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REPRESENTATION OF RHOTRIX TYPE A SEMIGROUPS[†]

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Abstract. This paper considers representation of rhotrix type A semigroups in terms of right ω-cosets of its closed rhotrix type A subsemigroup, which is a more general form of representation of rhotrix type A semigroup than the one given recently by Ndubuisi et al in [12].

Keywords: rhotrix type A subsemigroups; transitive representations; ω-cosets; partial ordering.

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1. INTRODUCTION AND PRELIMINARIES

In [1], the idea of rhotrix was introduced as an object whose elements are arranged in a rhomboidal nature which of course was an extension of matrix-tertions and matrix noitrets given by Atanassov and Shannon [9]. Suppose R and Q are two rotrices such that

$$R = \left\langle \begin{matrix} & a & \\ b & h(R) & d \\ & e & \end{matrix} \right\rangle, Q = \left\langle \begin{matrix} & f & \\ g & h(Q) & j \\ & k & \end{matrix} \right\rangle$$

where $h(R)$ and $h(Q)$ are the hearts of these rhotrices.

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It follows from [1] that

$$R+Q=\left\langle \begin{matrix} & a & \\ b & h(R) & d \\ & e & \end{matrix} \right\rangle + \left\langle \begin{matrix} & f & \\ g & h(Q) & j \\ & k & \end{matrix} \right\rangle = \left\langle \begin{matrix} & a+f & \\ b+g & h(R)+h(Q) & d+j \\ & e+k & \end{matrix} \right\rangle$$

and $$R\circ Q=\left\langle \begin{matrix} & ah(Q)+fh(R) & \\ bh(Q)+gh(R) & h(R)h(Q) & dh(Q)+jh(R) \\ & eh(Q)+kh(R) & \end{matrix} \right\rangle$$

An alternative multiplication method was given by Sani [5] as follows;

$$R\circ Q=\left\langle \begin{matrix} & af+dg & \\ bf+eg & h(R)h(Q) & aj+dk \\ & bj+ek & \end{matrix} \right\rangle.$$

Sani [6] also gave a generalization of this row-column multiplication of heart-oriented rhotrices as:

$$R_n\circ Q_n=\langle a_{i_1j_1},c_{l_ik_1}\rangle\circ\langle b_{i_2j_2},d_{l_2k_2}\rangle=\langle\sum_{i_2j_{1=1}}^{t}\left(a_{i_1j_1}b_{i_2j_2}\right),\sum_{l_2k_1=1}^{t-1}\left(c_{l_ik_1}d_{l_2k_2}\right)\rangle\ ,t=\frac{n+1}{2},$$

where R_n and Q_n denote n-dimensional rhotrices (with n rows and n columns).

Mohammed [2] and Isere [4] gave a new technique for expressing rhotrices in a general form. Another method of rhotrix representation was given by chinedu in [11]. Also in [3], some construction of rhotrix semigroup was given. The type A version of the rhotrix semigroup as well as its congruences was presented in [12]. Since it is known in [12] that there is a faithful representation of rhotrix type A semigroup with the matrix semigroup and from Howie [8] and Petrich [10] that there is a form of representation of an inverse semigroup in terms of the semigroup of one-to-one partial transformation of closed right ω-cosets of its inverse subsemigroup. It is natural to ask whether analogous results of [8] and [10] hold for rhotrix type A semigroups.

In particular, it is shown that for a rhotrix type A semigroup $S=(R_n(F),\ \circ)$ with a closed rhotrix type A subsemigroup K, there is a transitive representation θ^K from $S=(R_n(F),\ \circ)$ to the semigroup of one-to-one partial transformations of the closed right ω-cosets of K.

In section 2, we present results in partial order and ω-cosets of a rhotrix type A semigroup. The aforementioned transitive representation θ^K is presented in section 3.

For the notations and terminologies not mentioned in this paper, the reader is referred to [8], [10], [12], [13] and [14].

Let us now recall some definitions and known results.

Let a, b be elements of a semigroup S, we define $a\,\mathcal{R}^* b$ if and only if for all $x, y \epsilon S^1$, $xa = ya \Leftrightarrow xb = yb$. Dually we define the relation $\mathcal{L}^*$. Let S be a semigroup and $a \epsilon S$. The elements $a^\dagger$(resp. a^*) will denote an idempotent element in $\mathcal{R}^*$(resp. $\mathcal{L}^*$)-class R_a^* (resp. L_a^*).

A semigroup S with a semilattice of idempotents $E(S)$ is said to be an adequate semigroup if each $\mathcal{R}^*$-class and $\mathcal{L}^*$-class contain an idempotent.

With $E(S)$ being a semilattice such an idempotent is unique. A left adequate semigroup is said to be a left type A if for all $e \epsilon E(S)$ and $a \epsilon S$, $ae = (ae)^\dagger a$ (see [7]) and dually for right type A semigroups. A semigroup S is said to be a type A semigroup if it is both left and right type A.

It is important to note that every type A semigroup is essentially a special subsemigroup of an inverse semigroup through an embedding, thus several results in type A semigroups are analogous to those of an inverse semigroup. In particular, for $X = S$, where S is a type A semigroup, we have the following result adopted from [14].

Lemma 1.1. A type A semigroup S has a faithful representation with $I^*(X)$, the type A semigroup of one-to-one partial transformation on the set X.

The result below is analogous to Lemma 1.1.

Lemma 1.2 [12]. A rhotrix type A semigroup $S = (R_n(F), \circ)$ has a faithful representation with the matrix semigroup.

Suppose X be a set and K be a rhotrix type A subsemigroup of $I^*(X)$, K is said to be transitive if for any $\langle a_{ij}, c_{lk}\rangle, \langle b_{ij}, d_{lk}\rangle \epsilon S = (R_n(F), \circ)$ there exist $\mu \epsilon K$ such that $\langle a_{ij}, c_{lk}\rangle\mu = \langle b_{ij}, d_{lk}\rangle$. A representation of a rhotrix type A semigroup $S = (R_n(F), \circ)$ by $I^*(X)$ is said to be transitive if $K = S\theta = (R_n(F), \circ)\theta$ is a transitive rhotrix type A subsemigroup of $I^*(X)$.

From now henceforth, S will denote a rhotrix type A semigroup while $E(S)$ denotes its semilattice of idempotents.

2. PARTIAL ORDERING IN S AND ω-COSETS OF THE RHOTRIX TYPE A SUBSEMIGROUP OF S

Let S be a rhotrix type A semigroup with semilattice $E(S)$ of idempotents, a natural partial ordering denoted by $\leq$ will be defined on S as follows:

For $\langle a_{ij}, c_{lk}\rangle, \langle b_{ij}, d_{lk}\rangle \epsilon\ S$, $\langle a_{ij}, c_{lk}\rangle \leq \langle b_{ij}, d_{lk}\rangle$ if $\langle a_{ij}, c_{lk}\rangle = \langle I_{ij}, C_{lk}\rangle\langle b_{ij}, d_{lk}\rangle$ for some $\langle I_{ij}, C_{lk}\rangle \epsilon\ E(S)$.

It is important to note that $\langle a_{ij}, c_{lk}\rangle \leq \langle b_{ij}, d_{lk}\rangle$ if and only if $\langle a_{ij}, c_{lk}\rangle = \langle a_{ij}, c_{lk}\rangle^{\dagger}\langle b_{ij}, d_{lk}\rangle = \langle b_{ij}, d_{lk}\rangle\langle a_{ij}, c_{lk}\rangle^{*}$.

We have the following Lemma.

Lemma 2.1. Let S be a rhotrix type A semigroup and $\langle a_{ij}, c_{lk}\rangle, \langle b_{ij}, d_{lk}\rangle \epsilon\ S$. Then the following conditions are equivalent

i) $\langle a_{ij}, c_{lk}\rangle \leq \langle b_{ij}, d_{lk}\rangle$

ii) $\langle a_{ij}, c_{lk}\rangle = \langle b_{ij}, d_{lk}\rangle^{\dagger}\langle a_{ij}, c_{lk}\rangle = \langle a_{ij}, c_{lk}\rangle\langle b_{ij}, d_{lk}\rangle^{*}$

iii) $\langle a_{ij}, c_{lk}\rangle = \langle a_{ij}, c_{lk}\rangle^{\dagger}\langle b_{ij}, d_{lk}\rangle\langle a_{ij}, c_{lk}\rangle^{*}$

Proof. i) $\Rightarrow$ ii) Let $\langle a_{ij}, c_{lk}\rangle \leq \langle b_{ij}, d_{lk}\rangle$, then we have $\langle a_{ij}, c_{lk}\rangle = \langle a_{ij}, c_{lk}\rangle^{\dagger}\langle b_{ij}, d_{lk}\rangle$ so we have

$$\begin{aligned}\langle b_{ij}, d_{lk}\rangle^{\dagger}\langle a_{ij}, c_{lk}\rangle &= \langle b_{ij}, d_{lk}\rangle^{\dagger}\langle a_{ij}, c_{lk}\rangle^{\dagger}\langle b_{ij}, d_{lk}\rangle \\ &= \langle a_{ij}, c_{lk}\rangle^{\dagger}\langle b_{ij}, d_{lk}\rangle^{\dagger}\langle b_{ij}, d_{lk}\rangle \\ &= \langle a_{ij}, c_{lk}\rangle^{\dagger}\langle b_{ij}, d_{lk}\rangle = \langle a_{ij}, c_{lk}\rangle.\end{aligned}$$

Similarly, $\langle a_{ij}, c_{lk}\rangle\langle b_{ij}, d_{lk}\rangle^{*} = \langle a_{ij}, c_{lk}\rangle$. Thus ii) is true

ii) $\Rightarrow$ iii) is obvious.

iii) $\Rightarrow$ i). Let $\langle a_{ij}, c_{lk}\rangle = \langle a_{ij}, c_{lk}\rangle^{\dagger}\langle b_{ij}, d_{lk}\rangle\langle a_{ij}, c_{lk}\rangle^{*}$. Then we have that

$$\langle a_{ij}, c_{lk}\rangle^{\dagger}\langle b_{ij}, d_{lk}\rangle\langle a_{ij}, c_{lk}\rangle^{*} = \langle b_{ij}, d_{lk}\rangle\left(\langle a_{ij}, c_{lk}\rangle^{\dagger}\langle b_{ij}, d_{lk}\rangle\right)^{*}\langle a_{ij}, c_{lk}\rangle^{*}$$

$$= \langle b_{ij}, d_{lk}\rangle\langle a_{ij}, c_{lk}\rangle^{*}$$

and $\langle a_{ij}, c_{lk}\rangle^{\dagger}\langle b_{ij}, d_{lk}\rangle\langle a_{ij}, c_{lk}\rangle^{*} = \langle a_{ij}, c_{lk}\rangle^{*}\left(\langle b_{ij}, d_{lk}\rangle\langle a_{ij}, c_{lk}\rangle^{*}\right)^{\dagger}\langle b_{ij}, d_{lk}\rangle$

$$= \langle a_{ij}, c_{lk}\rangle^{\dagger}\langle b_{ij}, d_{lk}\rangle \ .$$

This shows that $\langle a_{ij}, c_{lk}\rangle = \langle a_{ij}, c_{lk}\rangle^{\dagger}\langle b_{ij}, d_{lk}\rangle = \langle b_{ij}, d_{lk}\rangle\langle a_{ij}, c_{lk}\rangle$. Thus $\langle a_{ij}, c_{lk}\rangle \leq \langle b_{ij}, d_{lk}\rangle$ and the proof is complete.

Lemma 2.2. Suppose S is a rhotrix type A semigroup and $\langle a_{ij}, c_{lk}\rangle, \langle b_{ij}, d_{lk}\rangle \in S$ with $\langle I_{ij}, c_{lk}\rangle, \langle a_{ij}, I_{lk}\rangle \in E(S)$ left and right units respectively of $\langle a_{ij}, c_{lk}\rangle\langle b_{ij}, d_{lk}\rangle$ and $\langle a_{ij}, c_{lk}\rangle$. Then $\langle I_{ij}, c_{lk}\rangle \leq \langle a_{ij}, I_{lk}\rangle$.

Proof. The proof is a routine check.

It is well known that idempotents commute in a rhotrix type A semigroup. So in S, we have that

$$\langle a_{ij}, I_{lk}\rangle\langle I_{ij}, c_{lk}\rangle = \langle I_{ij}, c_{lk}\rangle\langle a_{ij}, I_{lk}\rangle$$

$= \langle a_{ij}, c_{lk}\rangle$ (where $a_{ij} \in E(M_t(F))$ and $c_{lk} \in E(M_{t-1}(F))$), see [12].

At this point, it is worth doing to define a more general form of partial ordering ω instead of $\leq$.

Let S be a rhotrix type A semigroup and $\langle x_{ij}, y_{lk}\rangle \in S$.

Define a relation $\omega \in S$ as follows;

$$\langle x_{ij}, y_{lk}\rangle\, \omega = \left\{\langle a_{ij}, c_{lk}\rangle \in S : \left(\langle x_{ij}, y_{lk}\rangle, \langle a_{ij}, c_{lk}\rangle\right) \in \omega\right\} .$$

Now suppose that $\langle x_{ij}, y_{lk}\rangle, \langle a_{ij}, c_{lk}\rangle \in S$ such that $\left(\langle x_{ij}, y_{lk}\rangle, \langle a_{ij}, c_{lk}\rangle\right) \in \omega$ then obviously we have that $\left(\langle x_{ij}, y_{lk}\rangle^{\dagger}, \langle a_{ij}, c_{lk}\rangle^{\dagger}\right) \in \omega$ and $\left(\langle x_{ij}, y_{lk}\rangle^{*}, \langle a_{ij}, c_{lk}\rangle^{*}\right)$ and for all $\langle u_{ij}, v_{lk}\rangle, \langle p_{ij}, q_{lk}\rangle \in S$, $\left(\langle x_{ij}, y_{lk}\rangle\langle u_{ij}, v_{lk}\rangle,\ \langle a_{ij}, c_{lk}\rangle\langle u_{ij}, v_{lk}\rangle\right) \in \omega$ and $\left(\langle p_{ij}, q_{lk}\rangle\langle u_{ij}, v_{lk}\rangle, \langle p_{ij}, q_{lk}\rangle\langle a_{ij}, c_{lk}\rangle\right) \in \omega$.

Suppose $K = \langle m_{ij}, n_{lk}\rangle$ is a subset of S, then the closure of K in S is given by

$$\langle m_{ij}, n_{lk}\rangle\, \omega = \left\{\langle a_{ij}, c_{lk}\rangle \in S : \left(\langle m_{ij}, n_{lk}\rangle\langle a_{ij}, c_{lk}\rangle\right) \in \omega\right\}.$$

The following Lemma is evident

Lemma 2.3. Suppose $K = \langle m_{ij}, n_{lk}\rangle$ and $T = \langle s_{ij}, t_{lk}\rangle$ are subsets of S. Then we have the following

i) $\langle m_{ij}, n_{lk}\rangle \subseteq \langle m_{ij}, n_{lk}\rangle\, \omega$

ii) $\langle m_{ij}, n_{lk}\rangle\, \omega \subseteq \langle s_{ij}, t_{lk}\rangle\, \omega$ if $\langle m_{ij}, n_{lk}\rangle \subseteq \langle s_{ij}, t_{lk}\rangle$

iii) $\big(\langle m_{ij}, n_{lk}\rangle\, \omega\big)\langle x_{ij}, y_{lk}\rangle \subseteq \big(\langle m_{ij}, n_{lk}\rangle\langle x_{ij}, y_{lk}\rangle\big)\, \omega$ for $\langle x_{ij}, y_{lk}\rangle \in S$.

Proof. i) The proof is obvious

ii) The proof is straight forward

iii) Let $\langle b_{ij}, d_{lk}\rangle \in \big(\langle m_{ij}, n_{lk}\rangle\, \omega\big)\langle a_{ij}, c_{lk}\rangle$ so that $\langle b_{ij}, d_{lk}\rangle = \langle x_{ij}, y_{lk}\rangle\langle a_{ij}, c_{lk}\rangle$ where we know that

$\langle x_{ij}, y_{lk}\rangle \in \langle m_{ij}, n_{lk}\rangle\, \omega$. But $\big(\langle m_{ij}, n_{lk}\rangle\langle a_{ij}, c_{lk}\rangle\big) \in \omega$, so

$\big(\langle m_{ij}, n_{lk}\rangle\langle a_{ij}, c_{lk}\rangle, \langle x_{ij}, y_{lk}\rangle\langle a_{ij}, c_{lk}\rangle\big) \in \omega = \langle b_{ij}, d_{lk}\rangle$. Now $\langle b_{ij}, d_{lk}\rangle \in \big(\langle m_{ij}, n_{lk}\rangle\langle a_{ij}, c_{lk}\rangle\big)\, \omega$.

Thus we have that

$\big(\langle m_{ij}, n_{lk}\rangle\, \omega\big)\langle a_{ij}, c_{lk}\rangle \subseteq \big(\langle m_{ij}, n_{lk}\rangle\langle a_{ij}, c_{lk}\rangle\big)\, \omega$.

With ii) above, we have that $\Big(\big(\langle m_{ij}, n_{lk}\rangle\, \omega\big)\langle a_{ij}, c_{lk}\rangle\Big)\, \omega \subseteq \big(\langle m_{ij}, n_{lk}\rangle\langle a_{ij}, c_{lk}\rangle\big)\, \omega^2$.

But we know that $\langle m_{ij}, n_{lk}\rangle \subseteq \langle m_{ij}, n_{lk}\rangle\, \omega$. It then follows that

$\big(\langle m_{ij}, n_{lk}\rangle\langle a_{ij}, c_{lk}\rangle\big)\, \omega \subseteq \Big(\big(\langle m_{ij}, n_{lk}\rangle\, \omega\big)\langle a_{ij}, n_{lk}\rangle\Big)\, \omega = \big(\langle m_{ij}, n_{lk}\rangle\langle a_{ij}, n\rangle\big)\, \omega$.

Now suppose $K = \langle m_{ij}, n_{lk}\rangle$ is a rhotrix type A subsemigroup of S, an element $\langle x_{ij}, y_{lk}\rangle \in S$ is in $\langle m_{ij}, n_{lk}\rangle\langle x_{ij}, y_{lk}\rangle$ if and only if $\langle x_{ij}, y_{lk}\rangle^{\dagger} \in K$.

Suppose $\langle x_{ij}, y_{lk}\rangle^{\dagger} \in K$, then we have that

$\langle x_{ij}, y_{lk}\rangle^{\dagger}\langle x_{ij}, y_{lk}\rangle = \langle x_{ij}, I_{lk}\rangle\langle x_{ij}, y_{lk}\rangle = \langle x_{ij}, y_{lk}\rangle \in \langle m_{ij}, n_{lk}\rangle\langle x_{ij}, y_{lk}\rangle$. The set $\langle m_{ij}, n_{lk}\rangle\langle x_{ij}, y_{lk}\rangle$ is a right coset of $K = \langle m_{ij}, n_{lk}\rangle$ if $\langle x_{ij}, y_{lk}\rangle \in \langle m_{ij}, n_{lk}\rangle\langle x_{ij}, y_{lk}\rangle$.

In the same manner, we call the set $\big(\langle m_{ij}, n_{lk}\rangle\langle x_{ij}, y_{lk}\rangle\big)\, \omega$ a right ω-coset of $K = \langle m_{ij}, n_{lk}\rangle$ where $\langle x_{ij}, y_{lk}\rangle^{\dagger} \in K$.

Remark 2.4. It is important to note that the right ω-coset of K is analogous to that of inverse semigroups namely; $\big(\langle m_{ij}, n_{lk}\rangle\langle a_{ij}, c_{lk}\rangle\big)\, \omega = \big(\langle m_{ij}, n_{lk}\rangle\langle b_{ij}, d_{lk}\rangle\big)\, \omega$ and so on. In fact some properties of the right ω-coset of K are analogous to that of inverse semigroups [10] and type A semigroups [7].

3. REPRESENTATION OF RHOTRIX TYPE A SEMIGROUP

Now let $\mathcal{X}^* = \{(\langle m_{ij}, n_{lk}\rangle\langle a_{ij}, c_{lk}\rangle)\ \omega :\ \langle a_{ij}, c_{lk}\rangle^\dagger \epsilon\ K, \langle a_{ij}, c_{lk}\rangle\ \epsilon\ S\}$ be the set of all right ω-coset in S. We know from Lemma 2.3 (iii) that for $\langle u_{ij}, v_{lk}\rangle\ \epsilon\ S$, we have that

$(\langle m_{ij}, n_{lk}\rangle\langle a_{ij}, c_{lk}\rangle\omega)\langle u_{ij}, v_{lk}\rangle \subseteq (\langle m_{ij}, n_{lk}\rangle\langle a_{ij}, c_{lk}\rangle\langle u_{ij}, v_{lk}\rangle)\omega.$

So we have that

$(\langle a_{ij}, c_{lk}\rangle\langle u_{ij}, v_{lk}\rangle)^\dagger = (\langle a_{ij}, c_{lk}\rangle\langle u_{ij}, v_{lk}\rangle^\dagger)^\dagger$ and $\langle a_{ij}, c_{lk}\rangle^\dagger =$

$(\langle a_{ij}, c_{lk}\rangle^\dagger)^\dagger\langle a_{ij}, c_{lk}\rangle\omega\ \langle a_{ij}, c_{lk}\rangle,$

$(\langle a_{ij}, c_{lk}\rangle\langle u_{ij}, v_{lk}\rangle)^\dagger\omega\ \langle a_{ij}, c_{lk}\rangle^\dagger.$

Now suppose that $\langle a_{ij}, c_{lk}\rangle^\dagger\ \epsilon\ K$ then $(\langle a_{ij}, c_{lk}\rangle\langle u_{ij}, v_{lk}\rangle)^\dagger\ \epsilon\ K$.

Thus $(\langle m_{ij}, n_{lk}\rangle\langle a_{ij}, c_{lk}\rangle\langle u_{ij}, v_{lk}\rangle)\omega\ \epsilon\ \mathcal{X}^*$ for $(\langle m_{ij}, n_{lk}\rangle\langle a_{ij}, c_{lk}\rangle)\omega\ \epsilon\ \mathcal{X}^*$.

Let $I^*(\mathcal{X}^*)$ be the symmetric rhotrix type A semigroup associated with $\mathcal{X}^*$. It is obvious from [8] and [12] that $I^*(\mathcal{X}^*)$ is embeddable in a rhotrix inverse semigroup which implies that it is a subsemigroup of the rhotrix inverse semigroup.

Let $\langle u_{ij}, v_{lk}\rangle\ \epsilon\ S$ and define a mapping $\theta^K : S \to I^*(\mathcal{X}^*)$ by the rule that

$$\langle u_{ij}, v_{lk}\rangle\ \theta^K : (\langle m_{ij}, n_{lk}\rangle\langle x_{ij}, y_{lk}\rangle)\ \omega \to (\langle m_{ij}, n_{lk}\rangle\langle x_{ij}, y_{lk}\rangle\langle u_{ij}, v_{lk}\rangle)\ \omega$$

where the domain of the map is given by

$$dom\ \langle u_{ij}, v_{lk}\rangle\ \theta^K = \{(\langle m_{ij}, n_{lk}\rangle\langle x_{ij}, y_{lk}\rangle)\omega\ \epsilon\ \mathcal{X}^* :\ (\langle m_{ij}, n_{lk}\rangle\langle x_{ij}, y_{lk}\rangle\langle u_{ij}, v_{lk}\rangle)\ \omega\ \epsilon\ \mathcal{X}^*\}.$$

The Lemma below easily follows

Lemma 3.1. For each $\langle u_{ij}, v_{lk}\rangle\ \epsilon\ S,\ \langle u_{ij}, v_{lk}\rangle\ \theta^K$ is a one-to-one mapping

Proof. The proof is obvious

Lemma 3.2. Let $K = \langle m_{ij}, n_{lk}\rangle$ is a closed rhotrix type A semigroup of S. Then $\theta^K : S \to I^*(\mathcal{X}^*)$ such that $\{((\langle m_{ij}, n_{lk}\rangle\langle x_{ij}, y_{lk}\rangle)\omega, (\langle m_{ij}, n_{lk}\rangle\langle x_{ij}, y_{lk}\rangle\langle u_{ij}, v_{lk}\rangle)\omega) :$

$((\langle m_{ij}, n_{lk}\rangle\langle x_{ij}, y_{lk}\rangle)\ \omega, (\langle m_{ij}, n_{lk}\rangle\langle x_{ij}, y_{lk}\rangle\langle u_{ij}, v_{lk}\rangle))\ \omega\ \epsilon\ I^*(\mathcal{X}^*)\}$ is a representation of S.

Proof. That θ^K is a one-to-one mapping of S into $I^*(\mathcal{X}^*)$ and well defined is obvious.

Now let $\langle u_{ij}, v_{lk}\rangle, \langle t_{ij}, h_{lk}\rangle \epsilon\, S$,

$\Big((\langle m_{ij}, n_{lk}\rangle\langle x_{ij}, y_{lk}\rangle)\omega, (\langle m_{ij}, n_{lk}\rangle\langle x_{ij}, y_{lk}\rangle\langle u_{ij}, v_{lk}\rangle\langle t_{ij}, h_{lk}\rangle)\omega\Big)$, then $\langle x_{ij}, y_{lk}\rangle^{\dagger} \epsilon\, K$ and

$\left(\langle x_{ij}, y_{lk}\rangle\langle u_{ij}, v_{lk}\rangle\langle t_{ij}, h_{lk}\rangle\right)^{\dagger} \epsilon\, K$. It follows that

$\langle x_{ij}, y_{lk}\rangle\langle u_{ij}, v_{lk}\rangle\, \langle t_{ij}, h_{lk}\rangle^{\dagger}\omega\, \langle x_{ij}, y_{lk}\rangle\langle u_{ij}, v_{lk}\rangle$ since it is clear that $\langle x_{ij}, y_{lk}\rangle\langle u_{ij}, v_{lk}\rangle\, \langle t_{ij}, h_{lk}\rangle^{\dagger}$

= $\left(\langle x_{ij}, y_{lk}\rangle\langle u_{ij}, v_{lk}\rangle\, \langle t_{ij}, h_{lk}\rangle^{\dagger}\right)^{\dagger}\langle x_{ij}, y_{lk}\rangle\langle u_{ij}, v_{lk}\rangle$.

Using the property of ω, we have that

$\left(\langle x_{ij}, y_{lk}\rangle\langle u_{ij}, v_{lk}\rangle\, \langle t_{ij}, h_{lk}\rangle^{\dagger}\right)^{\dagger} = \left(\langle x_{ij}, y_{lk}\rangle\langle u_{ij}, v_{lk}\rangle\langle t_{ij}, h_{lk}\rangle\right)^{\dagger}\omega\, \left(\langle x_{ij}, y_{lk}\rangle\langle u_{ij}, v_{lk}\rangle\right)^{\dagger}$.

Hence $\left(\langle x_{ij}, y_{lk}\rangle\langle u_{ij}, v_{lk}\rangle\, \langle t_{ij}, h_{lk}\rangle^{\dagger}\right)^{\dagger} \epsilon\, K\omega$.

It is known that $K\omega = K$ so $\left(\langle x_{ij}, y_{lk}\rangle\langle u_{ij}, v_{lk}\rangle\right)^{\dagger}\epsilon\, K$, thus $(\langle m_{ij}, n_{lk}\rangle\langle x_{ij}, y_{lk}\rangle\langle u_{ij}, v_{lk}\rangle)\omega$ $\epsilon\, \mathcal{X}^{*}$.

Using the fact that $(\langle m_{ij}, n_{lk}\rangle\langle x_{ij}, y_{lk}\rangle)\omega\, \langle u_{ij}, v_{lk}\rangle\, \theta^{K} = (\langle m_{ij}, n_{lk}\rangle\langle x_{ij}, y_{lk}\rangle\langle u_{ij}, v_{lk}\rangle)\omega$, it now follows that $\Big((\langle m_{ij}, n_{lk}\rangle\langle x_{ij}, y_{lk}\rangle)\, \omega,\ (\langle m_{ij}, n_{lk}\rangle\langle x_{ij}, y_{lk}\rangle\langle u_{ij}, v_{lk}\rangle)\Big)\, \omega\, \epsilon\, \langle u_{ij}, v_{lk}\rangle\, \theta^{K}$.

Conversely, let $\Big((\langle m_{ij}, n_{lk}\rangle\langle x_{ij}, y_{lk}\rangle)\omega\,,\ (\langle m_{ij}, n_{lk}\rangle\langle d_{ij}, g_{lk}\rangle)\omega\Big)\, \epsilon\, \langle u_{ij}, v_{lk}\rangle\, \theta^{K}\, .\ \langle t_{ij}, h_{lk}\rangle\, \theta^{K}$.

So there exists $(\langle m_{ij}, n_{lk}\rangle\langle z_{ij}, p_{lk}\rangle)\omega\, \epsilon\, \mathcal{X}^{*}$ such that

$\Big((\langle m_{ij}, n_{lk}\rangle\langle x_{ij}, y_{lk}\rangle)\omega\, , (\langle m_{ij}, n_{lk}\rangle\langle z_{ij}, p_{lk}\rangle)\omega\,\Big)\ \epsilon\, \langle u_{ij}, v_{lk}\rangle\, \theta^{K}$ and

$\Big((\langle m_{ij}, n_{lk}\rangle\langle z_{ij}, p_{lk}\rangle)\omega\, , (\langle m_{ij}, n_{lk}\rangle\langle d_{ij}, g_{lk}\rangle)\omega\,\Big)\, \epsilon\, \langle t_{ij}, h_{lk}\rangle\, \theta^{K}$.

Since $\Big((\langle m_{ij}, n_{lk}\rangle\langle x_{ij}, y_{lk}\rangle)\omega\Big)\, \langle u_{ij}, v_{lk}\rangle\, \theta^{K} = (\langle m_{ij}, n_{lk}\rangle\langle x_{ij}, y_{lk}\rangle\langle u_{ij}, v_{lk}\rangle)\omega$

and $\Big((\langle m_{ij}, n_{lk}\rangle\langle z_{ij}, p_{lk}\rangle)\omega\Big)\, \langle t_{ij}, h_{lk}\rangle\, \theta^{K} = (\langle m_{ij}, n_{lk}\rangle\langle z_{ij}, p_{lk}\rangle\langle t_{ij}, h_{lk}\rangle)\omega$,

then we have that

$$(\langle m_{ij}, n_{lk}\rangle\langle z_{ij}, p_{lk}\rangle)\omega = (\langle m_{ij}, n_{lk}\rangle\langle x_{ij}, y_{lk}\rangle\langle u_{ij}, v_{lk}\rangle)\omega$$

and $\quad (\langle m_{ij}, n_{lk}\rangle\langle z_{ij}, p_{lk}\rangle\langle t_{ij}, h_{lk}\rangle)\omega = (\langle m_{ij}, n_{lk}\rangle\langle x_{ij}, y_{lk}\rangle\langle u_{ij}, v_{lk}\rangle\langle t_{ij}, h_{lk}\rangle)\omega$.

Thus $\quad (\langle u_{ij}, v_{lk}\rangle\langle t_{ij}, h_{lk}\rangle)\theta^{K} = \langle u_{ij}, v_{lk}\rangle\, \theta^{K}\langle t_{ij}, h_{lk}\rangle\, \theta^{K}$.

So that θ^{K} is a homomorphism and the proof is complete.

We will now show that the transitive property is embedded in θ^K.

Lemma 3.3. θ^K is transitive

Proof. Let $(\langle m_{ij}, n_{lk}\rangle\langle x_{ij}, y_{lk}\rangle)\omega$, $(\langle m_{ij}, n_{lk}\rangle\langle d_{ij}, g_{lk}\rangle)\omega$ be right ω-cosets of K. Let $\langle u_{ij}, v_{lk}\rangle \epsilon S$ and $\langle x_{ij}, y_{lk}\rangle\langle u_{ij}, v_{lk}\rangle = \langle x_{ij}, y_{lk}\rangle^{\dagger}\langle d_{ij}, g_{lk}\rangle$, then $(\langle x_{ij}, y_{lk}\rangle\langle u_{ij}, v_{lk}\rangle)^{\dagger} = (\langle x_{ij}, y_{lk}\rangle \langle u_{ij}, v_{lk}\rangle^{\dagger})^{\dagger}$ and $(\langle x_{ij}, y_{lk}\rangle^{\dagger}\langle d_{ij}, g_{lk}\rangle)^{\dagger} = (\langle x_{ij}, y_{lk}\rangle \langle u_{ij}, v_{lk}\rangle^{\dagger})^{\dagger}$.

But it is known that $(\langle x_{ij}, y_{lk}\rangle^{\dagger}\langle d_{ij}, g_{lk}\rangle)^{\dagger} = \langle x_{ij}, y_{lk}\rangle^{\dagger}\langle d_{ij}, g_{lk}\rangle^{\dagger}$. Since $\langle x_{ij}, y_{lk}\rangle^{\dagger}$, $\langle d_{ij}, g_{lk}\rangle^{\dagger} \epsilon K$, then $(\langle x_{ij}, y_{lk}\rangle^{\dagger}\langle d_{ij}, g_{lk}\rangle)^{\dagger} \epsilon K$.

Thus $(\langle x_{ij}, y_{lk}\rangle^{\dagger}\langle u_{ij}, v_{lk}\rangle)^{\dagger} \epsilon K$ and $(\langle m_{ij}, n_{lk}\rangle\langle x_{ij}, y_{lk}\rangle\langle u_{ij}, v_{lk}\rangle)\omega \ \epsilon \mathcal{X}^*$.

We have that

$$(\langle m_{ij}, n_{lk}\rangle\langle x_{ij}, y_{lk}\rangle\langle u_{ij}, v_{lk}\rangle)\omega = (\langle m_{ij}, n_{lk}\rangle\langle x_{ij}, y_{lk}\rangle^{\dagger}\langle d_{ij}, g_{lk}\rangle)\omega \leq (\langle m_{ij}, n_{lk}\rangle\langle d_{ij}, g_{lk}\rangle)\omega,$$

and $\langle x_{ij}, y_{lk}\rangle^{\dagger}\langle d_{ij}, g_{lk}\rangle = \langle d_{ij}, g_{lk}\rangle(\langle x_{ij}, y_{lk}\rangle^{\dagger}\langle d_{ij}, g_{lk}\rangle)^*$

$$= \langle d_{ij}, g_{lk}\rangle(\langle x_{ij}, y_{lk}\rangle\langle u_{ij}, v_{lk}\rangle)^* \epsilon (\langle m_{ij}, n_{lk}\rangle\langle x_{ij}, y_{lk}\rangle\langle u_{ij}, v_{lk}\rangle)\omega .$$

More so, we have that

$\langle x_{ij}, y_{lk}\rangle^{\dagger}\langle d_{ij}, g_{lk}\rangle \ \omega \ \langle d_{ij}, g_{lk}\rangle$ and so $\langle d_{ij}, g_{lk}\rangle \epsilon (\langle m_{ij}, n_{lk}\rangle\langle x_{ij}, y_{lk}\rangle\langle u_{ij}, v_{lk}\rangle)\omega$.

Consequently, we have that

$$(\langle m_{ij}, n_{lk}\rangle\langle d_{ij}, g_{lk}\rangle)\omega \leq (\langle m_{ij}, n_{lk}\rangle\langle x_{ij}, y_{lk}\rangle\langle u_{ij}, v_{lk}\rangle)\omega$$

and $(\langle m_{ij}, n_{lk}\rangle\langle x_{ij}, y_{lk}\rangle\langle u_{ij}, v_{lk}\rangle)\omega = (\langle m_{ij}, n_{lk}\rangle\langle d_{ij}, g_{lk}\rangle)\omega$.

Thus, $((\langle m_{ij}, n_{lk}\rangle\langle x_{ij}, y_{lk}\rangle)\omega, (\langle m_{ij}, n_{lk}\rangle\langle d_{ij}, g_{lk}\rangle)\omega) \ \epsilon \langle u_{ij}, v_{lk}\rangle \ \theta^K$.

Remark 3.4. It is important to note that suppose α is an element in the rhotrix type A semigroup $I^*(\mathcal{X}^*)$, then α may not have an inverse.

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CONFLICT OF INTEREST

The authors declare that there is no conflict of interests.

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