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COMPATIBLE SELF MAPS OF TYPE (A) IN A CONE METRIC SPACES

ALA'A MAZEN AL-MSADEEN*

Department of Mathematics, Faculty of Science, Tafila Technical University, Tafila 66110, Jordan

Abstract. In this paper the concept of compatibility for a pair of self maps in a cone metric space without assuming its normality was discussed, various types of compatibility, some definitions and theorems were studied. The purpose of this research is to obtain common fixed point theorems for compatible self maps of type (A), some results generalize some of the results in the literature.

Keywords: cone metric space; compatible; compatible of type A; self map.

2010 AMS Subject Classification: 47H10.

1. INTRODUCTION

Cone metric spaces were introduced in[5] by using partially ordering.In these spaces, they replaced the set of real numbers of a metric spaces by an ordered Banach space E with a closed cone P. On the other hand, a lot of researchers, as Jungck introduced the concept of compatible maps as a generalization of commuting maps. Murthy , Jungck [2] and Cho [3] introduced a new type of compatible maps and named as compatible map of type (A). In this present paper we will prove some theorems on compatibility for pair of self maps in a cone metric space. This paper shows that the compatible maps of type (A) in a cone metric space are equivalent to

*Corresponding author

E-mail address: alaa_almsadeen@ttu.edu.jo

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compatible maps under certain conditions. In the Sequel we will discuss some common fixed point theorems for compatible maps of type (A) in a cone metric spaces.

2. PRELIMINARIES

Definition 2.1. [5] *Let E be a real Banach space with* $\|.\|$ *as a norm , and let P be a subset of E . Then P is said to be a cone iff:*
i) P is closed, $P \neq \emptyset$ *and* $P \neq \{\theta\}$*, where* θ *is zero vector in E;*
ii) if $t,s \in \mathbb{R}^+$ *and* $x,y \in P$*, then* $tx+sy \in P$*;*
iii) if $x,-x \in P$*, then* $x = \theta$*.*

$\forall x,y \in E$, the space E with closed cone P can be partially ordered; that is $x \leq y$ iff $y-x \in P$. On the other hand; $x < y$ means that $y-x \in P$ and $x \neq y$, while $x \ll y$ means $y-x \in intP$.

Definition 2.2. [5] *The cone P subset of E is called normal if* $\exists \boldsymbol{K} > 0$ *s.t ,* $\|x\| \leq \boldsymbol{K}\|y\|$ *,* $\forall x,y \in E$*.*

Lemma 1. [10] *Let P be a cone with real Banach space E and* $< x_n >, < y_n >$ *be a sequences, then:*
i)if $x \leq y$ *and* $0 \leq a \leq b$*, then* $ax \leq by$*;*
ii)if $x \leq y$ *and* $t \leq s$*, then* $x+t \leq y+s$*;*
iii)if $< x_n > \leq < y_n >$ $\forall n \in N$ *with* $lim_{n\to\infty} < x_n >= x$ *and* $lim_{n\to\infty} < y_n >= y$*, then* $x \leq y$*.*

Lemma 2. [7] *If P is a cone,* $x \in P$*,* $\alpha \in [0,1)$ *and* $x \leq \alpha x$*, then* $x = \theta$*.*

Proof. :If $x \leq \alpha x$, then $(\alpha-1)x \in P$. Since $x \in p$ and $0 \leq \alpha < 1$, we have $(1-\alpha) \in P$, which implies that $x = \theta$. □

Definition 2.3. [8] *An ordered pair* (X,d) *with any non-empty set X is called cone metric space, where* $d : X \times X \longrightarrow E$ *is mapping satisfying:*
i)$d(x,y) > \theta, \forall x,y \in X$ *and* $d(x,y) = \theta$ *iff* $x = y$*;*
ii)$d(x,y) = d(y,x), \forall x,y \in X$*;*
iii)$d(x,y) \leq d(x,z)+d(z,y), \forall x,y,z \in X$*.*
Then d is called a cone metric on X and (X,d) *is called a cone metric space.*

Example 1. [5] *Let $\mathbb{R}^2$ be a banach space, $P = \{(x,y) \in \mathbb{R}^2 : x,y \geq 0\}$ be a cone, and $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^2$, such that $d(x,y) = (|x-y|, \beta|x-y|)$. Where $\beta \geq 0$. Then (X,d) is a cone metric space.*

i)$d(x,y) = (|x-y|, \beta|x-y|) > \theta$, and $d(x,y) = \theta$ iff $|x-y| = \theta$, which implies that $x = y$.

ii)$d(x,y) = (|x-y|, \beta|x-y|) = (|x-y|, \beta|y-x|) = d(y,x), \forall x,y \in R$.

$$\begin{aligned} iii) d(x,y) = (|x-y|, \beta|x-y|) &= (|x-z+z-y|, \beta|x-z+z-y|) \\ &\leq (|x-z|+|z-y|, \beta(|x-z|+|z-y|)) \\ &= (|x-z|, \beta|x-z|+|z-y|, \beta|z-y|) \\ &= d(x,z)+d(z,y) \qquad \forall x,y,z \in R \end{aligned}$$

Definition 2.4. [4] *Let (X,d) be a cone metric space with $x \in X$ and $< x_n >$ be a sequence in X, then:*

i)$< x_n > \longrightarrow x, \forall\ c \in E$, with $\theta \ll c, \exists$ a natural number N such that $d(x_n,x) \ll c$, $\forall n \geq N$, in other words $lim_{n\to\infty} x_n = x$

ii) $< x_n >$ is called Cauchy sequence whenever $\forall c \in E$ with $\theta \ll c, \exists$ a natural number N such that $d(x_n,x_m) \ll c, \forall n,m \geq N$.

iii) (X,d) is called complete cone metric space if every Cauchy sequence is convergent.

iv) (X,d) is called sequentially compact cone metric space if for any sequence $< x_n >$ in$X$, there is a subsequence $< x_{n_k} >$ of $< x_n >$ such that x_{n_k} is convergent in X.

Definition 2.5. [9] *A pair of self maps (Q,R) on a cone metric space (X,d) is called*

i) compatible if $d(QRx_n, RQx_n) \ll c$.

ii) Weakly compatible if $Qx = Rx$ implies $QRx = RQx$.

iii) compatible of type (A) if $d(RQx_n, QQx_n) \ll c$ and $d(QRx_n, RRx_n) \ll c$.

iv) commuting if $QRx = RQx, \forall x \in X$.

Example 2. *Let (X,d) be a complete cone metric space on a normed cone $P = \{(a,b) : a,b \geq 0\}$ with $d(x,y) = (|x-y|, \alpha|x-y|), \alpha \geq 0$ where $X = [0,3]$, R^2 be a real banach space. If Q_x, R_x be two self maps on X defined as:*

$$Q_x = \begin{cases} 2-x\ , & 0 \le x < 1 \\ \\ 2 & ,\ 1 \le x \le 3 \end{cases} \quad \textit{and} \quad R_x = \begin{cases} 2x\ , & 0 \le x < 1 \\ \\ x\ \ , & 1 \le x \le 3\ , x \ne \dfrac{4}{3} \\ \\ 2\ \ , & x = \dfrac{4}{3} \end{cases}$$

Show that (Q,R) *is not compatible.*

Take $< x_n >= \frac{2}{3} + \frac{1}{n}$ *in* X *for* $n \in N$ *s.t* $n \ge 4$. *Then* $Qx_n = 2 - (\frac{2}{3} + \frac{1}{n}) = \frac{4}{3} - \frac{1}{n}$ *and* $Rx_n = 2(\frac{2}{3} + \frac{1}{n}) = \frac{4}{3} - \frac{2}{n}$. *So* $Qx_n, Rx_n \longrightarrow \frac{4}{3}$ *in* (X,d).

Now,

$$d(Qx_n, \frac{4}{3}) = d(\frac{4}{3} - \frac{1}{n}, \frac{4}{3}) = (\frac{1}{n} - \alpha\frac{1}{n}) \longrightarrow \theta \ \ \textit{and}$$

$$\begin{aligned} d(QRx_n, RQx_n) &= d((\frac{4}{3} + \frac{2}{n}), S(\frac{4}{3} - \frac{2}{n})) \\ &= d(2, \frac{4}{3} - \frac{2}{n}) = d(Q(\frac{4}{3} + \frac{2}{n}), R(\frac{4}{3} - \frac{1}{n})) \\ &= (|\frac{2}{3} + \frac{1}{n}|, \alpha|\frac{2}{3} + \frac{1}{n}|) \to (\frac{2}{3} + \alpha\frac{2}{3}) \end{aligned}$$

So , (Q,R) *isn't compatible .*

Lemma 3. [9] *Every compatible self map on a cone metric space* (X,d) *is weakly compatible .*

Proof. : Let (Q,R) be a pair of self maps on (X,d).

Take $<x_n> = R, \forall n \in N. Qx_n, Rx_n \longrightarrow Qx = Rx$ and since (Q,R) is compatible, then $d(QRx_n, RQx_n) = d(QRx, RQx) \ll c$ for $c \gg \theta \in E$. Hence $QRx = RQx$. □

Theorem 2.6. [6] *For a compatible pair of self maps (Q,R) in a cone metric space (X,d), if* $RQx_n \longrightarrow Rx$*, then* $QRx_n \longrightarrow Rx$.

Proof. : Let $c \ll \theta$, and since (Q,R) compatible, so,$\exists M \in N$ s.t

$\frac{c}{2} - d(RQx_n, QRx_n)$ and $\frac{c}{2} - d(RQx_n, Rx) \in intP$, $\forall n > M$.

$c - d(RQx_n, QRx_n) - d(RQx_n, Rx) \in intP$, $\forall n > M$.

$d(QRx_n, Rx) \le d(RQx_n, QRx_n) + d(RQx_n, Rx)$

to get $d(QRx_n, RQx_n) + d(RQx_n, Rx) - d(QRx_n, Rx) \in P$, $\forall n > M$.

We have $c - d(QRx_n, Rx) \in intP$.

That means; $d(QRx_n, Rx) \ll c, \forall n > M$.

Which implies that $QRx_n \longrightarrow Rx$. □

Corollary 2.7. [6] *Let (X,d)be a complete cone metric space with four self maps Q,R,U,Von Xsuch that:*

i)$Q(x) \subset V(x)$ and $R(x) \subset U(x)$;

ii)Pair (Q,U) is compatible and (R,V) is weakly compatible;

iii)one of Q or U is continous;

iv)For some $\alpha,\gamma,\lambda,\delta,\eta \in [0,1)$, with $\alpha+\gamma+\lambda+\delta+\eta < 1, \forall x,y \in X$. We have

$$d(Qx,Ry) \leq \alpha d(Qx,Uy)+\gamma d(Ry,Vy)+\lambda d(Ux,Vy)+\eta d(Qx,Uy)+\delta d(Ux,Ry)$$

. Then $Q,R,V$and U have a unique common fixed point in X.

Theorem 2.8. [6] *Let (X,d) be a complete cone metric space with a cone metric P, and Q,R,U,V,W and L be self maps on X satisfying:*

i) $W(X) \subset UV(X), L(X) \subset VR(X)$;

ii) Pair (W,VR) is compatible and the pair (L,UV) is weakly compatible;

iii) One of W or VR is continuous;

iv) For some $\alpha,\gamma,\lambda,\eta,\delta \in [0,1)$ with $\alpha+\gamma+\eta+\lambda+\delta < 1, \forall x,y \in X$. We have

$$d(Wx,Ly) \leq \alpha d(Wx,VRx)+\gamma d(Ly,UVy)+\lambda d(VRx,UVy)+\eta d(Wx,UVy)+\delta d(VRx,Ly)$$

. Then Q,R,V,W, and L have unique common fixed point in X.

Corollary 2.9. [6] *Let (X,d) be a complete cone metric space with a cone P, and Q, R be self map on X satisfying:*

i) $Q(X) \subset R(X)$;

ii) Pair (Q,R) is compatible;

iii) One of Q or R is continuous;

iv) For some $\alpha,\gamma,\lambda,\eta,\delta \in [0,1)$ with $\alpha+\gamma+\eta+\lambda+\delta < 1, \ \forall x,y \in X$ .We have

$d(Qx,Ry) \leq \alpha d(Qx,Ry)+\gamma d(Qy,Ry)+\lambda d(Rx,Ry)+\eta d(Qx,Ry)+\delta d(Rx,Qy)$.

Then Q and R have unique common fixed point in X.

Lemma 4. [1] *Let (X,d) be a cone metric space and Q ,R be two continuous self maps. Then both maps are compatible of type (A) iff they are compatible .*

Proof. : $\Longleftarrow$) Let Q and R be continuous and $\lim_{n\to\infty} Qx_n = \lim_{n\to\infty} Rx_n = t$, then $\lim_{n\to\infty} QQx_n = \lim_{n\to\infty} QRx_n = Qt$ and $\lim_{n\to\infty} RRx_n = lim_{n\to\infty} RQx_n = Rt$.
For $c \in E$ with $c \ll \theta$, and from compatibility,
$d(QRx_n, RRx_n) \leq d(QRx_n, RQx_n) + d(RQx_n, RRx_n) \ll c$ and
$d(RQx_n, QQx_n) \leq d(RQx_n, QRx_n) + d(QRx_n, QQx_n) \ll c$.
$\Longrightarrow$) Let Q and R be continuous and $\lim_{n\to\infty} Qx_n = \lim_{n\to\infty} Rx_n = t$, then $\lim_{n\to\infty} QQx_n = \lim_{n\to\infty} QRx_n = Qt$.
For $c \in E$ with $c \ll \theta$, $d(RQx_n, QRx_n) \leq d(RQx_n, QQx_n) + d(QRx_n, QQx_n) \ll c$. □

Theorem 2.10. [1] *If Q and R are two compatible self maps of type (A) on a cone metric space (X,d), where $Qt = Rt$, then:*
i)$QRt = QQt$.
ii)$RQt = RRt$.

Proof. : Take$< t_n >$ as a sequence in X defined by $t_n = t, \forall n \in N$, and $Qt = Rt$.
So $\lim_{n\to\infty} Qt_n = Qt$ and $\lim_{n\to\infty} Rt_n = Rt$. Since Q and R compatible of type (A), we have
i)$d(QRt, QQt) = d(QRt, QRt) = d(QRt_n, QRt_n) = 0$. So $QRt = QQt$.
ii)$d(RQt, RRt) = d(RQt, RQt) = d(RQt_n, RQt_n) = 0$. So $RQt = RRt$. □

Theorem 2.11. [1] *Let Q and R be two compatible self maps of type (A) in cone metric space (X,d) , and $\lim_{n\to\infty} Qx_n = \lim_{n\to\infty} Rx_n = z$, for some $z \in X$. Then:*
i)$d(QRx_n, Rz) \ll c$ and $d(QQx_n, Rz) \ll c$, if R is continuous.
ii)$d(RQx_n, Qz) \ll c$ and $d(RRx_n, Qz) \ll c$, if R is continuous.

Proof. : i)Let R be continuous self map on X, then
$\lim_{n\to\infty} RQx_n = \lim_{n\to\infty} RRx_n = Rz$.
So we have
$d(QRx_n, Rz) \leq d(QRx_n, RRx_n) + d(RRx_n, RQx_n) + d(RQx_n, Rz) \ll \frac{c}{2} + \frac{c}{2} = c$ and
$d(QQx_n, Rz) \leq d(QQx_n, RQx_n) + d(RQx_n, RRx_n) + d(RRx_n, Rz) \ll \frac{c}{2} + \frac{c}{2} = c$.
ii) Let Q be continuous self map on X
,then $\lim_{n\to\infty} QRx_n = \lim_{n\to\infty} QQx_n = Qz$.

So we have

$d(RQx_n,Rz) \leq d(RQx_n,QQx_n)+d(QQx_n,QRx_n)+d(QRx_n,Qz) \ll \frac{c}{2}+\frac{c}{2}=c$ and
$d(RRx_n,Qz) \leq d(RRx_n,QRx_n)+d(QRx_n,QQx_n)+d(QQx_n,Qz) \ll \frac{c}{2}+\frac{c}{2}=c.$ □

Theorem 2.12. [1] *Let $Q : X \longrightarrow X$ be a mapping in a cone metric space (X,d) with normal constant* $\mathbf{K}$ *s.t the mapping satisfies the condition $d(Qx,Qy) \leq \frac{k}{1-k}d(x,y), \forall x,y \in X$, where $k \in [0,\frac{1}{2})$. Then, Q has a unique fixed point in X, and the sequence $<Q^n{}_x>$ converges to the fixed point for each $x \in X$.*

Proof. : Choose $x_0 \in X$, set $x_1 = Qx_0$,$x_2 = Qx_1 = Q^2x_0,\ldots,x_{n+1} = Qx_n = Q^{n+1}x_0,\ldots$ We have

$$\begin{aligned}
d(x_{n+1},x_n) &= d(Qx_n,Qx_{n-1}) \leq \frac{k}{1-k}d(x_n,x_{n-1}) \\
&= hd(x_n,x_{n-1}), \quad \text{where } h = \frac{k}{1-k} \\
\text{For} n > m, d(x_n,x_m) &\leq d(x_n,x_{n-1})+d(x_{n-1},x_{n-2})+...+d(x_{m+1},x_m) \\
&\leq (h^{n-1}+h^{n-2}+...+h^m)d(x_1,x_0) \leq \frac{h^m}{1-h}d(x_1,x_0)
\end{aligned}$$

To get $||d(x_n,x_m)|| \leq \frac{h^m}{1-h}\mathbf{K}\|\mathbf{d(x_1,x_0)}\|$.
This implies $d(x_n,x_m) \longrightarrow \theta$ as $n,m \longrightarrow \infty$. Hence x_n is a Cauchy sequence in (X,d) . By the completeness of $X, \exists x* \in X$ s.t $x_n \longrightarrow x*$ as $n \longrightarrow \infty$.
Since

$$\begin{aligned}
d(Qx*,x*) &\leq d(Qx*,Qx_n)+d(Qx_n,x*) \\
&\leq \frac{k}{1-k}d(x*,x_n)+d(x_{n+1},x*) \\
\|d(Qx*,x*)\| &\leq \mathbf{K}\big(\frac{k}{1-k}\|d(x*,x_n)\|+\|d(x_{n+1},x*)\|\big) \longrightarrow 0.
\end{aligned}$$

Hence, $\|d(Qx*,x*)\| = 0$ .This implies $Qx* = x*$. So $x*$ is a fixed point of Q. Now if $y*$is another fixed point of Q, then $d(x*,y*) = d(Qx*,Qy*) \leq \frac{k}{1-k}d(x*,y*)$
Hence , $\|d(x*,y*)\| = 0$ and $x* = y*$.
Therefore, the fixed point of Q is unique. □

Theorem 2.13. [1] *Let $(X,\|.\|_a)$ be a cone normed space, then $(X,\|.\|)$ with $\|x\| = \inf_{v\in A}\|v\|, \forall x \in X$, where $A = \{v \in P : v \geq \|x\|_a\}$ is a normed space.*

Proof. : i)$\|x\| = \inf_{v\in A}, \|v\| > 0$ and $\|x\| = 0 \Longleftrightarrow \inf_{v\in A} \|v\| = 0 \Longleftrightarrow \exists \langle v_n \rangle \subset P$, $v_n \geq \|x\|_a$ s.t $\|v_n\| < \frac{1}{n}$, for some $n \in N$. Since $v_n \geq \|x\|_a$ and $v_n \longrightarrow \theta$ as $n \longrightarrow \infty$, then $\|x\|_a \leq \theta$,implies that $\|x\|_a \in (P \cap -P)$. Therefore, $\|x\|_a = \theta \Longleftrightarrow x = 0$.

ii)Let $\alpha \in R$, $x \in X$. Then $\|\alpha x\| = inf_{v\in A_0}\|v\|$ where $A_0 =\{\ v \in P : v \geq \|\alpha x\|_a\}$ and Since $\|\alpha x\|_a = |\alpha|\|x\|_a$, so $v \geq |\alpha|\|x\|_a$, we have $\|\alpha x\| = |\alpha|\|x\|$.

iii)let $x, y \in X$, then $\|x+y\| = \inf_{v_1\in B} \|v_1\|$ where

$$B = \{v_1 \in P : v_1 \geq \|x+y\|_a\}, \|x\| = \inf_{v_2\in C} \|v_2\|$$

,

$$C = \{v_2 \in P : v_2 \geq \|x\|_a\} and \|y\| = \inf_{v_3\in D} \|v_3\|$$

$$D = \{v_3 \in P : v_3 \geq \|y\|_a\}.$$

for arbitrary $E = \{v_2, v_3 \in P : v_2 \geq \|x\|_a\}, \{v_3 \geq \|y\|_a\}$

we get $v_2 + v_3 \geq \|x\|_a + \|y\|_a \geq \|x+y\|_a$.

Then $E \supset B$ implies $\inf_{v_2,v_3\in E} \|v_2 + v_3\| \geq \inf_{v_1\in B} \|v_1\|$,

note that

$\inf_{v_2,v_3\in E} \|v_2 + v_3\| \leq \inf_{v_2,v_3\in E}(\|v_2\| + \|v_3\|) = \inf_{v_2\in C} \|v_2\| + \inf_{v_3\in D} \|v_3\|$,

Thus

$\inf_{v_2\in C} \|v_2\| + \inf_{v_3\in D} \|v_3\| \geq \inf_{v_1\in B} \|v_1\|$.

So $\|x\| + \|y\| \geq \|x+y\|$. Which implies $\|.\|$ is a norm on X. □

Corollary 2.14. [1] *Let $(X, \|.\|_a)$ be a cone normed space, and $\langle x_n \rangle$ be a sequence in a normed space $(X, \|.\|)$ with $\|x\| = \inf_{v\in A} \|v\|, \forall x \in X$, where $A = \{v \in P : v \geq \|x\|_a\}$, if $\langle x_n \rangle$is Cauchy (convergent) sequence in $(X, \|.\|_a)$ then it is also in $(X, \|.\|)$.*

Proof. : Let $\langle x_n \rangle$ be a Cauchy sequence in $(X, \|.\|_a)$ and $\varepsilon > 0$ be real number. Then for $c \in E$ with $c \gg \theta$, we have $0 \leq \|\frac{c}{k}\| < \varepsilon$, for some $k \in N$.

Hence $\frac{c}{k} \gg \theta$, and since $\langle x_n \rangle$ is a Cauchy in $(X, ||.||_a), \exists n_0 \in N s.t$

$\|x_n - x_m\|_a \ll \frac{c}{k}, \forall m, n \geq n_0$.

But $\|x_n - x_m\| = \inf_{u\in A|} \|u\| \leq \frac{\|c\|}{k} < \varepsilon, \forall m, n \geq n_0$, where

$$A = \{u \in P : u \geq ||x_n - x_m||_a\}$$

So $<x_n>$ is a Cauchy in $(X, \|.\|)$. □

Theorem 2.15. [1] *Let $B \subset X$ be a compact subset of a cone metric space (X,d) , then B is:*
i)countably compact space .
ii)Lindelof space .

Proof. : i)Let $B \subset X$ be a compact set and $B = \{B_i\}_{i \in I}$ be a countable open cover for B s.t $B = \bigcup_{i \in I} B_i$. From compactness of B,$\exists B' \subset B$ finite subcover for B where $B' = \{B_\alpha\}_{\alpha=1}^{n}$. So we have a finite subcover from a countable open cover for B, which means that B is countably compact space.
ii)Let $B \subset X$ be a compact set and $B = \{B_i\}_{i \in I}$ be an open cover for B s.t $B = \bigcup_{i \in I} B_i$, and since B is compact, then $\exists B' \subset B$ finite subcover for B where $B' = \{B_\alpha\}_{\alpha=1}^{n}$. But every finite set is countable, so $B' \subset B$ is countable. Therefore, B is lindelof space. □

Corollary 2.16. [1] *A subset $B \subset X$ of a cone metric space (X,d) is compact if it is lindelof and countably compact space.*

Theorem 2.17. *Every sequentially compact cone metric space (X, d) is complete .*

Proof. : Let $< x_n >$ be a cauchy sequence in a cone metric space (X,d) .

From sequentially compact of (X,d);∃ subesquence $< x_{n_k} >$ in X for some $n_k \in N$, which converges to x.
That means $d(x_{n_k}, x) \ll \frac{c}{2}$ for $\frac{c}{2} \ll \theta$ and
since $< x_n >$ is cauchy;$d(x_n, x_{n_k}) \ll \frac{c}{2}$ for $\frac{c}{2} \ll \theta$, $n_k \in N$.
Now, $d(x_n, x) + d(x_{n_k}, x) \leq \frac{c}{2} + \frac{c}{2} \ll \theta$
Hence, $d(x_n, x) = 0$
$< x_n >$ converges to x.
So, (X,d) is complete cone metric space. □

Conflict of Interests

The author(s) declare that there is no conflict of interests.

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