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RATIONAL COHOMOLOGY OF CLASSIFYING SPACES AND HILALI CONJECTURE

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Abstract. Let X be a simply connected space with finite-dimensional rational homotopy groups. Denote by $\mathrm{Bau}_1(X)$ the classifying space of fibrations with fiber X and $\max \pi_*(X) = \max\{i; \pi_i(X) \neq 0\}$. In this paper, we show that the dimensional rational cohomology and the Lusternik-Schnirelmann category of $\mathrm{Bau}_1(X)$ are infinite if $\max \pi_*(X)$ is odd. Our results apply, in particular, when X is elliptic. As a consequence, we prove the Hilali conjecture for classifying space.

Keywords: rational homotopy theory; classifying space; Sullivan minimal model; Lusternik-Schnirelmann category; Hilali conjecture.

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1. Introduction

Given a simply connected CW complex X of finite type, let $\mathrm{aut}_1(X)$ denote the space of self-homotopy equivalences of X that are homotopic to the identity. This a monoid with multiplication given by composition of maps and topologized as a sub-space of $\mathrm{Map}(X,X)$; the space of all continuous functions with the compact-open topology. So, by applying the Dold-Lashof

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construction to the monoid $\mathrm{aut}_1(X)$, we obtain the classifying space $\mathrm{Baut}_1(X)$ [4]. Recall that $\mathrm{Baut}_1(X)$ classifies orientable fibrations with fiber X [[1] and [12]].

The identity $\Omega\mathrm{Baut}_1(X) \simeq \mathrm{aut}_1(X)$ [[3], satz, 7.3] gives the isomorphism

$$\pi_{*+1}\left(Baut_1(X);*\right) \cong \pi_*\left(\mathrm{aut}_1(X);1\right).$$

Then, the space $\mathrm{Baut}_1(X)$ is, in turn, a simply connected CW complex and so admits a rationalization $\mathrm{Baut}_1(X)_{\mathbb{Q}}$. We observe that, as the space $\mathrm{Baut}_1(X)$ is quite complicated, calculations and other descriptions will be difficult to obtain.

The calculations of $H^*\left(\mathrm{Baut}_1(X);\mathbb{Q}\right)$ and $\pi_*\left(\mathrm{Baut}_1(X)\right)\otimes\mathbb{Q}$ are the subject of a long line of celebrated structure theorems. In appendix 1 of [11], Milnor computed $H^*\left(\mathrm{Baut}_1(\mathbb{S}^n);\mathbb{Q}\right)$ to be a polynomial algebra with a single positive degree generator.

We turn now to describe the conjecture referred in the title. We say a space X is π-finite if X is a simply connected CW complex and dim $\pi_*(X)\otimes\mathbb{Q} < \infty$. A π-finite space X is said to be elliptic if dim $H^*(X;\mathbb{Q}) < \infty$. One of the most famous and most influential conjectures for these spaces is:

Conjecture 1. *(Hilali) Let X be a simply connected elliptic space, then*

$$dim\ \pi_*(X)\otimes\mathbb{Q} \leq dim\ H^*(X;\mathbb{Q}).$$

Generally, speaking about cohomology is delicate, invariant, and difficult to compute. Until now, the Hilali conjecture has been established in various special cases, but in general it remains open [[6], [2], [14] and [15]].

With this long and extensive history, it is surprising that the verification question of Hilali conjecture for classifying space has never been addressed.

Now we briefly summarize the continent of this paper. In section 2, we recall the necessary definitions and preliminaries from rational homotopy theory, namely the Sullivan minimal model and derivation. Section 3 is devoted to our results. Finally, we conclude this section with examples and suggestion for further work.

2. SOME PRELIMINARIES IN RATIONAL HOMOTOPY THEORY

All our spaces will be simply connected with the homotopy type of CW complexes with rational cohomology of finite type. We will work with $\mathbb{Q}$ as ground field and our principal tools are Sullivan minimal models. A detailed description of these and the standard tools of rational homotopy theory can be found in [5]. For our purposes, we recall the following:

D. Sullivan in [13] defined a contravariant functor A_{PL} which associates to each space X a commutative graded differential algebra (hereafter cgda) $A_{PL}(X)$ which represents the rational homotopy type of X. He also constructed, for each simply connected cgda (A,d) (i.e., satisfying $H^0(A,d) = H^1(A,d) = 0$), another cgda $(\Lambda V,d)$ and a map

$$(\Lambda V,d)\overset{\cong}{\to}(A,d)$$

which induces an isomorphism in cohomology, where ΛV denotes the free commutative graded algebra on the graded vector space $V = \oplus_n V^n$, which has a well ordered, homogeneous basis $\{x_\alpha\}$ such that, if $V_{<\alpha}$ denotes span $\left\{x_\beta \mid \beta < \alpha\right\}$, we have $dx_\alpha \in \Lambda^{\geq 2}(V_{<\alpha})$. The cgda $(\Lambda V,d)$ is called a Sullivan minimal model of (A,d) or a Sullivan minimal model of X if $A = A_{PL}(X)$.

In particular, if $(\Lambda V,d)$ is a Sullivan minimal model of X, there are isomorphism's:

$$H^*(\Lambda V,d) \cong H^*(X;\mathbb{Q}) \text{ as commutative graded algebras.}$$

$$V \cong \pi_*(X) \otimes \mathbb{Q} \text{ as graded vector spaces.}$$

Therefore, we can also characterize an elliptic space in terms of its Sullivan minimal model. A space X with Sullivan minimal model $(\Lambda V,d)$ is elliptic if V and $H^*(\Lambda V,d)$ are both finite dimensional. There is a remarkable sub-class of elliptic spaces called pure spaces.

Definition 1. *An elliptic Sullivan minimal model $(\Lambda V,d)$ is called pure if*

$$dV^{\text{even}} = 0 \textit{ and } dV^{\text{odd}} \subset \Lambda V^{\text{even}}.$$

Also, a simply connected elliptic space X is called pure if its Sullivan minimal model is so.

Examples of such spaces is given by finite products of spheres and finite products of complex projective spaces.

The space $\mathrm{Baut}_1(X)$ was among the first geometric objects described in rational homotopy theory. In his foundational paper [13], Sullivan gave a model for this simply connected space in terms of derivation of Sullivan minimal model. For the convenience of the reader, we recall the definition of derivation.

Definition 2. *A derivation of degree i of a cgda $(\Lambda V, d)$ will mean a linear map lowering degrees by i and satisfying the product law:*

$$\theta\left(V^j\right) \subseteq (\Lambda V)^{j-i} \text{ and } \theta(xy) = \theta(x)y + (-1)^{i|x|}x\theta(y) \text{ for } x, y \in \Lambda V.$$

We write $\mathrm{Der}_i(\Lambda V, d)$ for the vector space of all degree i derivations. The boundary operator $D : \mathrm{Der}_i(\Lambda V, d) \to \mathrm{Der}_{i-1}(\Lambda V, d)$ is defined by

$$D(\theta) = d\theta - (-1)^i \theta d.$$

Let $\mathrm{Der}_*(\Lambda V, d)$ be the set of all positive degree derivations with the restriction that $\mathrm{Der}_1(\Lambda V, d)$ is the vector space of derivations of degree one which commute with the differential d. Then $(\mathrm{Der}_*(\Lambda V, d), D)$ has the structure of a differential graded Lie algebra with the commutator bracket

$$[\theta_1, \theta_2] = \theta_1\theta_2 - (-1)^{|\theta_1||\theta_2|}\theta_2\theta_1$$

for $\theta_1, \theta_2 \in \mathrm{Der}_*(\Lambda V, d)$.

We use the following theorem to prove our results.

Theorem 1. *(Sullivan) Let X be a simply connected CW complex of finite type with Sullivan minimal model $(\Lambda V, d)$, then*

$$\pi_*(\Omega Baut_1(X)) \otimes \mathbb{Q} \cong H_*(Der(\Lambda V, d), D)$$

as graded Lie algebras, in which the left hand side has the Samelson bracket.

Calculations of the rational homotopy type of $\mathrm{Baut}_1(X)$ follow the process:

$$\begin{array}{ccccccc} X & \to & (\Lambda V, d) & \to & (Der_*(\Lambda V, d), D) & \to & \mathrm{Baut}_1(X) \\ \text{space} & \to & \text{minimal model} & \to & \text{dg Lie agebra} & \to & \text{classifying space} \end{array}$$

To well illustrate this process, we propose the following example.

Example 1. *For $n \geq 1$, we have*

$$Baut_1(\mathbb{S}^{2n+1}) \simeq K(\mathbb{Q}, 2n+2).$$

Write the Sullivan minimal model of $\mathbb{S}^{2n+1}$ *as* $(\Lambda(y), 0)$ *with* $|y| = 2n+1$. *Then,* $(\mathrm{Der}_*(\Lambda(y), 0), D)$ *is the abelian Lie algebra* $\langle y^* \rangle$ *where* y^* *denotes the dual derivation to* y *with* $D = 0$. *So* $(\mathrm{Der}_*(\Lambda(y), 0), 0)$ *has homology isomorphic, as a graded vector space, to* $\langle y^* \rangle$. *This is exactly a differential graded Lie algebra model for* $K(\mathbb{Q}, 2n+2)$.

In the sequel, we assume that all spaces appearing in this paper are *rational simply connected spaces*, i.e., $X \simeq X_{\mathbb{Q}}$.

3. Main Results

Given a π-finite space X, write

$$\max\ \pi_*(X) = \max\ \{i; \pi_i(X) \neq 0\}.$$

In other words,

$$\max\ \pi_*(X) = \max\ \left\{i; V^i \neq 0\right\},$$

where $(\Lambda V, d)$ is the Sullivan minimal model of X. For example,

$$\max\ \pi_*(\mathbb{S}^{2n}) = 4n-1 \text{ and } \max\ \pi_*(\mathbb{S}^{2n+1}) = 2n+1.$$

In this section, we prove that, when X is π-finite and $\max\ \pi_*(X)$ is odd, the dimensional rational cohomology and the Lusternik-Schnirelmann category of $\mathrm{Baut}_1(X)$ are both infinite. For these spaces, we deduce that $\mathrm{Baut}_1(X)$ satisfies the Hilali conjecture.

We begin by the following which plays a key role in the sequel:

Lemma 1. *(cf.* [8], *Prop 2.3) Let X be a π-finite space with*

$$\pi_i(X) = \begin{cases} 0 & \text{for } i > N \\ \mathbb{Q}^r \text{ some } r \geq 1 & \text{for } i = N \end{cases}$$

Then

$$\pi_i(Baut_1(X)) = \begin{cases} 0 & \text{for } i > N+1 \\ \mathbb{Q}^r & \text{for } i = N+1. \end{cases}$$

Proof. Denote by $(\Lambda V, d)$ the Sullivan minimal model of X with V non-zero only in degrees $\leq N$ and dim $V^N = r$. Then the differential graded Lie algebra of derivations $(Der_*(\Lambda V, d), D)$ is non-zero only in degrees $\leq N$. Therefore, we have

$$H_i(Der(\Lambda V, d), D) = 0 \quad \text{for } i > N.$$

By Theorem 1, it follows that

$$\pi_i(\mathrm{Baut}_1(X)) = 0 \quad \text{for } i > N+1.$$

Furthermore, in degree N, for each $\theta \in Hom_{\mathbb{Q}}\left(V^N, \mathbb{Q}\right)$, we obtain a derivation in $(Der_*(\Lambda V, d), D)$ of degree N by setting $\theta\left(V^N\right) = 1$ and extending as a derivation. Any such derivation is a D-cycle, since the elements of V^N do not occur in the differential of any other generators. There are no non-zero boundaries of degree N, since $(Der_*(\Lambda V, d), D)$ is zero in degree $N+1$ and higher. So, we have

$$H_N(Der(\Lambda V, d), D) = Hom_{\mathbb{Q}}\left(V^N, \mathbb{Q}\right).$$

It follows that by using Theorem 1

$$\pi_{N+1}(\mathrm{Baut}_1(X)) = Hom_{\mathbb{Q}}\left(V^N, \mathbb{Q}\right).$$

□

Now, let us prove the main result of this paper.

Theorem 2. *Let X be a π-finite space with* max $\pi_*(X)$ *is odd. Then* $\mathrm{Baut}_1(X)$ *has infinite dimensional rational cohomology.*

Proof. First, write $(\Lambda V, d)$ the Sullivan minimal model of X with V is finite dimensional. Further, as stated in the preliminaries $\mathrm{Hom}(V, \mathbb{Q}) \cong \pi_*(X)$, the condition " max $\pi_*(X)$ is odd" is equivalent to the top degree of V is odd. Thus, the Sullivan minimal model of X takes the form $(\Lambda V, d) = (\Lambda(U \oplus \mathbb{Q}(y)), d)$ with $\mathbb{Q}(y)$ is a graded supplementary of U in V and y is of odd degree. Define a derivation θ of ΛV by $\theta(y) = 1$ and $\theta(z) = 0$ if $z \in \Lambda U$. Then for degree reasons, we have $D(\theta) = 0$. Otherwise, it is easy to see that the derivation θ cannot bound.

Further, denote by x the class $[\theta] \in H_{|y|}(\mathrm{Der}(\Lambda V)) \cong \pi_{|y|+1}(\mathrm{Baut}_1(X))$. Thus from Lemma 1, $\mathrm{Baut}_1(X)$ has a Sullivan minimal model of the form

$$M(\mathrm{Baut}_1(X)) \cong (\Lambda(W \oplus \mathbb{Q}(x)), \delta)$$

with W is finite dimensional and max $\pi_*(\mathrm{Baut}_1(X)) = |x| = |y| + 1$.

Second, we appeal to some results of Halperin concerning elliptic Sullivan minimal models. First of all, to any Sullivan minimal model $(\Lambda V, d)$ with V is finite dimensional , there is an associated pure model, denoted $(\Lambda V, d_\sigma)$, which is defined by adjusting the differential d to d_σ as follows: We set $d_\sigma = 0$ on each even degree generator of V, and on each odd degree generator $v \in V$, we set $d_\sigma(v)$ equal to the part of $d(v)$ contained in $\Lambda(V^{even})$. One checks that this defines a differential d_σ on ΛV, and thus we obtain a pure Sullivan minimal model $(\Lambda V, d_\sigma)$. Applying all this to the Sullivan minimal model $(\Lambda(W \oplus \mathbb{Q}(x)), \delta)$, we obtain for $n \geq 1$, $[x^n] \in H^*(\Lambda(W \oplus \mathbb{Q}(x)), \delta_\sigma)$. Thus, by Proposition 32.4 in [5], we have dim $H^*(\Lambda(W \oplus \mathbb{Q}(x)), \delta)$ is infinite, and the result follows. □

The following result aim to verify Conjecture 1 for classifying space.

Theorem 3. *Let X be as in Theorem 2. Then* $\mathrm{Baut}_1(X)$ *satisfies the Hilali conjecture.*

Proof. First by Theorem 2, we obtain

$$\dim H^*(\mathrm{Baut}_1(X)) = \infty$$

Otherwise, since $\pi_*(X)$ is finite dimensional and from Lemma 1, we get

$$\dim \pi_*(\mathrm{Baut}_1(X)) < \infty$$

Combining all the above, we deduce that

$$\dim H^*(\mathrm{Baut}_1(X)) > \dim \pi_*(\mathrm{Baut}_1(X)).$$

□

We now mention a second consequence of Theorem 2. For this, let us recall the definition of Lusternik-Schnirelmann category. Recall that this is an old and well known numerical invariant of the homotopy type of spaces which may be defined as follows: The Lusternik-Schnirelmann category of X, $\mathrm{cat}(X)$, is the least integer n such that X can be covered by $(n+1)$ open subsets contractible in X and is ∞ if no such n exists. The rational category of a simply connected space X, $\mathrm{cat}_0(X)$, is defined by $\mathrm{cat}_0(X) = \mathrm{cat}(X_{\mathbb{Q}})$. For example $\mathrm{cat}_0(\mathbb{S}^n) = 1$ and $\mathrm{cat}_0(\mathbb{C}P^n) = n$ for $n \geq 2$.

In [[9] and [10]], Gatsinzi giving many classes of spaces X for which the rational Lusternik-Schnirelmann category of $\mathrm{Baut}_1(X)$ is infinite.

We generalize one result of his here:

Theorem 4. *Let X be as in Theorem 2. Then $\mathrm{Baut}_1(X)$ has infinite Lusternik-Schnirelmann category.*

Proof. It is an immediate consequence of Theorem 2 and Proposition 32.4 in [5]. □

A much stronger consequence of Theorem 2 and Theorem 4 follows if we restrict the space X as follows.

Corollary 1. *Let X be an elliptic space. Then, we have*

$$\dim H^*(\mathrm{Baut}_1(X);\mathbb{Q}) = \infty \text{ and } cat(\mathrm{Baut}_1(X)) = \infty.$$

Remark 1. *Let X be a π-finite space. If X is elliptic, then max $\pi_*(X)$ is odd, but in general, the converse is false. Indeed, consider the following space $X = K(\mathbb{Q},2p)\times K(\mathbb{Q},2q+1)$ where $q \geq p \geq 1$. Thus max $\pi_*(X) = 2q+1$ though X is not elliptic.*

We conclude this section with examples and suggestion for future work. Our focus here is more on the Hilali conjecture. We begin with the following:

Example 2. *Let* $X = \overbrace{K(\mathbb{Q},2p)\times \ldots \times K(\mathbb{Q},2p)}^{n\ times}$. *Hence, from* [7], *we have*

$$\mathrm{Baut}_1(X) = \overbrace{K(\mathbb{Q},2p+1)\times \ldots \times K(\mathbb{Q},2p+1)}^{n\ times}$$

It follows immediately,

$$\begin{aligned} dim\, H^*(Baut_1(X)) &= 2^n \\ &\geq n \\ &\geq dim\, \pi_*(Baut_1(X)). \end{aligned}$$

Thus, $\mathrm{Baut}_1(X)$ *satisfies the Hilali conjecture.*

Example 3. *For $r \geq 1$ and $s \geq 2r-1$, let X be a π-finite space with Sullivan minimal model $(\Lambda(y,z,x,t,u,v), d)$, where degrees of the generators given by $|y| = |z| = 2r$, $|x| = 2s$, $|t| = |u| = 2r+2s-1$ and $|v| = 4r+2s-2$. The differential is defined as follows: $dy = dz = 0$, $dx = 0$, $dt = xy$, $du = xz$, $dv = yu - zt$. Then, from Theorem 32 of* [5]*, we have*

$$\mathrm{Baut}_1(X) \simeq K(\mathbb{Q}, 4r-1) \times K(\mathbb{Q}, 4r+2s-1)$$

Now, it is easy to verify that $\mathrm{Baut}_1(X)$ *satisfies the Hilali conjecture though* $\max \pi_*(X)$ *is even.*

From the above examples, it would be interesting to have a generalization of Theorem 2. We suggest the following as a specific question in this area.

Question 1. *Let X be a π-finite space. Does* $\mathrm{Baut}_1(X)$ *satisfy the Hilali conjecture?*

Conflict of Interests

The author(s) declare that there is no conflict of interests.

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