Available online at http://scik.org
J. Math. Comput. Sci. 2022, 12:133
https://doi.org/10.28919/jmcs/7266
ISSN: 1927-5307

SOME RATIONAL CONTRACTION AND APPLICATIONS OF FIXED POINT THEOREMS TO $\mathscr{F}$-METRIC SPACE IN DIFFERENTIAL EQUATIONS

MOHAMMED M.A. TALEB[1,*], V.C. BORKAR[2]

[1]Department of Mathematics, Science College, Swami Ramanand Teerth Marathwada University, Nanded-431606, India

[2]Department of Mathematics, Yeshwant Mahavidyalaya, Swami Ramanand Teerth Marathwada University, Nanded-431606, India

Abstract. In this article, we present define generalized $(\alpha\theta-\psi)-$ rational contraction in $\mathscr{F}$-metric spaces and find a new fixed point results, and apply the results we obtained for study existence and uniqueness solution of nonlinear neutral differential equation with an unbounded delay.

Keywords: fixed point; $\mathscr{F}$-metric space; generalized $(\alpha\theta-\psi)$-rational contraction; nonlinear neutral differential equation.

2010 AMS Subject Classification: 54H25, 47H10.

1. Introduction

Since 1922, when the Polish mathematician Banach presented his theorem known as the Banach Contraction Principle (see [6]), the fixed point theorems have witnessed rapid and significant development Bakhtin [7] or Czerwik, Stefan [8] in many fields. Studies of the fixed point theorems have developed with the introduction of many generalized metric space such as b-metric space and $\mathscr{F}$-metric space, which was introduced by Jalili and Samet [4] in (2018). For

*Corresponding author
E-mail address: mohaayedtaleb@gmail.com
Received February 13, 2022

more details see ([2], [3], [4], [9]) etc. On the other hand several researchers stated various contraction conditions for the fixed point theorem, like convex contraction , F-contraction,$(\alpha-\psi)$-contraction, $(\alpha\theta-\psi)$- contraction ([1], [5], [15], [16], [17], [18]) etc.

2. PRELIMINARIES

We will first start by providing a definition of the set $\mathscr{F}$ (see Jleli [4]).

Definition 1.[4]. Consider the family $\mathscr{F}$ consisting of each function $\mathfrak{f}$ from $(0,+\infty)$ to $\mathbb{R}$, such that:

$(\mathscr{F}_1)$ $0<\mathfrak{s}<\mathfrak{c}$ implies $\mathfrak{f}(\mathfrak{s})\leq\mathfrak{f}(\mathfrak{c})$, this means $\mathfrak{f}$ is non decreasing .
$(\mathscr{F}_2)$ for all a sequence $\{\mathfrak{c}_\mathfrak{n}\}\subset(0,+\infty)$, we have

$$\lim_{\mathfrak{n}\to+\infty}\mathfrak{c}_\mathfrak{n}=0 \; if \; and \; only \; if \lim_{\mathfrak{n}\to+\infty}\mathfrak{f}(\mathfrak{c}_\mathfrak{n})=-\infty.$$

Now the generalized definition of metric space is as follows:

Definition 2.[4]. Let Z be a set that is not empty, and let the function $\rho:Z\times Z\to[0,\infty)$ be a given. Assume that $\exists$ $(\mathfrak{f},\gamma)\in\mathscr{F}\times[0,\infty)$, such that,
$(\mathfrak{D}_1)$ $(\kappa,\tilde{\kappa})\in Z\times Z$, $\rho(\kappa,\tilde{\kappa})=0$ if and only if $\kappa=\tilde{\kappa}$.
$(\mathfrak{D}_2)$ $\rho(\kappa,\tilde{\kappa})=\rho(\tilde{\kappa},\kappa)$ for all $(\kappa,\tilde{\kappa})\in Z\times Z$.
$(\mathfrak{D}_3)$ For every $(\kappa,\tilde{\kappa})\in Z\times Z$, $\forall$ $\mathfrak{N}$ in $\mathbb{N}$ and $\mathfrak{N}\geq 2$, and also for each $(\mathfrak{v}_j)_{j=1}^{\mathfrak{N}}\subset Z$ with $(\mathfrak{v}_1,\mathfrak{v}_\mathfrak{N})=(\kappa,\tilde{\kappa})$, we have

$$\rho(\kappa,\tilde{\kappa})>0\Rightarrow\mathfrak{f}(\rho(\kappa,\tilde{\kappa}))\leq\mathfrak{f}\Big(\sum_{j=1}^{\mathfrak{N}-1}\rho(\mathfrak{v}_j,\mathfrak{v}_{j+1})\Big)+\gamma.$$

Then we say that ρ is an $\mathscr{F}$-metric on Z, and (Z,ρ) is called $\mathscr{F}$-metric space.

Example 1.[4] Let $Z=\mathbb{N}$, and let $\rho:Z\times Z\to[0,\infty)$ be the mapping define by

$$\rho(\kappa,\tilde{\kappa})=\begin{cases}0 & if \;\kappa=\tilde{\kappa}\\ e^{|\kappa-\tilde{\kappa}|} & if \;\kappa\neq\tilde{\kappa}\end{cases}$$

for all $(\kappa, \tilde{\kappa}) \in Z \times Z$. Then ρ is an $\mathscr{F}$-metric on Z with $f(\mathfrak{c}) = \frac{-1}{\mathfrak{c}}$, $\mathfrak{c} > 0$ and $\gamma = 1$.

Definition 3. [4]: Let (Z, ρ) be an $\mathscr{F}$-metric space.

1. We can say that the sequence $\{\kappa_{\mathfrak{n}}\} \subset Z$ is $\mathscr{F}$-convergent to κ if :

$$\lim_{\mathfrak{n}\to\infty} \rho(\kappa_{\mathfrak{n}}, \kappa) = 0,$$

2. we say that $\{\kappa_{\mathfrak{n}}\}$ is $\mathscr{F}$-Cauchy if:

$$\lim_{\mathfrak{n},\mathfrak{m}\to+\infty} \rho(\kappa_{\mathfrak{n}}, \kappa_{\mathfrak{m}}) = 0,$$

3. also we say that (Z, ρ) is $\mathscr{F}$-complete, if each $\mathscr{F}$-Cauchy sequence in Z is $\mathscr{F}$-convergent to a certain element in Z.

Let Ψ the family consisting of all nondecreasing functions ψ from $[0,\infty)$ to $[0,\infty)$ such that $\sum_{\mathfrak{n}=1}^{\infty} \psi^{\mathfrak{n}}(\kappa) < \infty$ for all $\kappa > 0$, where $\psi^{\mathfrak{n}}$ is the $\mathfrak{n}$-th iterate of ψ. Also $\psi(\kappa) < \kappa$, $\forall\ \kappa > 0$ and $\psi(\kappa) = 0 \Longleftrightarrow \kappa = 0$.

The concept of $\alpha - \psi$-contractions and α-admissible mapping was introduced by, Samet et al, in 2012.[13] They defined the notion of α-admissible mappings as follows:

Definition 4.[13] Let $H : Z \to Z$ and $\alpha : Z \times Z \to [0,\infty)$. be a mapping. Then H is called α-admissible mapping if :

$$\alpha(\kappa, \tilde{\kappa}) \geq 1 \Rightarrow \alpha(H\kappa, H\tilde{\kappa}) \geq 1,$$

$\forall\ \kappa,\ \tilde{\kappa} \in Z$.

The extended concept of α-admissible mapping introduced by Hussain et al. [12] As follows:

Definition 5.[12] Let $H : Z \to Z$ and $\alpha,\ \theta : Z \times Z \to [0,\infty)$. Then H is called α-admissible mapping with respect to θ if:

$$\alpha(\kappa,\tilde{\kappa}) \geq \theta(\kappa,\tilde{\kappa}) \Rightarrow \alpha(H\kappa,H\tilde{\kappa}) \geq \theta(H\kappa,H\tilde{\kappa}),$$

$\forall\ \kappa,\ \tilde{\kappa} \in Z$.

In 2020 Al-Mezel et al [1]. introduced definition of generalized $(\alpha\theta - \psi)$-contraction in $\mathscr{F}$-metric spaces and established fixed point theorem for such mappings in $\mathscr{F}$-metric spaces. In this section, we define the concept of generalized $(\alpha\theta - \psi)$-rational contraction and establish a new fixed point theorem in the context of $\mathscr{F}$-metric spaces.

3. Main Results

Definition 6. : Let (Z,ρ) be an $\mathscr{F}$-metric space and $H : Z \to Z$. Then H is said to be generalized $(\alpha\theta - \psi)$- rational contraction if there exists α , $\theta : Z \times Z \to [0,\infty)$ and $\psi \in \Psi$ such that

$$\alpha(\kappa,H\kappa)\alpha(\tilde{\kappa},H\tilde{\kappa}) \geq \theta(\kappa,H\kappa)\theta(\tilde{\kappa},H\tilde{\kappa})$$

implies

$$(1)\quad \rho(H\kappa,H\tilde{\kappa}) \leq \psi\Big(\max\Big\{\rho(\kappa,\tilde{\kappa}),\min\{\frac{\rho(\kappa,H\kappa)\rho(\tilde{\kappa},H\tilde{\kappa})}{1+\rho(\kappa,\tilde{\kappa})},\frac{\rho(\tilde{\kappa},H\tilde{\kappa})[1+\rho(\kappa,H\kappa)]}{1+\rho(\kappa,\tilde{\kappa})}\}\Big\}\Big)$$

$\forall \kappa\ ,\tilde{\kappa} \in Z$.

Theorem 1. : Let (Z,ρ) be an $\mathscr{F}$-complete $\mathscr{F}$-metric space and $H : Z \to Z$ be a generalized $(\alpha\theta - \psi)$- rational contraction such that H satisfies the condition in definition 5, and suppose that:

(i) $\exists\ \kappa_0 \in Z$ such that $\alpha(\kappa_0,H\kappa_0) \geq \theta(\kappa_0,H\kappa_0)$.

(ii) If $\{\kappa_n\}$ is a sequence in Z such that $\kappa_n \to \kappa,\ \alpha(\kappa_n,\kappa_{n+1}) \geq \theta(\kappa_n,\kappa_{n+1})\ \forall n \in \mathbb{N}$, then $\alpha(\kappa,H\kappa) \geq \theta(\kappa,H\kappa)$. Then H has a unique fixed point $\kappa^* \in Z$.

Proof: Define $\{\kappa_{\mathfrak{n}}\}$ in Z by $\kappa_{\mathfrak{n}+1} = H^{\mathfrak{n}}\kappa_0 = H\kappa_{\mathfrak{n}}$, $\forall \mathfrak{n} \in \mathbb{N}$. And let $\kappa_0 \in Z$ such that (i) is hold . Since H satisfies the condition in definition 5, then we have:

$$\alpha(\kappa_0, \kappa_1) = \alpha(\kappa_0, H\kappa_0) \geq \theta(\kappa_0, H\kappa_0) = \theta(\kappa_0, \kappa_1).$$

Continuing in this way, we get

$$\alpha(\kappa_{\mathfrak{n}-1}, \kappa_{\mathfrak{n}}) = \alpha(\kappa_{\mathfrak{n}-1}, H\kappa_{\mathfrak{n}-1}) \geq \theta(\kappa_{\mathfrak{n}-1}, H\kappa_{\mathfrak{n}-1}) = \theta(\kappa_{\mathfrak{n}-1}, \kappa_{\mathfrak{n}}), \tag{2}$$

and

$$\alpha(\kappa_{\mathfrak{n}}, \kappa_{\mathfrak{n}+1}) = \alpha(\kappa_{\mathfrak{n}}, H\kappa_{\mathfrak{n}}) \geq \theta(\kappa_{\mathfrak{n}}, H\kappa_{\mathfrak{n}}) = \theta(\kappa_{\mathfrak{n}}, \kappa_{\mathfrak{n}+1}). \tag{3}$$

from (2)and (3) we get

$$\alpha(\kappa_{\mathfrak{n}-1}, H\kappa_{\mathfrak{n}-1})\alpha(\kappa_{\mathfrak{n}}, H\kappa_{n}) \geq \theta(\kappa_{\mathfrak{n}-1}, H\kappa_{\mathfrak{n}-1})\theta(\kappa_{\mathfrak{n}}, H\kappa_{\mathfrak{n}}) \tag{4}$$

$\forall \mathfrak{n} \in \mathbb{N}$. Now if there is natural number $\mathfrak{n}_0$, $\kappa_{\mathfrak{n}_0+1} = \kappa_{\mathfrak{n}_0}$, then $H\kappa_{\mathfrak{n}_0} = \kappa_{\mathfrak{n}_0}$ and hence $\kappa_{\mathfrak{n}_0}$ is a fixed point of H. In this case, the proof is finished. Suppose that $\kappa_{\mathfrak{n}+1} \neq \kappa_{\mathfrak{n}}$ $\forall \mathfrak{n} \in \mathbb{N}$, and let $\mathfrak{f} \in \mathscr{F}$, $\gamma \in [0, \infty)$ be such that $(\mathscr{D}_3)$ is satisfied.
Let $\varepsilon > 0$ then by $(\mathscr{F}_2)$ there exists $\delta > 0$ such that,

$$0 < \mathfrak{c} < \delta \Rightarrow \mathfrak{f}(\mathfrak{c}) < \mathfrak{f}(\varepsilon) - \gamma. \tag{5}$$

Now by (1) we have:

$$\begin{aligned}
\rho(\kappa_{\mathfrak{n}}, \kappa_{\mathfrak{n}+1}) &= \rho(H\kappa_{\mathfrak{n}-1}, H\kappa_{\mathfrak{n}}) \\
&\leq \psi\Big(\max\Big\{\rho(\kappa_{\mathfrak{n}-1}, \kappa_{\mathfrak{n}}), \\
&\min\{\frac{\rho(\kappa_{\mathfrak{n}-1}, H\kappa_{\mathfrak{n}-1})\rho(\kappa_{\mathfrak{n}}, H\kappa_{\mathfrak{n}})}{1+\rho(\kappa_{\mathfrak{n}-1}, \kappa_{\mathfrak{n}})}, \frac{\rho(\kappa_{\mathfrak{n}}, H\kappa_{\mathfrak{n}})[1+\rho(\kappa_{\mathfrak{n}-1}, H\kappa_{\mathfrak{n}-1})]}{1+\rho(\kappa_{\mathfrak{n}-1}, \kappa_{\mathfrak{n}})}\}\Big\}\Big) \\
&\leq \psi\Big(\max\Big\{\rho(\kappa_{\mathfrak{n}-1}, \kappa_{\mathfrak{n}}), \\
&\min\{\frac{\rho(\kappa_{\mathfrak{n}-1}, \kappa_{\mathfrak{n}})\rho(\kappa_{\mathfrak{n}}, \kappa_{\mathfrak{n}+1})}{1+\rho(\kappa_{\mathfrak{n}-1}, \kappa_{\mathfrak{n}})}, \frac{\rho(\kappa_{\mathfrak{n}}, \kappa_{\mathfrak{n}+1})[1+\rho(\kappa_{\mathfrak{n}-1}, \kappa_{\mathfrak{n}})]}{1+\rho(\kappa_{\mathfrak{n}-1}, \kappa_{\mathfrak{n}})}\}\Big\}\Big) \\
&\leq \psi\Big(\max\Big\{\rho(\kappa_{\mathfrak{n}-1}, \kappa_{\mathfrak{n}}), \min\{\rho(\kappa_{\mathfrak{n}}, \kappa_{\mathfrak{n}+1}), \rho(\kappa_{\mathfrak{n}}, \kappa_{\mathfrak{n}+1})\}\Big\}\Big) \\
&\leq \psi\Big(\max\Big\{\rho(\kappa_{\mathfrak{n}-1}, \kappa_{\mathfrak{n}}), \rho(\kappa_{\mathfrak{n}}, \kappa_{\mathfrak{n}+1})\Big\}\Big).
\end{aligned}$$

Now if $\max\Big\{\rho(\kappa_{\mathfrak{n}-1},\kappa_{\mathfrak{n}}),\rho(\kappa_{\mathfrak{n}},\kappa_{\mathfrak{n}+1})\Big\}=\rho(\kappa_{\mathfrak{n}},\kappa_{\mathfrak{n}+1})$ then $\rho(\kappa_{\mathfrak{n}},\kappa_{\mathfrak{n}+1})\le\psi(\rho(\kappa_{\mathfrak{n}},\kappa_{\mathfrak{n}+1}))<\rho(\kappa_{\mathfrak{n}},\kappa_{\mathfrak{n}+1})$ which is a contradiction, and hence $\max\Big\{\rho(\kappa_{\mathfrak{n}-1},\kappa_{\mathfrak{n}}),\rho(\kappa_{\mathfrak{n}},\kappa_{\mathfrak{n}+1})\Big\}=\rho(\kappa_{\mathfrak{n}-1},\kappa_{\mathfrak{n}})$, then we get:

$$\rho(\kappa_{\mathfrak{n}},\kappa_{\mathfrak{n}+1})\le\psi(\rho(\kappa_{\mathfrak{n}-1},\kappa_{\mathfrak{n}}))\le\psi(\psi(\rho(\kappa_{\mathfrak{n}-2},\kappa_{\mathfrak{n}-1}))\le\ldots\le\psi^{\mathfrak{n}}(\rho(\kappa_0,\kappa_1)).\tag{6}$$

Let $\mathfrak{n}(\varepsilon)\in\mathbb{N}$ such that $0<\sum_{\mathfrak{n}\ge\mathfrak{n}(\varepsilon)}\psi^{\mathfrak{n}}(\rho(\kappa_0,\kappa_1))<\delta$. By (5),(6) and $(\mathscr{F}_1)$ we get:

$$\mathfrak{f}\Big(\sum_{j=\mathfrak{n}}^{\mathfrak{m}-1}\sigma(\kappa_j,\kappa_{j+1})\Big)\le\mathfrak{f}\Big(\sum_{j=\mathfrak{n}}^{\mathfrak{m}-1}\psi^{j}(\rho(\kappa_0,\kappa_1))\Big)\le\mathfrak{f}\Big(\sum_{\mathfrak{n}\ge\mathfrak{n}(\varepsilon)}\psi^{\mathfrak{n}}(\rho(\kappa_0,\kappa_1))\Big)<\mathfrak{f}(\varepsilon)-\gamma,\tag{7}$$

for $\mathfrak{m}>\mathfrak{n}\ge\mathfrak{n}(\varepsilon)$ with $\rho(\kappa_{\mathfrak{n}},\kappa_{\mathfrak{m}})>0$. Using $(\mathfrak{D}_3)$ and (7) we obtain:

$$\mathfrak{f}(\rho(\kappa_{\mathfrak{n}},\kappa_{\mathfrak{m}}))\le\mathfrak{f}\Big(\sum_{j=\mathfrak{n}}^{\mathfrak{m}-1}\rho(\kappa_j,\kappa_{j+1})\Big)+\gamma<\mathfrak{f}(\varepsilon).$$

By $(\mathscr{F}_1)$ we get $\rho(\kappa_{\mathfrak{n}},\kappa_{\mathfrak{m}})<\varepsilon$. This means $\{\kappa_{\mathfrak{n}}\}$ is $\mathscr{F}$-Cauchy sequence, and since (Z,ρ) is $\mathscr{F}$-complete, then there exists $\kappa^*\in Z$ such that

$$\lim_{\mathfrak{n}\to\infty}\rho(\kappa_{\mathfrak{n}},\kappa^*)=0.\tag{8}$$

Since $\kappa_{\mathfrak{n}}\to\kappa^*$ and $\alpha(\kappa_{\mathfrak{n}},\kappa_{\mathfrak{n}+1})\ge\theta(\kappa_{\mathfrak{n}},\kappa_{\mathfrak{n}+1})$, then by (ii) $\alpha(\kappa^*,H\kappa^*)\ge\theta(\kappa^*,H\kappa^*)$. Thus

$$\alpha(\kappa^*,H\kappa^*)\alpha(\kappa_{\mathfrak{n}},H\kappa_{\mathfrak{n}})\ge\theta(\kappa^*,H\kappa^*)\theta(\kappa_{\mathfrak{n}},H\kappa_{\mathfrak{n}}).\tag{9}$$

Now we prove that κ^* is a fixed point of H. Suppose that $\rho(H\kappa^*,\kappa^*)>0$ then by $(\mathfrak{D}_3)$ we have

$$\mathfrak{f}(\rho(H\kappa^*,\kappa^*))\le\mathfrak{f}(\rho(H\kappa^*,H\kappa_{\mathfrak{n}})+\rho(H\kappa_{\mathfrak{n}},\kappa^*))+\gamma\,\forall\mathfrak{n}\in\mathbb{N}.$$

Using (1) we get

$$\begin{aligned}
&\mathfrak{f}(\rho(H\kappa^*,\kappa^*))\\
\leq\ & \mathfrak{f}\Big(\psi\Big(\max\Big\{\rho(\kappa^*,\kappa_{\mathfrak{n}}),\min\{\frac{\rho(\kappa^*,H\kappa^*)\rho(\kappa_{\mathfrak{n}},H\kappa_{\mathfrak{n}})}{1+\rho(\kappa^*,\kappa_{\mathfrak{n}})},\frac{\rho(\kappa_{\mathfrak{n}},H\kappa_{\mathfrak{n}})[1+\rho(\kappa^*,H\kappa^*)]}{1+\rho(\kappa^*,\kappa_{\mathfrak{n}})}\}\Big\}\Big)\\
&+\rho(H\kappa_{\mathfrak{n}},\kappa^*)\Big)+\gamma\\
\leq\ & \mathfrak{f}\Big(\psi\Big(\max\Big\{\rho(\kappa^*,\kappa_{\mathfrak{n}}),\min\{\frac{\rho(\kappa^*,H\kappa^*)\rho(\kappa_{\mathfrak{n}},\kappa_{\mathfrak{n}+1})}{1+\rho(\kappa^*,\kappa_{\mathfrak{n}})},\frac{\rho(\kappa_{\mathfrak{n}},\kappa_{\mathfrak{n}+1})[1+\rho(\kappa^*,H\kappa^*)]}{1+\rho(\kappa^*,\kappa_{\mathfrak{n}})}\}\Big\}\Big)\\
&+\rho(\kappa_{\mathfrak{n}+1},\kappa^*)\Big)+\gamma\\
<\ & \mathfrak{f}\Big(\max\Big\{\rho(\kappa^*,\kappa_{\mathfrak{n}}),\min\{\frac{\rho(\kappa^*,H\kappa^*)\rho(\kappa_{\mathfrak{n}},\kappa_{\mathfrak{n}+1})}{1+\rho(\kappa^*,\kappa_{\mathfrak{n}})},\frac{\rho(\kappa_{\mathfrak{n}},\kappa_{\mathfrak{n}+1})[1+\rho(\kappa^*,H\kappa^*)]}{1+\rho(\kappa^*,\kappa_{\mathfrak{n}})}\}\Big\}\\
&+\rho(\kappa_{\mathfrak{n}+1},\kappa^*)\Big)+\gamma.
\end{aligned}$$

Now either (a)

$$\min\{\frac{\rho(\kappa^*,H\kappa^*)\rho(\kappa_{\mathfrak{n}},\kappa_{\mathfrak{n}+1})}{1+\rho(\kappa^*,\kappa_{\mathfrak{n}})},\frac{\rho(\kappa_{\mathfrak{n}},\kappa_{\mathfrak{n}+1})[1+\rho(\kappa^*,H\kappa^*)]}{1+\rho(\kappa^*,\kappa_{\mathfrak{n}})}\}=\frac{\rho(\kappa^*,H\kappa^*)\rho(\kappa_{\mathfrak{n}},\kappa_{\mathfrak{n}+1})}{1+\rho(\kappa^*,\kappa_{\mathfrak{n}})}.$$

Or (b)

$$\min\{\frac{\rho(\kappa^*,H\kappa^*)\rho(\kappa_{\mathfrak{n}},\kappa_{\mathfrak{n}+1})}{1+\rho(\kappa^*,\kappa_{\mathfrak{n}})},\frac{\rho(\kappa_{\mathfrak{n}},\kappa_{\mathfrak{n}+1})[1+\rho(\kappa^*,H\kappa^*)]}{1+\rho(\kappa^*,\kappa_{\mathfrak{n}})}\}=\frac{\rho(\kappa_{\mathfrak{n}},\kappa_{\mathfrak{n}+1})[1+\rho(\kappa^*,H\kappa^*)]}{1+\rho(\kappa^*,\kappa_{\mathfrak{n}})}.$$

If (a) satisfies, then

$$\mathfrak{f}(\rho(H\kappa^*,\kappa^*))<\mathfrak{f}\Big(\max\Big\{\rho(\kappa^*,\kappa_{\mathfrak{n}}),\frac{\rho(\kappa^*,H\kappa^*)\rho(\kappa_{\mathfrak{n}},\kappa_{\mathfrak{n}+1})}{1+\rho(\kappa^*,\kappa_{\mathfrak{n}})}\Big\}+\rho(\kappa_{\mathfrak{n}+1},\kappa^*)\Big)+\gamma.$$

In this case if

$$\max\Big\{\rho(\kappa^*,\kappa_{\mathfrak{n}}),\frac{\rho(\kappa^*,H\kappa^*)\rho(\kappa_{\mathfrak{n}},\kappa_{\mathfrak{n}+1})}{1+\rho(\kappa^*,\kappa_{\mathfrak{n}})}\Big\}=\rho(\kappa^*,\kappa_{\mathfrak{n}}),$$

then

$$\mathfrak{f}(\rho(H\kappa^*,\kappa^*))<\mathfrak{f}\Big(\rho(\kappa^*,\kappa_{\mathfrak{n}})+\rho(\kappa_{\mathfrak{n}+1},\kappa^*)\Big)+\gamma.$$

Taking the limit and by (8) and ($\mathscr{F}_2$) we get,

$$\lim_{\mathfrak{n}\to\infty}\mathfrak{f}(\rho(H\kappa^*,\kappa^*))\leq\lim_{\mathfrak{n}\to\infty}\mathfrak{f}\Big(\rho(\kappa^*,\kappa_{\mathfrak{n}})+\rho(\kappa_{\mathfrak{n}+1},\kappa^*)\Big)+\gamma=-\infty.$$

Which is a contradiction, and hence $\rho(H\kappa^*,\kappa^*)=0$. And if

$$\max\Big\{\rho(\kappa^*,\kappa_{\mathfrak{n}}),\frac{\rho(\kappa^*,H\kappa^*)\rho(\kappa_{\mathfrak{n}},\kappa_{\mathfrak{n}+1})}{1+\rho(\kappa^*,\kappa_{\mathfrak{n}})}\Big\}=\frac{\rho(\kappa^*,H\kappa^*)\rho(\kappa_{\mathfrak{n}},\kappa_{\mathfrak{n}+1})}{1+\rho(\kappa^*,\kappa_{\mathfrak{n}})},$$

then

$$\mathfrak{f}(\rho(H\kappa^*,\kappa^*)) < \mathfrak{f}\Big(\frac{\rho(\kappa^*,H\kappa^*)\rho(\kappa_{\mathfrak{n}},\kappa_{\mathfrak{n}+1})}{1+\rho(\kappa^*,\kappa_{\mathfrak{n}})} + \rho(\kappa_{\mathfrak{n}+1},\kappa^*)\Big) + \gamma.$$

Also taking the limit and by (8) and ($\mathscr{F}_2$) we get

$$\lim_{\mathfrak{n}\to\infty} \mathfrak{f}(\rho(H\kappa^*,\kappa^*)) \leq \lim_{\mathfrak{n}\to\infty} \mathfrak{f}\Big(\frac{\rho(\kappa^*,H\kappa^*)\rho(\kappa_{\mathfrak{n}},\kappa_{\mathfrak{n}+1})}{1+\rho(\kappa^*,\kappa_{\mathfrak{n}})} + \rho(\kappa_{\mathfrak{n}+1},\kappa^*)\Big) + \gamma = -\infty.$$

Also which is a contradiction, and hence $\rho(H\kappa^*,\kappa^*) = 0$. Through applying the same steps in (b) we get $\rho(H\kappa^*,\kappa^*) = 0$ i.e. $H\kappa^* = \kappa^*$. Now we prove that κ^* is a unique fixed point of H, so suppose that H has another fixed point ζ^* such that $H\zeta^* = \zeta^*$. Since $\kappa_{\mathfrak{n}} \to \kappa^*$ and $\alpha(\kappa_{\mathfrak{n}},\kappa_{\mathfrak{n}+1}) \geq \theta(\kappa_{\mathfrak{n}},\kappa_{\mathfrak{n}+1})$ then

$$\alpha(\kappa^*,H\kappa^*) \geq \theta(\kappa^*,H\kappa^*). \tag{10}$$

And $\zeta_n \to \zeta^*$ and $\alpha(\zeta_n,\zeta_{n+1}) \geq \theta(\zeta_n,\zeta_{n+1})$ then

$$\alpha(\zeta^*,H\zeta^*) \geq \theta(\zeta^*,H\zeta^*). \tag{11}$$

By (10) and (11) for $\kappa^*,\zeta^* \in Z$ we have,

$$\alpha(\kappa^*,H\kappa^*)\theta(\zeta^*,H\zeta^*) \geq \theta(\kappa^*,H\kappa^*)\theta(\zeta^*,H\zeta^*). \tag{12}$$

Using (1) we get

$$\begin{aligned}
\rho(\kappa^*,\zeta^*) &= \rho(H\kappa^*,H\zeta^*) \\
&\leq \psi\Big(\max\Big\{\rho(\kappa^*,\zeta^*), \\
&\quad \min\Big\{\frac{\rho(\kappa^*,H\kappa^*)\rho(\zeta^*,H\zeta^*)}{1+\rho(\kappa^*,\zeta^*)}, \frac{\rho(\zeta^*,H\zeta^*)[1+\rho(\kappa^*,H\kappa^*)]}{1+\rho(\kappa^*,\zeta^*)}\Big\}\Big\}\Big),
\end{aligned}$$

$\Rightarrow \rho(\kappa^*,\zeta^*) \leq \psi(\rho(\kappa^*,\zeta^*)) < \rho(\kappa^*,\zeta^*)$, which is a contradiction. Hence H has unique fixed point in Z.

Example 2. Let $Z = \mathbb{R}$ and ρ be an $\mathscr{F}$-metric given in example 1. Define $H : Z \to Z$ by

$$H\kappa = \begin{cases} 4\kappa, & if\ \kappa > 0 \\ \frac{\kappa}{4}, & if\ 0 \leq \kappa \leq 1 \\ 0, & otherwise. \end{cases}$$

And define $\alpha, \theta : Z \times Z \to [0,\infty)$ by

$$\alpha(\kappa,\tilde{\kappa}) = \theta(\kappa,\tilde{\kappa}) \begin{cases} 1, & if\ \kappa,\tilde{\kappa} \in [0,1] \\ 0, & otherwise. \end{cases}$$

Then H is generalized $(\alpha\theta - \psi)$- rational contraction mapping with $\psi(\mathfrak{c}) = k\mathfrak{c}$, $\forall \mathfrak{c} \geq 0$ and $k \in (0,1)$ that is

$$\rho(H\kappa,H\tilde{\kappa}) \leq k\Big(\max\Big\{\rho(\kappa,\tilde{\kappa}),\min\{\frac{\rho(\kappa,H\kappa)\rho(\tilde{\kappa},H\tilde{\kappa})}{1+\rho(\kappa,\tilde{\kappa})},\frac{\rho(\tilde{\kappa},H\tilde{\kappa})[1+\rho(\kappa,H\kappa)]}{1+\rho(\kappa,\tilde{\kappa})}\}\Big\}\Big).$$

All the condition of theorem 1 are satisfied, and hence H has unique fixed point $0 \in Z$.

4. APPLICATION

We will using the theorem (1) to prove that there exists a solution to the following differential equations. And also we will prove that this solution is unique.

$$z'(\ell) = -a(\ell)z(\ell) + b(\ell)\mathfrak{I}(z(\ell - \varsigma(\ell))) + c(\ell)z'(\ell - \varsigma(\ell)), \tag{13}$$

where $a(\ell)$, $b(\ell)$ are continuous, $c(\ell)$ is continuously differentiable and $\varsigma(\ell) > 0$ for all $\ell \in \mathbb{R}$ and is twice continuously differentiable. For more information in this direction, (see[10]-[11]).

Lemma 1.[14]. Suppose that $\varsigma'(\ell) \neq 1$ for all $\ell \in \mathbb{R}$. Then $z(\ell)$ is a solution of (13) if and only if

$$\begin{aligned} z(\ell) &= \Big(z(0) - \frac{c(0)}{1-\varsigma'(0)}z(-\varsigma(0))\Big)e^{-\int_0^\ell a(\mathfrak{s})d\mathfrak{s}} + \frac{c(\ell)}{1-\varsigma'(\ell)}z(\ell-\varsigma(\ell)) \\ &\quad - \int_0^\ell (\mathfrak{h}(\mathfrak{v})z(\mathfrak{v}-\varsigma(\mathfrak{v}))) - b(\mathfrak{v})\mathfrak{I}(z(\mathfrak{v}-\varsigma(\mathfrak{v})))e^{-\int_0^\ell a(\mathfrak{s})d\mathfrak{s}}d\mathfrak{v}, \end{aligned} \tag{14}$$

where

$$\mathfrak{h}(\mathfrak{v}) = \frac{\varsigma''(\mathfrak{v})c(\mathfrak{v}) + (c'(\mathfrak{v}) + c(\mathfrak{v})a(\mathfrak{v}))(1-\varsigma'(\mathfrak{v}))}{(1-\varsigma'(\mathfrak{v}))^2}. \tag{15}$$

Now let $\xi : (-\infty,0] \to \mathbb{R}$ be given continuous bounded initial function.Then $z(\ell) = z(\ell,0,\xi)$ is a solution of (13) if $z(\ell) = \xi(\ell)$ for $\ell \leq 0$ and satisfies (13) for $\ell \geq 0$. Let $\mathscr{C}$ be the space consisting of all continuous functions ν from $\mathbb{R}$ to $\mathbb{R}$. Now we will define the following set

$$Z_{\xi} = \{\nu : \mathbb{R} \to \mathbb{R},\ \nu(\ell) = \xi(\ell)\ if\ \ell \leq 0,\ \nu(\ell) \to 0\ as\ \ell \to \infty,\ \nu \in \mathscr{C}\}. \tag{16}$$

Then Z_{ξ} is a Banach space equipped with the supremum norm $\| \,.\, \|$.

Lemma 2. :[5] The Banach space $(Z_{\xi}, \| \,.\, \|)$ endowed with the metric ρ defined by $\rho(\ell, \ell^*) = \| \ell - \ell^* \| = sup \mid \ell(z) - \ell^*(z) \mid$, where $\ell, \ell^* \in Z_{\xi}$, is an $\mathscr{F}$-metric space.

Theorem 2 : Let $H : Z_{\xi} \to Z_{\xi}$ be a mapping defined by

$$\begin{aligned}(H\nu)(\ell) &= \Big(\nu(0) - \frac{c(0)}{1-\varsigma'(0)}\nu(-\varsigma(0))\Big)e^{-\int_0^{\ell} a(\mathfrak{s})d\mathfrak{s}} + \frac{c(\ell)}{1-\varsigma'(\ell)}\nu(\ell - \varsigma(\ell)) \\ &\quad - \int_0^{\ell} (\mathfrak{h}(\mathfrak{v})\nu(\mathfrak{v} - \varsigma(\mathfrak{v}))) - b(\mathfrak{v})\Im(\nu(\mathfrak{v} - \varsigma(\mathfrak{v})))e^{-\int_{\mathfrak{v}}^{\ell} a(\mathfrak{s})d\mathfrak{s}} d\mathfrak{v},\end{aligned} \tag{17}$$

$\forall\ \nu \in Z_{\xi}$. Suppose that these assertions are satisfied:

(i) there exists $\mu \geq 0$ and $\psi \in \Psi$ such that

$$\int_0^{\ell} \mid (\mathfrak{h}(\mathfrak{v})\nu(\mathfrak{v} - \varsigma(\mathfrak{v}))) - \kappa(\mathfrak{v} - \varsigma(\mathfrak{v})) \mid e^{-\int_{\mathfrak{v}}^{\ell} a(\mathfrak{s})d\mathfrak{s}} d\mathfrak{v}$$

$$\leq \frac{\mu}{2}\psi\Big(\max\Big\{\|\nu - \kappa\|, \min\{\frac{\|\nu - H\nu\|\|\kappa - H\kappa\|}{1 + \|\nu - \kappa\|}, \frac{\|\kappa - H\kappa\|[1 + \|\nu - H\nu\|]}{1 + \|\nu - \kappa\|}\}\Big\}\Big) \tag{18}$$

and

$$\int_0^{\ell} \mid (b(\mathfrak{v})\Im(\nu(\mathfrak{v} - \varsigma(\mathfrak{v}))) - \Im(\kappa(\mathfrak{v} - \varsigma(\mathfrak{v}))) \mid e^{-\int_{\mathfrak{v}}^{\ell} a(\mathfrak{s})d\mathfrak{s}} d\mathfrak{v}$$

$$\leq \frac{\mu}{2}\psi\Big(\max\Big\{\|\nu - \kappa\|, \min\{\frac{\|\nu - H\nu\|\|\kappa - H\kappa\|}{1 + \|\nu - \kappa\|}, \frac{\|\kappa - H\kappa\|[1 + \|\nu - H\nu\|]}{1 + \|\nu - \kappa\|}\}\Big\}\Big) \tag{19}$$

$\forall\ \nu, \kappa \in Z_{\xi}$.

(ii)

$$\Big|\frac{c(\ell)}{1-\varsigma'(\ell)}\Big| + \mu \leq 1,\ \ell \geq 0. \tag{20}$$

Then H has unique fixed point in Z_ξ.

Proof: Define $\alpha, \theta : \mathscr{C} \times \mathscr{C} \to \mathbb{R}$ by

$$\alpha(\nu,\kappa) = \theta(\nu,\kappa) = \begin{cases} 1, & if\ \nu,\kappa \in Z_\xi \\ 0, & otherwise. \end{cases}$$

Next for ν , $\kappa \in Z_\xi$ such that $\alpha(\nu,\kappa) = \theta(\nu,\kappa) \geq 1$. It follows from (17) that $H\nu$, $H\kappa \in Z_\xi$ and hence $\alpha(H\nu,H\kappa) = \theta(H\nu,H\kappa) \geq 1$. As (18)-(20) hold, then for $\nu,\kappa \in Z_\xi$, we have

$$\begin{aligned} \mid (H\nu)(\ell) - (H\kappa)(\ell) \mid &\leq \left|\tfrac{c(\ell)}{1-\varsigma'(\ell)}\right| \|\nu - \kappa\| \\ &+ \int_0^\ell \mid (\mathfrak{h}(\mathfrak{v})\nu(\mathfrak{v}-\varsigma(\mathfrak{v}))) - \kappa(\mathfrak{v}-\varsigma(\mathfrak{v})) \mid e^{-\int_{\mathfrak{v}}^{\ell} a(\mathfrak{s})d\mathfrak{s}} d\mathfrak{v} \\ &+ \int_0^\ell \mid (b(\mathfrak{v})\Im(\nu(\mathfrak{v}-\varsigma(\mathfrak{v}))) - \Im(\kappa(\mathfrak{v}-\varsigma(\mathfrak{v}))) \mid e^{-\int_{\mathfrak{v}}^{\ell} a(\mathfrak{s})d\mathfrak{s}} d\mathfrak{v} \\ &\leq \left|\frac{c(\ell)}{1-\varsigma'(\ell)}\right| \|\nu-\kappa\| + \mu\psi\Big(\max\Big\{\|\nu-\kappa\|, \min\{\frac{\|\nu - H\nu\|\|\kappa - H\kappa\|}{1+\|\nu-\kappa\|}, \frac{\|\kappa - H\kappa\|[1+\|\nu - H\nu\|]}{1+\|\nu-\kappa\|}\}\Big\}\Big) \\ &\leq \left\{\left|\frac{c(\ell)}{1-\varsigma'(\ell)}\right| + \mu\right\}\psi\Big(\max\Big\{\|\nu-\kappa\|, \min\{\frac{\|\nu - H\nu\|\|\kappa - H\kappa\|}{1+\|\nu-\kappa\|}, \frac{\|\kappa - H\kappa\|[1+\|\nu - H\nu\|]}{1+\|\nu-\kappa\|}\}\Big\}\Big) \\ &\leq \psi\Big(\max\Big\{\|\nu-\kappa\|, \min\{\frac{\|\nu - H\nu\|\|\kappa - H\kappa\|}{1+\|\nu-\kappa\|}, \frac{\|\kappa - H\kappa\|[1+\|\nu - H\nu\|]}{1+\|\nu-\kappa\|}\}\Big\}\Big). \end{aligned}$$

And hence

$$\rho(H\nu,H\kappa) \leq \psi\Big(\max\Big\{\rho(\nu,\kappa), \min\Big\{\frac{\rho(\nu,H\nu)\rho(\kappa,H\kappa)}{1+\rho(\nu,\kappa)}, \frac{\rho(\kappa,H\kappa)[1+\rho(\nu,H\nu)]}{1+\rho(\nu,\kappa)}\Big\}\Big\}\Big).$$

This means that H is generalized $(\alpha\theta - \psi)$- rational contraction. Thus by Theorem 1, H has a unique fixed point in Z_ξ which solves (13).

5. Conclusions

In this paper we defined generalized $(\alpha\theta - \psi)$-rational contraction in $\mathscr{F}$-metric space and achieved novel fixed point results . And application of our findings, we looked into the existence of a solution for the nonlinear neutral differential equation with an unbounded delay and provided that this solution is unique.

Conflict of Interests

The author(s) declare that there is no conflict of interests.

References

[1] S.A. Al-Mezel, J. Ahmad, G. Marino, Fixed point theorems for generalized $(\alpha\beta-\psi)$-contractions in F-metric spaces with applications, Mathematics 8 (2020), 584.

[2] S. Czerwik, Contraction mappings in *b*-metric spaces, Acta Math. Inform. Univ. Ostrav. 1 (1993), 5-11.

[3] H. Huang, G. Deng, S. Radenovic, Fixed point theorems in b-metric spaces with applications to differential equations, J. Fixed Point Theory Appl. 20 (2018), 1-24.

[4] M. Jleli, B. Samet, On a new generalization of metric spaces, J. Fixed Point Theory Appl. 20 (2018), 1-20.

[5] A. Hussain, T. Kanwal, Existence and uniqueness for a neutral differential problem with unbounded delay via fixed point results, Trans. A. Razmadze Math. Inst. 172 (2018), 481-490.

[6] S. Banach, Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales, Fund. Math. 3 (1922), 133-181.

[7] I.A. Bakhtin, The contraction mapping principle in quasimetric spaces, Func. An., Gos. Ped. Inst. Unianowsk. 30 (1989), 26-37.

[8] S. Czerwik, Contraction mappings in *b*-metric spaces, Acta Math. Inform. Univ. Ostrav. 1 (1993), 5-11.

[9] L.A. Alnaser, et al. New fixed point theorems with applications to non-linear neutral differential equations, Symmetry 11 (2019), 602.

[10] M. Bachar, M.A. Khamsi, Delay differential equations: a partially ordered sets approach in vectorial metric spaces, Fixed Point Theory Appl. 2014 (2014), 193.

[11] J.K. Hale, Retarded functional differential equations: basic theory, In: Theory of Functional Differential Equations, pp. 36-56. Springer, New York, NY, 1977.

[12] N. Hussain, P. Salimi Salimi, Suzuki-wardowski type fixed point theorems for α-gf-contractions, Taiwan. J. Math. 18 (2014), 1879-1895.

[13] B. Samet, C. Vetro, P. Vetro, Fixed point theorems for α–ψ-contractive type mappings, Nonlinear Anal.: Theory Meth. Appl. 75 (2012), 2154-2165.

[14] Djoudi, Ahcene, and Rabah Khemis."Fixed point techniques and stability for neutral nonlinear differential equations with unbounded delays, Georgian Math. J. 13 (2006), 25-34.

[15] V.I. Istraţescu, Some fixed point theorems for convex contraction mappings and mappings with convex diminishing diameters.—I. Ann. Mat. Pura Appl. 130 (1982), 89-104.

[16] D. Wardowski, Fixed points of a new type of contractive mappings in complete metric spaces, Fixed Point Theory Appl. 2012 (2012), 94.

[17] D.W. Boyd, J.S.W. Wong, On nonlinear contractions, Proc. Amer. Math. Soc. 20 (1969), 458-464.

[18] E. Ameer, H. Aydi, M. Arshad, H. Alsamir, M.S. Noorani, Hybrid multivalued type contraction mappings in αK-complete partial b-metric spaces and applications, Symmetry 11 (2019), 86.