{2(k-1)}(x)=\delta$ and $\Box^k R^H_{2k}(v)=\delta$ where δ is the Dirac-delta distribution, that is $R^H_{2k}(v)$ and $(-1)^{k-1}R^e_{2(k-1)}(x)$ are the elementary solutions of the operators $\Box^k$ and $\triangle^{k-1}$, respectively, see[8, p.11] and see[2, p.118]. The functions $R^H_{2k}(v)$ and $R^e_{2(k-1)}(x)$ are defined by (9) and (11), respectively, with $\beta=2(k-1),\alpha=2k$. And by Lemma 2.3, the function $T_{2k}(z)*S_{2k}(w)$ is an elementary solutions of the operator L^k, are defined by equation (14) and (15), respectively, with $\gamma=\nu=2k$. Thus, convolving both sides of (33) by

$$(-1)^{k-1}R^e_{2(k-1)}(x) * R^H_{2k}(v) * T_{2k}(z) * S_{2k}(w),$$

we obtain

$$\begin{aligned}&[(-1)^{k-1}R^e_{2(k-1)}(x) * R^H_{2k}(v) * T_{2k}(z) * S_{2k}(w)] * \triangle^{k-1}\Box^k L^k u(x)\\ &\quad= [(-1)^{k-1}R^e_{2(k-1)}(x) * R^H_{2k}(v) * T_{2k}(z) * S_{2k}(w)] * W(x).\end{aligned}$$

By properties of convolution, we obtain

$$[\triangle^{k-1}(-1)^{k-1}R^e_{2(k-1)}(x)] * [\Box^k R^H_{2k}(v)] * [L^k T_{2k}(z) * S_{2k}(w)] * u(x) =$$

$$[(-1)^{k-1}R^e_{2(k-1)}(x) * R^H_{2k}(v) * T_{2k}(z) * S_{2k}(w)] * W(x),$$

$$\delta * \delta * \delta * u(x) =$$

$$[(-1)^{k-1}R^e_{2(k-1)}(x) * R^H_{2k}(v) * T_{2k}(z) * S_{2k}(w)] * W(x).$$

Thus

$$u(x) = (-1)^{k-1}R^e_{2(k-1)}(x) * R^H_{2k}(v) * T_{2k}(z) * S_{2k}(w) * W(x) \tag{37}$$

as required. Consider $\triangle^{k-1}\Box^k L^k u(x) = 0, \ for\ x \in \partial\Omega$. By Lemma 2.1, we have

$$\Box^k L^k u(x) = (-1)^{k-2}(R^e_{2(k-2)}(x))^{(m)}.$$

Convolving both sides of the above equation by $R^H_{2k}(v) * T_{2k}(z) * S_{2k}(w)$, we obtain

$$R^H_{2k}(v) * T_{2k}(z) * S_{2k}(w) * \Box^k L^k u(x)$$
$$= R^H_{2k}(v) * T_{2k}(z) * S_{2k}(w) * (-1)^{k-2}(R^e_{2(k-2)}(x))^{(m)},$$

$$[\Box^k R^H_{2k}(v)] * [L^k * T_{2k}(z)S_{2k}(w)] * u(x)$$
$$= R^H_{2k}(v) * T_{2k}(z) * S_{2k}(w) * (-1)^{k-2}(R^e_{2(k-2)}(x))^{(m)},$$

$$\delta * \delta * u(x)$$
$$= R^H_{2k}(v) * T_{2k}(z) * S_{2k}(w) * (-1)^{k-2}(R^e_{2(k-2)}(x))^{(m)},$$

$$u(x) = R^H_{2k}(v) * T_{2k}(z) * S_{2k}(w) * (-1)^{k-2}(R^e_{2(k-2)}(x))^{(m)},$$

$for\ x \in \partial\Omega$ and $k = 2, 3, 4, \ldots$.

Moreover, for $q = 0$ then (25) becomes

$$\triangle^{4k}_p u(x) = f(x, \triangle^{4k-1}u(x)), \tag{38}$$

with boundary condition

$$\triangle^{4k-1}u(x) = 0, \ for\ x \in \partial\Omega,$$

where $\triangle_p^{4k}$ is the Laplacian of p-dimension iterated $4k$-times. we have

$$u(x) = (-1)^{k-1} R^e_{2(k-1)}(x) * R^e_{6k}(x) * W(x) \tag{39}$$

as a solution of (38) where $|x| = x_1^2 + x_2^2 + ... + x_p^2$.

On the other hand, we can also find (39) from (37), since $q = 0$, we have $R^H_{2k}(v)$ reduces to $R^e_{2(k)}(x)$, $S_{2k}(w)$ reduces to $R^e_{2(k)}(x)$ and $T_{2k}(z)$ reduces to $R^e_{2(k)}(x)$, where $|x| = x_1^2 + x_2^2 + ... + x_p^2$.

Thus, by (37) for $q = 0$, we obtain

$$\begin{aligned} u(x) &= (-1)^{k-1} R^e_{2(k-1)}(x) * R^e_{2k}(x) * R^e_{2k}(x) * R^e_{2k}(x) * W(x) \\ &= (-1)^{k-1} R^e_{2(k-1)}(x) * R^e_{2k+2k+2k}(x) * W(x) \\ &= (-1)^{k-1} R^e_{2(k-1)}(x) * R^e_{6k}(x) * W(x). \end{aligned}$$

This completes the proof. □

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