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THE ABOODH REDUCED DIFFERENTIAL TRANSFORM METHOD FOR THE HIROTA-SATSUMA COUPLED KDV AND MKDV EQUATIONS

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Abstract. This paper presents the validity and efficiency of the coupled Aboodh and Reduced Differential Transform Methods (ABRDTM). This method has been used in solving the coupled mKdV and KdV equations involving two different types of initial conditions. The Reduced Differential Transform Method which is a modified form of differential transform method (DTM) and emanated from the Taylors expansion is incorporated into the scheme of the Aboodh transform method and calculated in an iterative procedure which converged quickly to a closed form solution. In this work, the examples illustrated showed that the scheme provides a series of function which converged to the analytical solutions of the system earlier mentioned.

Keywords: KdV; mKdV; Aboodh transform; reduced differential transform method.

2010 AMS Subject Classification: 34C20.

1. INTRODUCTION

The concept of finding analytical solution to nonlinear equations are very essential in the understanding of many nonlinear phenomenon. For Example, the wave phenomenon observed in plasma-optical fibres and fluid dynamics are mostly modelled by some kink-shape tanh solution and bell-shape sech solutions. In this work, the generalized Hirota-satsuma equation coupled

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KdV and mKdV equations introduced by kangalgil and Ayaz is considered [15];

$$
\begin{aligned}
u_t &= \tfrac{1}{2}u_{xxx} - 3uu_x + 3(vw)_x \\
v_t &= -v_{xxx} + 3uv_x, \\
w_t &= -w_{xxx} + 3uw_x
\end{aligned} \tag{1}
$$

and coupled mKdV equation;

$$
\begin{aligned}
u_t &= \tfrac{1}{2}u_{xxx} - 3u^2u_x + \tfrac{3}{2}v_{xx} + 3uv_x + 3u_xv - 3\lambda u_x \\
v_t &= -v_{xxx} - 3vv_x - 3u_xv_x + 3u^2v_x + 3\lambda v_x
\end{aligned} \tag{2}
$$

where λ is an arbitrary constant and the subscripts x and t denote the differentials with respect to x and t respectively. Equation(1) is thus reduced to a new system of coupled KdV equation when w=v* and w=v [14]. Also,when u=0 Equation (2)then becomes a generalized KdV and an mKdV for v=0 [4]. These coupled equations have been studied by numerous researchers through distinct methods. Jiang and Zhu[41] provided a suggestion on the solitary wave solution for coupled mKdV systems by using the Homotopy Perturbation Method(HPM), the miura transformation was worked on by Wu *et al.*[26]and He *et al.* solved the solitary wave solution for the Hirota-Satsuma Coupled mKdV and KdV equations with the variational iteration method[18]. Zayed *et al* [31] also used the Jacobi elliptic function to solve these equations,Yong and Zhang [32] employed the projective Riccati equations method, Assas [33] solved the coupled-KdV equations with the variational iteration method. Raslan[34] and Kaya[39] solved the Hirota Satsuma equations using the Adomian's decomposition method, Ganji and Rafei [35] the homotopy pertubation method, Abbasbandy[36] the homotopy analysis method, Cao *et al*.[37] the trigonometric function transform method,Yong *et al*.[38] the homogenous balance method and Zayed *et al.*[30] used the Jacobian and rational methods to explain the solitary wave solution for the nonlinear coupled KdV system of equations.

In this paper, a reliable procedure to solve coupled mKdV and KdV equations is introduced. This procedure involves merging of two transforms; the Aboodh Transform method and Reduced Differential Transform Method. The theory of the Reduced Differential Transform method for defining sets of transformation rules to overcome the complex calculations of the traditional Differential Transform Method(DTM)was introduced by Keskin and Oturanc[40].

Also, the Aboodh transform method has recently been used to solve varieties of problems in applied sciences.

In this work, the combined technique (ABRDTM)is highlighted first and then the application to the Hirota-Satsuma coupled mKdV and KdV equations has been evaluated for different initial conditions. In the result section, the closed form solutions have been obtained and compared with [15] and the analytical solution listed in tabular and graphical forms.

2. Properties of Aboodh Transform

The Aboodh Transform [1] [2] [3] [22] [23] [24] defined for functions of exponential order and is given in a set A as;

$$A = \left\{ f(\tau) : M, k_1, k_2 > 0, |f(\tau)| < Me^{-v\tau} \right\} \tag{3}$$

Where M is a constant that must be an infinite number and k_1,k_2 may be either finite or infinite. The Aboodh transform defined by Aboodh *et al.* [6] is denoted by the operator A(.) and defined by the integral;

$$A\left[f(\tau)\right] = K(v) = \frac{1}{v}\int_0^\infty f(\tau)e^{-v\tau}d\tau, \tau \geq 0, k_1 \leq v \leq k_2 \tag{4}$$

Given that, K(v) is the Aboodh transform of $f(\tau)$ such that

$$A[(f(\tau))] = K(v)$$

then,the integral transform:

(1) $A[f^{'}(\tau)] = vk(v) - \frac{1}{v}f(0)$
(2) $A[f^{''}(\tau)] = v^2k(v) - \frac{1}{v}f^{'}(0) - f(0)$
(3) $A[f^m(\tau)] = v^{(m)}k(v) - \sum_{k=0}^{m-1}\frac{f^{(k)}(0)}{v^{2-m+k}}$

where $f(\tau)$ which is the inverse Aboodh transform of K(v)is define as:

$$f(\tau) = A^{-1}[K(v)]$$

3. Aboodh Reduced Differential Transform Method(ABRDTM)

This paper focuses on the solution of the Hirota-Satsuma coupled mKdV and KdV using the merger of the Aboodh Transform Method and Reduced Differential Transform Method(ABRDTM)

As given by[5],[14] considering;

$$\frac{\delta^m u(x,\tau)}{\delta\tau^m} + Ru(x,\tau) + Nu(x,\tau) = f(x,\tau) \tag{5}$$

where m=1,2,3, with the initial conditions;

$$\frac{\delta^{m-1} u(x,\tau)}{\delta\tau^{m-1}} = g_{m-1}(x) \tag{6}$$

Partial derivative of the function $u(x,\tau)$ of the m^{th} is given as $\frac{d^m u(x,\tau)}{d\tau^m}$, where R defines the linear differential equation, N is the nonlinear terms of the differential equations and $f(x,\tau)$ the source terms. Applying the Aboodh transform into equation(5) then;

$$A\left[\frac{\delta^m u(x,\tau)}{\delta\tau^m}\right] + A\left[Ru(x,\tau)\right] + A\left[Nu(x,\tau)\right] = A\left[f(x,\tau)\right] \tag{7}$$

where,

$$A\left[\frac{\delta^m u(x,\tau)}{\delta\tau^m}\right] = A\left[u(x,\tau)\right] v^{(m)} - \sum_{k=0}^{m-1} \frac{1}{v^{2-m+k}} \frac{\delta^k u(x,0)}{\delta\tau^k} \tag{8}$$

Substituting equation (8)into (7) this gives;

$$A\left[u(x,\tau)\right] v^{(m)} - \sum_{k=0}^{m-1} \frac{1}{v^{2-m+k}} \frac{\delta^k u(x,0)}{\delta\tau^k} + A\left[Ru(x,\tau)\right] + A\left[Nu(x,\tau)\right] = A\left[f(x,\tau)\right] \tag{9}$$

Thus,

$$A\left[u(x,\tau)\right] v^{(m)} = A\left[f(x,\tau)\right] + \sum_{k=0}^{m-1} \frac{1}{v^{2-m+k}} \frac{\delta^k u(x,0)}{\delta\tau^k} - \left\{A\left[Ru(x,\tau)\right] + A\left[Nu(x,\tau)\right]\right\} \tag{10}$$

Thus, Simplifying equation(10) we have;

$$A\left[u(x,\tau)\right] = \frac{1}{v^m} A\left[f(x,\tau)\right] + \sum_{k=0}^{m-1} \frac{1}{v^{2+k}} \frac{\delta^k u(x,0)}{\delta\tau^k} - \frac{1}{v^m}\left\{A\left[Ru(x,\tau)\right] + A\left[Nu(x,\tau)\right]\right\} \tag{11}$$

By applying the inverse Aboodh transform on equation (11)we obtain;

$$u(x,\tau) = A^{-1}\left[\frac{1}{v^m} A\left[f(x,\tau)\right] + \sum_{k=0}^{m-1} \frac{1}{v^{2+k}} \frac{\delta^k u(x,0)}{\delta\tau^k} - \frac{1}{v^m}\left\{A\left[Ru(x,\tau)\right] + A\left[Nu(x,\tau)\right]\right\}\right] \tag{12}$$

Equation (12) can then be written as;

$$u(x,\tau) = F(x,\tau) - A^{-1}\left[\frac{1}{v^m}\left\{A\left[Ru(x,\tau)\right] + A\left[Nu(x,\tau)\right]\right\}\right] \tag{13}$$

where $F(x,\tau)$ is the expression that arises from the initial conditions given and the source terms after it has been simplified. The solution will be expressed as:

$$u(x,\tau)=\sum_{r=0}^{\infty}u_r(x,\tau) \tag{14}$$

The nonlinear part is reduced as follows;

$$Nu(x,\tau)=\sum_{r=0}^{\infty}A_r \tag{15}$$

where A_r is expressed as the reduced polynomial which can be gotten from the below formula;

$$A_r=U_r(x)U_{k-r}(x),\quad r=0,1,...$$

Substituting equations(14)and (15)into equation(13) to obtain;

$$\sum_{r=0}^{\infty}u_r(x,\tau)=F(x,\tau)-A^{-1}\left[\frac{1}{v^m}\{A\left[R\sum_{r=0}^{\infty}u_r(x,\tau)\right]+A\left[\sum_{r=0}^{\infty}A_r\right]\}\right] \tag{16}$$

From equation(16)we have;

$$u_r(x,\tau)=F(x,\tau),\ \ r=0 \tag{17}$$

Also, the recursive relation as;

$$u_{r+1}=-A^{-1}\left[\frac{1}{v^m}\{A\left[Ru_r(x,\tau)\right]+A\left[A_r\right]\}\right]$$

where m=1,2,3 and $r\geq 0$

The exact solution $u(x,\tau)$ can be approximated by the truncated series;

$$u(x,\tau)=\lim_{N\to\infty}\sum_{r=0}^{N}u_r(x,\tau)$$

4. Application to the KdV Equation

Considering Equation (1) with the following conditions;

$$u(x,0)=\frac{1}{3}(\beta-8k^2)+4k^2\tanh^2(kx) \tag{18}$$

$$v(x,0)=-\frac{4(3k^4c_0-2\beta k^2c_1+4k^4c_1)}{3c_1^2}+\frac{4k^2}{c_1}\tanh^2(kx) \tag{19}$$

$$w(x,0) = c_0 + c_1 tanh^2(kx) \tag{20}$$

where $\beta, c_0, c_1 \neq 0$, and k are the arbitrary constants.

Taking the Aboodh transform of Equation (1);

$$vk(v) - \frac{1}{v}u(0) = A\left[\frac{1}{2}u_{xxx} - 3uu_x + 3(vw)_x\right]$$

$$vk(v) - \frac{1}{v}v(0) = A\left[-v_{xxx} + 3uv_x\right] \tag{21}$$

$$vk(v) - \frac{1}{v}w(0) = A\left[-w_{xxx} + 3uw_x\right]$$

where;

$$A[u_t] = vu(x,t) - \frac{1}{v}u(x,0)$$

$$A[v_t] = vv(x,t) - \frac{1}{v}v(x,0)$$

$$A[w_t] = vw(x,t) - \frac{1}{v}w(x,0)$$

Taking the inverse Aboodh transform of Equation (21)alongside the given conditions to obtain;

$$u = A^{-1}\{\frac{1}{v^2}\left[\frac{1}{3}(\beta - 8k^2) + 4k^2\tanh^2(kx)\right] + \frac{1}{v}A\left[\frac{1}{2}u_{xxx} - 3uu_x + 3(vw)_x\right]\} \tag{22}$$

$$v = A^{-1}\{\frac{1}{v^2}\left[-\frac{4(3k^4c_0 - 2\beta k^2c_1 + 4k^4c_1)}{3c_1^2} + \frac{4k^2}{c_1}\tanh^2(kx)\right] + \frac{1}{v}A\left[-v_{xxx} + 3uv_x\right]\} \tag{23}$$

$$w = A^{-1}\{\frac{1}{v^2}[c_0 + c_1 tanh^2(kx)] + \frac{1}{v}A\left[-W_{xxx} + 3uw_x\right]\} \tag{24}$$

The first iterate is given as;

$$u_0 = \frac{1}{3}(\beta - 8k^2) + 4k^2\tanh^2(kx) \tag{25}$$

$$v_0 = -\frac{4(3k^4c_0 - 2\beta k^2c_1 + 4k^4c_1)}{3c_1^2} + \frac{4k^2}{c_1}\tanh^2(kx) \tag{26}$$

$$w_0 = c_0 + c_1 tanh^2(kx) \tag{27}$$

The recursive relation is given as:

$$u_{n+1} = A^{-1}\left\{\frac{1}{v}A\left[\frac{1}{2}u_{n,xxx} - 3u_nu_{n,x} + 3(vw)_{n,x}\right]\right\} \tag{28}$$
$$v_{n+1} = A^{-1}\left\{\frac{1}{v}A\left[-v_{n,xxx} + 3u_nv_{n,x}\right]\right\}$$
$$`w_{n+1} = A^{-1}\left\{\frac{1}{v}A\left[-w_{n,xxx} + 3u_nw_{n,x}\right]\right\}$$

when n=0 then Equation (28) becomes;

$$u_1 = A^{-1}\left\{\frac{1}{v}A\left[\frac{1}{2}u_{0,xxx} - 3u_0u_{0,x} + 3(vw)_{0,x}\right]\right\} \tag{29}$$
$$v_1 = A^{-1}\left\{\frac{1}{v}A\left[-v_{0,xxx} + 3u_0v_{0,x}\right]\right\}$$
$$w_1 = A^{-1}\left\{\frac{1}{v}A\left[-w_{0,xxx} + 3u_0w_{0,x}\right]\right\}$$

Thus,

(30)
$$u_1 = A^{-1}\left\{\frac{1}{v}A\left[\frac{\left(8sinh(kx)(-3k^2c_0cosh(kx)^2 - 6cosh(kx)^2k^2c_0 + \beta c_1cosh(kx)^2 + \ldots\right)k^3}{cosh(kx)^5c_1}\right]\right\}$$
$$v_1 = A^{-1}\left\{\frac{1}{v}A\left[\frac{8sinh(kx)k^3\beta}{cosh(kx)^3c_1}\right]\right\}$$
$$w_1 = A^{-1}\left\{\frac{1}{v}A\left[16c_1(1 - tanh(kx)^2)^2k^3tanh(kx) - 8c_1tanh(kx)^3(1 - tanh(kx)^2)k^3 + \ldots\right]\right\}$$

Hence,

$$u_1 = \frac{\left(8sinh(kx)(-3k^2c_0cosh(kx)^2 - 6cosh(kx)^2k^2c_0 + \beta c_1cosh(kx)^2 + \ldots\right)k^3t}{cosh(kx)^5c_1} \tag{31}$$
$$v_1 = \frac{8sinh(kx)k^3t\beta}{cosh(kx)^3c_1}$$
$$w_1 = \frac{2tsinh(kx)c_1k\beta}{cosh(kx)^3}$$

when n=1

$$u_2 = A^{-1}\left\{\frac{1}{v}A\left[\frac{1}{2}u_{1,xxx} - 3(u_0u_{1,x} + u_1u_{0,x}) + 3(v_0w_1 + v_1w_0)_x\right]\right\} \tag{32}$$

$$v_2 = A^{-1}\left\{\frac{1}{v}A\left[-v_{1,xxx} + 3(u_0v_{1,x} + u_1v_{0,x}\right]\right\}$$

$$w_2 = A^{-1}\left\{\frac{1}{v}A\left[-w_{1,xxx} + 3(u_0w_{1,x} + u_1w_{0,x})\right]\right\}$$

Hence,

(33)
$$u_2 = -\frac{\left(12cosh(kx)^6k^4c_0 + 24cosh(kx)^6k^4c_1 + 2cosh(kx)^6\beta^2c_1 - 12cosh(kx)^6k^2c_0 - \ldots\right)4t^2k^4}{cosh(kx)^8c_1}$$

(34)
$$v_2 = -\frac{\left(2cosh(kx)^6\beta^2c_1 + 72cosh(kx)^4k^4c_0 + 144cosh(kx)^4k^4c_1 - 3cosh(kx)^4\beta^2c_{-}\ldots\right)4t^2k^4}{cosh(kx)^8c_1^2}$$

(35)
$$w_2 = -\frac{\left(2cosh(kx)^6\beta^2c_1 + 24cosh(kx)^4k^4c_0 + 48cosh(kx)^4k^4c_1 + 16cosh(kx)^4\beta k^2c_1 - \ldots\right)4t^2k^4}{cosh(kx)^8}$$

when n=2

$$u_3 = A^{-1}\left\{\frac{1}{v}A\left[\frac{1}{2}u_{2,xxx} - 3(u_0u_{2,x} + u_1u_{1,x} + u_2u_{0,x}) + 3(v_0w_2 + v_1w_1 + v_2w_0)_x\right]\right\} \tag{36}$$

$$v_3 = A^{-1}\left\{\frac{1}{v}A\left[-v_{2,xxx} + 3(u_0v_{2,x} + u_1v_{1,x} + u_2v_{0,x}\right]\right\}$$

$$w_3 = A^{-1}\left\{\frac{1}{v}A\left[-w_{2,xxx} + 3(u_0w_{2,x} + v_1w_{1,x} + u_2w_{0,x})\right]\right\}$$

Hence,

$$u_3 = \frac{\left(-9720k^6c_1^2 + 10440k^4c_1^2 - 720k^2c_1^2 + 45coshcosh(kx)^4\beta^2c_1^2 - \ldots\right)16t^3sinh(kx)}{3cosh(kx)^{11}c_1^2} \tag{37}$$

$$v_3 = \frac{\left(-21708k^4c_1 + 5292k^4c_0cosh(kx)^2 + 36000cosh(kx)^2k^4c_1 - \ldots\right)16t^3sinh(kx)}{3cosh(kx)^{11}c_1^2} \tag{38}$$

$$w_3 = \frac{\left(-7236k^4c_1 + 1764k^4c_0cosh(kx)^2 + 12000cosh(kx)^2k^4c_1 - \ldots\right)4t^3sinh(kx)}{3cosh(kx)^{11}} \tag{39}$$

Approximating the series in equations (25-39),then $u(x,t)$,$v(x,t)$, $w(x,t)$ can be determined by the Equations given below as:

$$u(x,t) = u_0 + u_1 + u_2 + u_3 + \ldots \tag{40}$$

Thus,

$$u(x,t) = \frac{8tsinh(kx)(-3k^2c_0cosh(kx)^2 - 6cosh(kx)^2 - 6cosh(kx)^2k^2c_1)}{cosh(kx)^5c_1} + \ldots$$

$$v(x,t) = v_0 + v_1 + v_2 + v_3 + \ldots \tag{41}$$

$$v(x,t) = \frac{8tsinh(kx)k^3\beta}{cosh(kx)^3c_1} - \frac{1}{cosh(kx)^8c_1^2}\left(4t^2k^4(2coshkx)^6\beta^2c_1\right)) + \ldots$$

$$w(x,t) = w_0 + w_1 + +w_2 + w_3 + \ldots \tag{42}$$

$$w(x,t) = \frac{2tsinh(kx)c_1k\beta}{cosh(kx)^3} - \frac{1}{cosh(kx)^8}\left(t^2k^2(2cosh(kx)^6)\beta^2c_1\right) + \ldots$$

Table 1. Comparisons between the numerical solution(ABRDTM) and analytical solution of the coupled KdV equation with the initial condition in Equations(18-20) where $c_0 = c_1 = \beta = 1, k = 0.1, t = 1$.

x	u_{num}	u_{analy}	v_{num}	v_{analy}	w_{num}	w_{analy}
-40	.3465052773	.3466011627	.06566783502	.06566782943	1.998362507	1.998362402
-30	.3454868399	.3461851777	.06525197191	.06525184445	1.987964060	1.987962778
-20	.3386008827	.3432422696	.06231369975	.06230893638	1.914421730	1.914390076
-10	.3108142225	.3271899722	.04632619381	.04625663889	1.513454720	1.513082639
0	.3071854666	.3070640150	.02613333333	.02613068170	1.010000000	1.009933709
10	.3486048769	.3322986140	.05141988761	.05136528069	1.641103062	1.640798684
20	.3489927303	.3443377450	.06340740135	.06340441178	1.941798252	1.941776961
30	.3470443375	.3463432716	.06541002353	.06540993837	1.991916122	1.991915126
40	.3467190185	.3466227462	.06568941744	.06568941289	1.998902081	1.998901989

Table 1b. Error between the numerical solution(ABRDTM) and analytical solution of the coupled KdV equation with the initial conditions in Equations(18-20) where $c_0 = c_1 = \beta = 1, k = 0.1, t = 1$.

x	u_{error}	v_{error}	w_{error}
-40	0.0000958854	$5.59(10^{-9})$	$1.05(10^{-7}$
-30	0.0006983378	$1.2746(10^{-7})$	0.000001282
-20	0.0046413869	0.00000476337	0.000031654
-10	0.0163757497	0.00006955492	0.000372081
0	0.0001214516	0.00000265163	0.000066291
10	0.0163062629	0.00005460692	0.000304378
20	0.0046549853	0.00000298957	0.000021291
30	0.0007010659	$8.516(10^{-8})$	$9.96(10^{-7})$
40	0.0000962723	$4.55(10^{-9}))$	$9.2(10^{-8})$

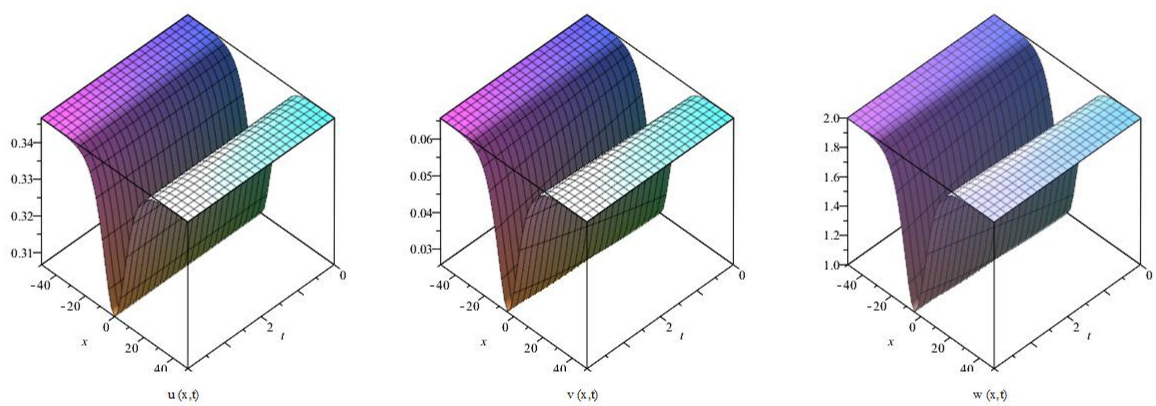


FIGURE 1. The numerical solutions for u(x,t),v(x,t) and w(x,t)

Equation (1) was solved again subject to the initial condition:

$$u(x,0) = \frac{1}{3}(\beta - 2k^2) + 2k^2 \tanh^2(kx) \tag{43}$$

$$v(x,0) = -\frac{4k^2c_0(\beta+k^2)}{3c_1^2} + \frac{4k^2(\beta+k^2)}{3c_1}\tanh(kx) \tag{44}$$

$$w(x,0) = c_0 + c_1 tanh(kx) \tag{45}$$

Table 2. Comparisons between the numerical and analytical solution of the coupled KdV equation with the initial condition in Equations(43-45) where $c_0 = k = \beta = 1, k = 0.1, t = 2$.

x	u_{num}	u_{analy}	v_{num}	v_{analy}	w_{num}	w_{analy}
-40	.3466267012	.3466266705	-.02691987165	-.02691986126	.00099963453	.0010004022
-30	.3463732202	.3463730133	-.02683417850	-.02683410448	.00736323657	.0073684798
-20	.3445961109	.3445954991	-.02621739376	-.02621698765	.05317622369	.0531939872
-10	.3354751638	.3354855632	-.02240821623	-.02240902851	.3363338590	.3359632297
0	.3274709759	.3274458069	-.01080924445	-.01080867902	1.197107200	1.197375320
10	.3405590059	.3405662667	-.002240532021	-.00224011796	1.833504318	1.833654607
20	.3457089092	.3457081597	-.0003262940575	-.00032665918	1.975767238	1.975743130
30	.3465343642	.3465341826	-.000044613562	-.00004467704	1.996687054	1.996682398
40	.3466487115	.3466486853	-.6046197462E-5	-.605506E-5	1.999551024	1.999550366

Table 2b. Comparisons between the exact and numerical errors of the coupled KdV equations with the initial conditions in Equations(38-40) where $c_0 = k = \beta = 1, k = 0.1, t = 2$.

x	u_{error}	v_{error}	w_{error}
-40	$3.07(10^{-8})$	$1.039(10^{-8})$	$-7.6767(10)^{-7}$
-30	$2.069(10^{-7})$	$7.402(10^{-8})$	0.000005243221
-20	$6.118(10^{-7})$	$4.0611(10^{-7})$	0.00001776351
-10	0.3354751638	$8.1228(10^{-7})$	0.0003706293
0	0.0000251690	0.02161792347	0.000268120
10	0.0000072608	$4.14061(10^{-7})$	0.000150289
20	$7.495(10^{-7})$	$3.651225(10^{-7})$	0.000024108
30	$1.816(10^{-7})$	$6.3478(10^{-8})$	0.000004656
40	$2.62(10^{-8})$	0.000012101257	$6.58(10^{-7})$

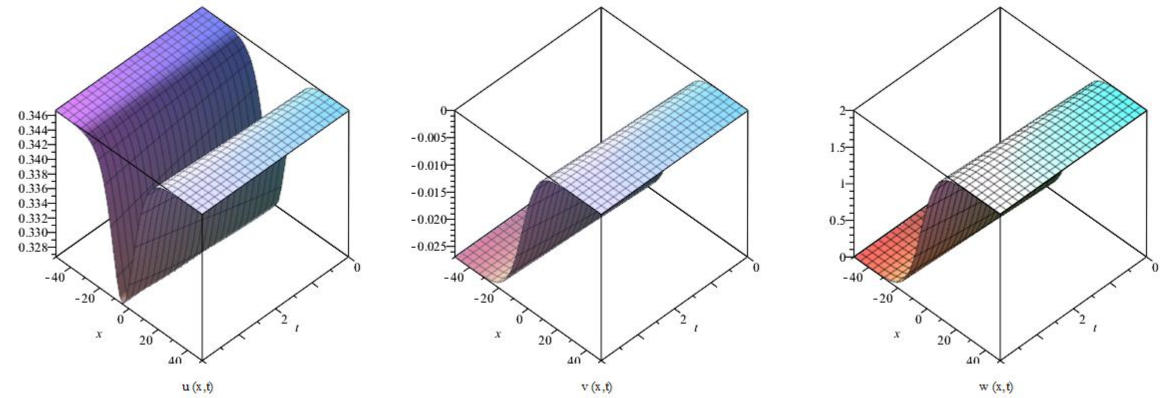


FIGURE 2. The numerical solutions for u(x,t),v(x,t) and w(x,t)

In the tables above, comparisons have been made between the ABRDTM and the analytical solution of the system considered.

5. APPLICATION TO THE MKDV EQUATION

Considering equation (2) with the initial condition;

$$u(x,0)=\frac{b}{2k}+k\tanh(kx) \tag{46}$$

$$v(x,0)=\frac{\lambda}{2}\left(1+\frac{k}{b}\right)+b\tanh(kx) \tag{47}$$

where $\beta, c_0, c_1 \neq 0$, and k are arbitrary constant.

Taking the Aboodh transform of Equation (2)we obtained;

$$vk(v)-\frac{1}{v}u(0)=A\left[\frac{1}{2}u_{xxx}-3u^2u_x+\frac{3}{2}v_{xx}+3(uv)_x-3\lambda u_x\right] \tag{48}$$

$$vk(v)-\frac{1}{v}v(0)=A\left[-v_{xxx}+3vv_x+3(vu)_x-3u^2v_x-3\lambda v_x\right] \tag{49}$$

Taking the inverse Aboodh transform of Equation (49) alongside the given conditions to obtain;

(50)

$$u(x,t)=A^{-1}\{\frac{1}{v^2}\left[\frac{b}{2k}+k\tanh(kx)\right]+\frac{1}{v}A\left[\frac{1}{2}u_{xxx}-3u^2u_x+\frac{3}{2}v_{xx}+3uv_x+3u_xv-3\lambda u_x\right]\}$$

(51)
$$v(x,t)=A^{-1}\{\frac{1}{v^2}\left[\frac{\lambda}{2}\left(1+\frac{k}{b}\right)+b\tanh(kx)\right]+\frac{1}{v}A\left[-v_{xxx}+3vv_x+3(vu)_x-3u^2v_x-3\lambda v_x\right]\}$$

The first iterate is given as;

$$u(x,0)=\frac{b}{2k}+k\tanh(kx) \tag{52}$$

$$v(x,0)=\frac{\lambda}{2}\left(1+\frac{k}{b}\right)+b\tanh(kx) \tag{53}$$

The Recursive relation is expressed as:

$$u_{n+1}(x,t)=A^{-1}\left\{\frac{1}{v}A\left[\frac{1}{2}u_{n,xxx}-3u_n^2u_{n,x}+\frac{3}{2}v_{n,xx}+3u_nv_{n,x}+3u_{n,x}v_n-3\lambda u_{n,x}\right]\right\} \tag{54}$$

$$v_{n+1}(x,t)=A^{-1}\left\{\frac{1}{v}A\left[-v_{n,xxx}+3v_nv_{n,x}+3(vu)_{n,x}-3u_n^2v_{n,x}-3\lambda v_{n,x}\right]\right\}$$

when n=0 then Equation (54) becomes;

$$u_1=\frac{1}{4}\frac{\left(-4bk^4-6bk^2\lambda+6k^3\lambda+3b^3\right)t}{cosh(kx)^2b} \tag{55}$$

$$v_1=-\frac{1}{4}\frac{\left(28b^2k^4cosh(kx)^2+6\lambda b^2k^2cosh(kx)^2-12bk^3\lambda cosh(kx)^2-6k^4\lambda cosh(kx)^2-...\right)t}{cosh(kx)^4kb}$$

When n=1,

$$u_2=\frac{\left(-16cosh(kx)^3sinh(kx)b^2k^8+144cosh(kx)^3sinh(kx)b^2k^6\lambda-...\right)t^2}{16kcosh(kx)^6b^2} \tag{56}$$

$$v_2=\frac{\left(-396cosh(kx)^3b^3k^5\lambda+576cosh(kx)^3b^2k^6+108cosh(kx)^3bk^7\lambda+...\right)t^2}{16k^2cosh(kx)^7b^2}$$

When n=2,

$$u_3=\frac{\left(-16cosh(kx)^3sinh(kx)b^2k^8+144cosh(kx)^3sinh(kx)b^2k^6\lambda-...\right)t^2}{24kcosh(kx)^6b^2} \tag{57}$$

$$v_3=\frac{\left(-396cosh(kx)^3b^3k^5\lambda+576cosh(kx)^3b^2k^6+108cosh(kx)^3bk^7\lambda+...\right)t^2}{24k^2cosh(kx)^7b^2}$$

Hence, the closed form is obtained as;

$$u(x,t) = \frac{b}{2k} + ktanh\left[k\left(x + \frac{1}{4}\left(\frac{6k\lambda}{b} + \frac{3b^2}{k^2} - 6\lambda - 4k^2\right)t\right)\right] \tag{58}$$

$$v(x,t) = \frac{\lambda}{2}\left(1 + \frac{k}{b}\right) + btanh\left[k\left(x + \frac{1}{4}\left(\frac{6k\lambda}{b} + \frac{3b^2}{k^2} - 6\lambda - 4k^2\right)t\right)\right]$$

Table 3. Comparisons between the numerical and analytical solutions of the coupled mKdV equation with the initial conditions in Equations(46-47) where $k = 0.1, b = \lambda = 0.1$ *and* $t = 0.5$.

x	u_{num}	u_{analy}	v_{num}	v_{analy}
-40	0.4000717889	0.4000097769	.00006970239290	.00007221960
-30	0.4005293006	0.4000722196	.0005140739471	.00053240638
-20	0.4038491249	0.4005324064	.003746666718	.00386819089
-10	0.4254250682	0.4254387962	.02508781610	.02543879620
0	0.5038818750	0.5036983125	.1055875000	.1036983125
10	0.5775699574	0.5776700629	.1790583310	.1776700629
20	0.5966265235	0.5966550519	.1969009038	.1966550519
30	0.5995364223	0.5995406680	.1995752020	.1995406680
40	0.5999371305	0.5999377125	.1999424098	.1999377125

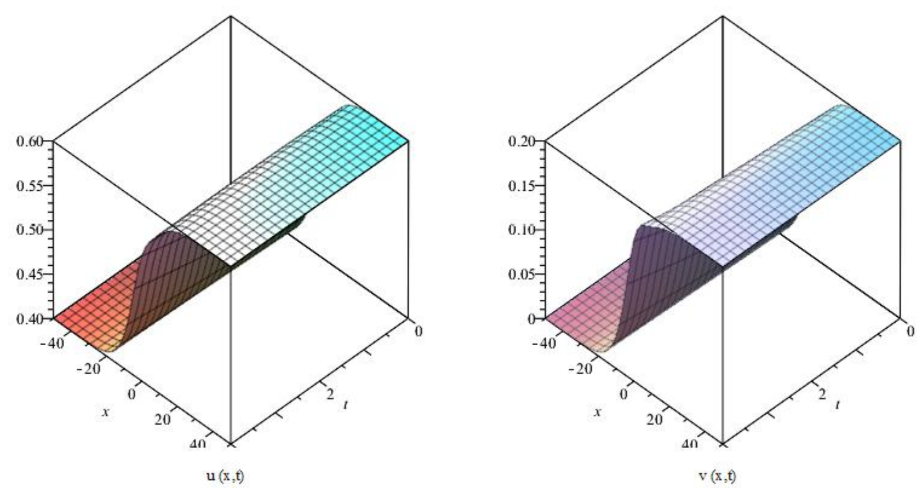


FIGURE 3. The numerical solutions for u(x,t) and v(x,t)

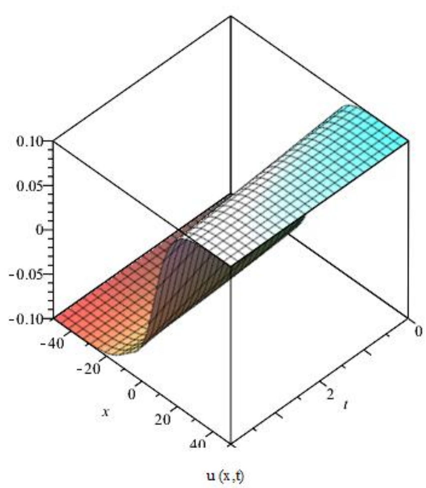
Equation (2) was further considered with a different initial condition given as:

$$u(x,0) = ktanh(kx) \tag{59}$$

$$v(x,0) = \frac{1}{2}(4k^2+\lambda) - 2k^2tanh^2(kx) \tag{60}$$

Table 4. Comparisons between the exact and numerical solutions of the coupled mKdV equation with the initial conditions in Equations(59-60) where $\lambda = 1, k = 0.1, t = 0.5$

x	u_{num}	u_{analy}	v_{num}	v_{analy}
-40	-0.09994178131	-0.09999219290	0.5000326720	0.5000031227
-30	-0.09957065168	-0.09957463673	0.5002401648	0.5001697834
-20	-0.09687218682	-0.09689910298	0.5017084899	0.5012211277
-10	-0.07904650037	-0.07915243723	0.5097287399	0.5074697834
0	-0.007550000000	-0.007535687005	0.5198112667	0.5198864268
10	0.07270488781	0.07280192976	0.5071908057	0.5093997580
20	0.09580535936	0.09582864870	0.5011645610	0.5016337402
30	0.09942167452	0.09942510121	0.5001616755	0.500229285
40	0.09992153295	0.09992200205	0.5000219565	0.5000311870



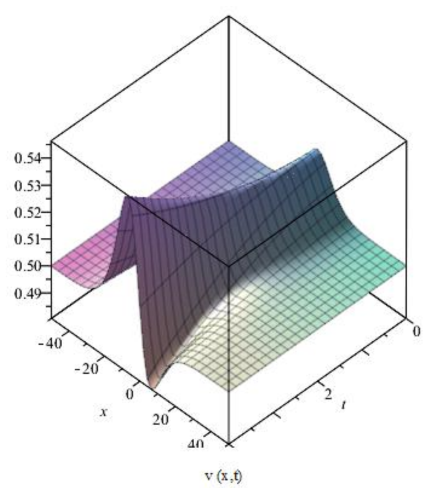


FIGURE 4. The numerical solutions for u(x,t) and v(x,t)

In Table 4, we made comparisons between the exact and the numerical solutions of $u(x,t)$ and $v(x,t)$ and noted that the results obtained include small errors.

6. Discussion of Results

The Aboodh Transform method has been effectively combined with reduced differential transform polynomials to handle the nonlinear aspect of the partial differential equation considered. The scheme was used to solve Equations (1) and (2) using four initial conditions. The result obtained is in series solutions which is in agreement with the analytical solution and thus shows the efficacy of the scheme. The Equations considered showed that the technique is effective for solving coupled nonlinear partial differential equation as the obtained results agrees with those in the highlighted references. Also, figures 1, 2, 3 and 4 show the graph of the problems considered at different initial conditions in order to give detail explanation on the behavior and shape of the coupled equations considered at any particular time which can be of interest to the engineers in case of control analysis and other physical problems. The small computational size which is also not affected by discretisation error will hence make the Aboodh reduced differential method a suitable and applicable scheme for other nonlinear evolution equation.

7. Conclusion

In this work, the ABRDTM has been used to obtain solutions of the Hirota-Satsuma coupled mKdV and KdV equations with two different initial conditions. Consequently, for Equations (1) and(2),the results computed have been presented in tabular form. Tables 1,2,3 and 4 compare the ABRDTM and analytical values for the initial conditions considered. It is expedient to say that the result gotten agreed with the closed form solutions [4]. However, the ABRDTM has an edge over the Differential Transform Method(DTM) [15] in terms of easier computability. Hence, the ABRDTM has many advantages and it is a strong tool for solving systems of nonlinear partial diferential equations with broad applications in physics and engineering fields.

Conflict of Interests

The authors declare that there is no conflict of interests.

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