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A GENERALIZATION OF GP-METRIC SPACE AND GENERALIZED G_b-METRIC SPACE AND RELATED FIXED POINT RESULTS

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Abstract. In the present article, a generalization of GP-metric space and generalized G_b-metric space has been introduced. In newly defined space, we study some properties and introduce some interesting and new concepts. Also, we present some fixed point results for various contraction mappings. Some consequences of these results are deduced in generalized G_b-metric spaces. We furnish multiple examples in support of new concepts, main results, and consequences.

Keywords: fixed point; k-contraction; G-metric space; G_b-metric space; GP-metric space.

2010 AMS Subject Classification: 47H10, 54H25.

1. INTRODUCTION

As an extension of usual distance, the notion of metric space was introduced by Frechet [1] in 1906. As a generalization of metric space, the concept of partial metric space was initiated in 1992 by Matthews [2], where self distance may be positive.

Definition 1.1. [2] Let U be a non-empty set. Then a mapping $d : U \times U \to [0,+\infty)$ is called a partial metric if for all $u,v,w \in U$,

$(p1)$ $u = v \Leftrightarrow d(u,u) = d(u,v) = d(v,v)$;

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$(p2)$ $d(u,u) \leq d(u,v)$;

$(p3)$ $d(u,v) = d(v,u)$;

$(p4)$ $d(u,w) \leq d(u,v) + d(v,w) - d(v,v)$.

Then, the pair (U,d) is called a partial metric space.

By weakening the triangle inequality in metric space, Bakhtin [3] and Czerwik [4] have introduced the idea of b-metric space with $s = 2$. In 1998, Czerwik [5] presented the notion of b-metric space in the following form:

Definition 1.2. [5] Let U be a non-empty set. Then a mapping $d : U \times U \to [0, +\infty)$ is called a b-metric if there exists a number $s \geq 1$ such that for all $u, v, w \in U$,

$(b1)$ $d(u,v) = 0$ if and only if $u = v$;

$(b2)$ $d(u,v) = d(v,u)$;

$(b3)$ $d(u,w) \leq s(d(u,v) + d(v,w))$.

Then pair (U,d) is called a b-metric space. Clearly, the family of b-metric spaces is larger than the family of metric spaces. But, for $s = 1$, b-metric space is a metric space.

In 2012, Harandi [6] extended the notion of partial metric by introducing metric-like-space as follows.

Definition 1.3. [6] Let U be a non-empty set. Then a mapping $d : U \times U \to [0, +\infty)$ is called a metric-like space if for all $u, v, w \in U$,

(m_l1) $d(u,v) = 0 \Rightarrow u = v$;

(m_l2) $d(u,v) = d(v,u)$;

(m_l3) $d(u,w) \leq d(u,v) + d(v,w)$.

Then, the pair (U,d) is called a metric-like space. It is noticed that in metric-like space, distance between two points may be less than self distance of one of the point.

Alghamdi *et al.* [8], in 2013 presented the concept b-metric-like spaces that generalized the notions of metric-like space and partial b-metric space. The notion of partial b-metric, as an extension of partial metric and b-metric, was given by Shukla [7] in 2014.

Definition 1.4. [7] Let U be a non-empty set. Then a mapping $d: U \times U \to [0,+\infty)$ is called a partial b-metric if there exists a number $s \geq 1$ such that for all $u,v,w \in U$,

(p_b1) $u = v \Leftrightarrow d(u,u) = d(u,v) = d(v,v)$;

(p_b2) $d(u,u) \leq d(u,v)$;

(p_b3) $d(u,v) = d(v,u)$;

(p_b4) $d(u,w) \leq s(d(u,v)+d(v,w)) - d(v,v)$.

Then, the pair (U,d) is called a partial b-metric space.

Definition 1.5. [8] Let U be a non-empty set. Then a mapping $d: U \times U \to [0,+\infty)$ is called a b-metric-like if there exists a number $s \geq 1$ such that for all $u,v,w \in U$,

$(b_{ml}1)$ $d(u,v) = 0 \Rightarrow u = v$;

$(b_{ml}2)$ $d(u,v) = d(v,u)$;

$(b_{ml}3)$ $d(u,w) \leq s(d(u,v)+d(v,w))$.

Then, the pair (U,d) is called a b-metric-like space.

Definition 1.6. [8] Let (U,d) be a b-metric-like space. Then

(i) a sequence $\{u_n\}$ *in* U converges to $u \in U$ if $\lim\limits_{n\to+\infty} d(u_n,u) = d(u,u)$.

(ii) A sequence $\{u_n\}$ in U is called Cauchy sequence if $\lim\limits_{n,m\to+\infty} d(u_n,u_m)$ exists and is finite.

(iii) (U,d) is said to be complete if every Cauchy sequence $\{u_n\}$ in U converges to $u \in U$ so that

$$\lim_{n,m\to+\infty} d(u_n,u_m) = d(u,u) = \lim_{n\to+\infty} d(u_n,u).$$

For comparison, see the following diagram (Sen *et al.* [9]):

$$\begin{array}{ccccc}
\text{Metric space} & \Rightarrow & \text{Partial Metric space} & \Rightarrow & \text{Metric-like space} \\
\Downarrow & & \Downarrow & & \Downarrow \\
b\text{-Metric space} & \Rightarrow & \text{Partial } b\text{-Metric space} & \Rightarrow & b\text{-Metric-like space}
\end{array}$$

After these extensions, in 2015, Jleli and Samet [10] introduced another interesting extension of metric space as shown below.

Definition 1.7. [10] Let U be a nonempty set and $D : U \times U \to [0,+\infty]$ be a given mapping. For every $u \in U$, define the set

$$\mathscr{C}(D,U,u) = \left\{ \{u_n\} \subset U \,|\, \lim_{n\to+\infty} D(u_n,u) = 0 \right\}.$$

Then D is a generalized metric on U if there exists $C > 0$ such that the following conditions holds:

$(D1)$ $(u,v) \in U \times U,\ D(u,v) = 0$ implies $u = v$;

$(D2)$ $D(u,v) = D(v,u),$ for all $(u,v) \in U \times U$;

$(D3)$ if $\{u_n\} \in \mathscr{C}(D,U,u)$, then $D(u,v) \leq C \limsup_{n\to+\infty} D(u_n,v)$.

In this case, the pair (U,D) is a generalized metric space.

Example 1.8. [11] Let $U = \{0,1\}$ and $D : U \times U \to [0,+\infty]$ be a mapping defined by $D(0,0) = 0$ and $D(0,1) = D(1,0) = D(1,1) = +\infty$. Then (U,D) is a generalized metric space.

Definition 1.9. [10] Let (U,D) be a generalized metric space. Then

(i) a sequence $\{u_n\}$ *in* U converges to $u \in U$ if $\{u_n\} \in \mathscr{C}(D,U,u)$, *i.e.*, $\lim_{n\to+\infty} D(u_n,u) = 0$.

(ii) a sequence $\{u_n\}$ *in* U is called a Cauchy sequence if $\lim_{n,m\to+\infty} D(u_n,u_m) = 0$.

(iii) (U,D) is said to be complete if every Cauchy sequence in U is convergent to some element in U.

In 2019, Asim and Imdad [12] introduced partial JS-metric space as a generalization of partial metric space as follows.

Definition 1.10. [12] Let U be a nonempty set and $D_p : U \times U \to [0,+\infty]$ be a given mapping. For every $u \in U$, define the set

$$\mathscr{K}(D_p,U,u) = \left\{ \{u_n\} \subset U \,|\, \lim_{n\to+\infty} D_p(u_n,u) = D_p(u,u) \right\}.$$

Then D_p is a partial JS-metric on U if (for all $u,v \in U$) it satisfies the following conditions:

(D_p1) if $D_p(u,u) = D_p(v,v) = D_p(u,v),$ then $u = v$;

(D_p2) $D_p(u,u) \leq D_p(u,v)$;

(D_p3) $D_p(u,v) = D_p(v,u)$;

(D_p4) there exists $C > 0$ such that if $(u,v) \in U \times U,\ \{u_n\} \in \mathscr{K}(D,U,u)$, then $D_p(u,v) \leq C \limsup_{n\to+\infty} D_p(u_n,v) + (C-1)D_p(u,u)$.

The pair (U,D_p) is said to be partial JS-metric space.

Remark 1.11. We have noticed that partial JS-metric space is not a generalization of generalized metric space (i.e. JS-metric space). See the following example.

Example 1.12. Let $U=[0,1]$ and $D: U\times U\to[0,+\infty]$ be a mapping defined by

$$D(u,v)=\begin{cases} 4, & \text{if } u,v\in(0,1],\\ |u-v|, & \text{otherwise.}\end{cases}$$

Then $(D1)$ and $(D2)$ are obvious. For $(D3)$, let $(u,v)\in U\times U$ and $\{u_n\}\in\mathscr{C}(D,U,u)$, First we prove that $u=0$. Suppose if $u>0$, then $\{u_n\}\in\mathscr{C}(D,U,u)$ implies $\lim\limits_{n\to+\infty} D(u_n,u)=0$. Now as $u>0$, therefore, there exists $n_0\in\mathbb{N}$ such that $u_n=0$ for all $n\geq n_0$.
Hence, for every $n\geq n_0$, $D(u_n,u)=|u_n-u|=|u|$, and since $u>0$, therefore, we have that $\lim\limits_{n\to+\infty} D(u_n,u)\neq 0$, a contradiction. Thus, we have $u=0$. Next, we prove $(D3)$ by considering the following cases:
Case 1: If $v=0$, then $D(u,v)=0$ and therefore $(D3)$ holds in this case.
Case 2: If $v\neq 0$, then $D(u,v)=|0-v|=|v|$ and

$$D(u_n,v)=\begin{cases} |v|, & \text{if } u_n=0,\\ 4, & \text{if } u_n\neq 0.\end{cases}$$

Therefore, $D(u,v)\leq \limsup\limits_{n\to+\infty} D(u_n,v)$, that is, $(D3)$ holds in this case for $C=1$.
Hence, (U,D) is a generalized metric space. But for $u=\frac{1}{2}$ and $v=0$, $D(u,u)=4\nleq\frac{1}{2}=D(u,v)$. Thus (U,D) is a not a partial JS-metric space.

Remark 1.13. Also, we have noticed that generalized metric space is itself an extension of b-metric-like space (and hence an extension of metric space, b-metric space, partial metric space, partial b-metric space and metric-like space).

Remark 1.14. Although, generalized metric space is an extension of b-metric-like space, but the concepts of convergent sequence and Cauchy sequence in generalized metric space (see Definition 1.9) do not reduce to the concepts of convergent sequence and Cauchy sequence in b-metric-like space (see Definition 1.6).

On the other hand, in 2006, Mustafa and Sims [13] introduced the notion called G-metric space as an alternative of D-metric space which was introduced by Dhage [14] in 1992. Some of basic concepts introduced by Dhage in [14] were proved inappropriate by Mustafa and Sims [15], Naidu *et al.* [16, 17]. The definition of G-metric space is as follows.

Definition 1.15. [13] Let U be a non-empty set and $G : U \times U \times U \to [0, +\infty)$ be a function satisfying:

$(G1)$ $G(u,v,w) = 0$ if $u = v = w$;

$(G2)$ $0 < G(u,u,v)$ for all $u, v \in U,$ with $u \neq v$;

$(G3)$ $G(u,u,v) \leq G(u,v,w)$ for all $u, v, w \in U,$ with $w \neq v$;

$(G4)$ $G(u,v,w) = G(u,w,v) = G(v,w,u) = \ldots$(symmetric in all three variables);

$(G5)$ $G(u,v,w) \leq G(u,a,a) + G(a,v,w)$ for all $u, v, w, a \in U,$ (rectangle inequality).

Then the mapping G is called a generalized metric or a G-metric on U, and the pair (U, G) is a G-metric space.

More details on G-metric space and various fixed point results in G-metric space can be found in [18–24].

In 2011, Zand and Nezhad [25] have introduced GP-metric space as a generalization of partial metric space and G-metric space.

Definition 1.16. [25] Let U be a non-empty set. Let $G : U \times U \times U \to [0, +\infty)$ be a function such that the following conditions hold:

(G_p1) $u = v = w$ if $G(u,v,w) = G(u,u,u) = G(v,v,v) = G(w,w,w)$;

(G_p2) $G(u,u,u) \leq G(u,u,v) \leq G(u,v,w)$ for all $u, v, w \in U$;

(G_p3) $G(u,v,w) = G(u,w,v) = G(v,w,u) = \ldots$(symmetric in all three variables);

(G_p4) $G(u,v,w) \leq G(u,a,a) + G(a,v,w) - G(a,a,a)$ for all $u, v, w, a \in U$.

Then the function G is called a GP-metric on U, and the pair (U, G) is a GP-metric space.

Later on, in 2013, Parvaneh *et al.* [26] have noticed that GP-metric spaces are symmetric due to (G_p2). Thus, GP-metric spaces are not generalization of those G-metric spaces which are nonsymmetric (for example, see Example 1 in [13]). In view of this, Parvaneh *et al.* [26] have redefined GP-metric space by changing the inequality (G_p2) as:

$$(G_p2') \quad G(u,u,u) \leq G(u,u,v) \leq G(u,v,w) \text{ for all } u, v, w \in U \textit{ with } v \neq w.$$

More on GP-metric spaces can be studied in [27–35]. Result in Proposition 2.10(2) in [33] is proved for symmetric GP-metric space. We now prove that result in both symmetric and nonsymmetric GP-metric space.

Proposition 1.17. *Let (U,G) be a GP-metric space and $\{u_n\}$ be any sequence in U. Then the following statements are equivalent:*
(I) $\lim_{n,m\to+\infty} G(u_n,u_m,u_m) = r < +\infty$.
(II) $\lim_{n,m,l\to+\infty} G(u_n,u_m,u_l) = r < +\infty$.

Proof. (II) implies (I) obviously. Now we prove that (I) implies (II). For given $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$, such that

$$|G(u_n,u_m,u_m) - r| < \varepsilon \; for\ all\ n,m \geq n_0. \tag{1}$$

Define a set $A = \{(n,l) \mid n,l \in \mathbb{N}\ with\ u_n \neq u_l\}$ and for each $k \in \mathbb{N}$, define

$$A_k = \{(n,l) \mid n,l \in \mathbb{N}\ with\ n,l \geq k\}.$$

Now, if for every $k \in \mathbb{N}$, there exists infinitely many pairs $(n,l) \in A_k \cap A$, then considering all pairs $(n,l) \in A_1 \cap A$ and $m \in \mathbb{N}$, we have, using (G_p4) and (G_p2'),

$$G(u_n,u_m,u_l) \leq G(u_n,u_m,u_m) + G(u_m,u_m,u_l) - G(u_m,u_m,u_m)$$

which implies

$$0 \leq G(u_m,u_n,u_l) - G(u_m,u_m,u_l) \leq G(u_n,u_m,u_m) - G(u_m,u_m,u_m).$$

Taking limit $n,m,l \to +\infty$ with $(n,l) \in A_1 \cap A$, on both sides, we have

$$0 \leq \lim_{n,m,l\to+\infty,((n,l)\in A_1\cap A)} G(u_m,u_n,u_l) - r \leq r - r = 0.$$

Thus, there exists $n_0' \geq n_0$ such that

$$|G(u_m,u_n,u_l) - r| < \varepsilon \; for\ all\ m \geq n_0'\ and\ (n,l) \in A_{n_0'} \cap A.$$

Also, for $(n,l) \in A_{n_0'} - A$ and $m \geq n_0'$, using (1),

$$|G(u_m,u_n,u_l) - r| < \varepsilon.$$

Thus (II) holds. $\square$

Also, in the same year, Aghajani *et al.* [36] (2013) initiated the concept of G_b-metric space by combining the concepts of G-metric spaces and b-metric space as:

Definition 1.18. [36] Let U be a non-empty set and $s \geq 1$ be a real number. Let $G : U \times U \times U \to [0,+\infty)$ be a function such that:
(G_b1) $G(u,v,w) = 0$ if $u = v = w$;
(G_b2) $0 < G(u,u,v)$ for all $u,v \in U,$ with $u \neq v$;
(G_b3) $G(u,u,v) \leq G(u,v,w)$ for all $u,v,w \in U,$ with $w \neq v$;
(G_b4) $G(u,v,w) = G(u,w,v) = G(v,w,u) = \ldots$(symmetric in all three variables);
(G_b5) $G(u,v,w) \leq s[G(u,a,a)+G(a,v,w)]$ for all $u,v,w,a \in U$.
Then the function G is called a generalized b-metric or a G_b-metric on U, and the pair (U,G) is a generalized b-metric space or G_b-metric space. Every G-metric space is a G_b-metric space with $s = 1$, but the converse is not true in general. See, the following example.

Example 1.19. [36] Let $U = \mathbb{R}$ be the set of real numbers. Define $G : U \times U \times U \to [0,+\infty)$ as

$$G(u,v,w) = \frac{1}{9}(|u-v|+|v-w|+|w-u|)^2 \; for \; all \; u,v,w \in U.$$

Then G is a generalized b-metric on U.

After this, many researchers proved numerous interesting results in generalized-b metric spaces (see, for detail, [37–48]). In [36], authors introduced the concept of G_b-metric space. In [49], Jain and Kaur have also used the name G_b-metric space, but for another abstract space. We now rename it as 'generalized G_b-metric space' and is defined as:

Definition 1.20. [49] Let U be a non-empty set. Let $G : U \times U \times U \to [0,+\infty)$ be a function such that there exists a real $s \geq 1$ with the following conditions:
(gG_b1) $G(u,v,w) = 0$ if $u = v = w$;
(gG_b2) $0 < G(u,u,v)$ for all $u,v \in U,$ with $u \neq v$;
(gG_b3) $G(u,u,v) \leq s\, G(u,v,w)$ for all $u,v,w \in U,$ with $w \neq v$;
(gG_b4) $G(u,v,w) = G(u,w,v) = G(v,w,u) = \ldots$(symmetric in all three variables);
(gG_b5) $G(u,v,w) \leq s[G(u,a,a)+G(a,v,w)]$ for all $u,v,w,a \in U$.
Then the function G is called a generalized G_b-metric on U, and the pair (U,G) is a generalized

G_b-metric space. Clearly, every generalized b-metric space is a generalized G_b-metric space, but converse is not true, see the following example:

Example 1.21. Define a mapping $G : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to [0, +\infty)$ as :

$$G(u, v, w) = |u - v|^2 + |v - w|^2 + |w - u|^2, \ for\ all\ u, v, w \in \mathbb{R}.$$

Then $(\mathbb{R}, G)$ is a generalized G_b-metric space with $s = 2$, but $(\mathbb{R}, G)$ is not a generalized b-metric space. For this, Let $u = 1,\ v = 3,\ w = 2$, then $G(u, v, w) = |1-3|^2 + |3-2|^2 + |2-1|^2 = 6$ and $G(u, v, v) = 2|1-3|^2 = 8$. Thus, $G(u, v, v) \nleq G(u, v, w)$, *i.e.,* (G_b3) does not hold.

Now, inspired by the work of Jleli and Samet [10], we define a generalization of G-metric space as in the following section.

2. G^*-Metric Space

Definition 2.1. Let U be a set having at least one element and let $G : U \times U \times U \to [0, +\infty]$ be a mapping. We say that G is a G^*-metric on U if there exists $\alpha > 0$ such that for all $u, v, w \in U$, the following conditions holds:

$(Gg1)$ $G(u, v, w) = 0$ implies $u = v = w$;

$(Gg2)$ $G(u, v, w) = G(u, w, v) = G(v, w, u) = \ldots$(symmetric in its variables);

$(Gg3)$ if $\{u_n\} \in C_U(G, u)$, then $G(u, v, w) \leq \alpha \left(\limsup\limits_{n \to +\infty} G(u_n, v, w) + G(u, u, u) \right)$,

$$where\ C_U(G, u) = \left\{ \{u_n\} \subset U \mid \lim_{n,m \to +\infty} G(u_n, u_m, u) = G(u, u, u) < +\infty \right\}.$$

In this case, we call the pair (U, G) a G^*-metric space with constant α.

Remark 2.2. Every generalized G_b-metric space is a G^*-metric space. Consider a generalized G_b-metric space (U, G). Then $(Gg1)$ and $(Gg2)$ are obvious. For $(Gg3)$, let $(u, v, w) \in U \times U \times$

U and $\{u_n\} \in C_U(G,u)$, then by (gG_b5),

$$\begin{aligned} G(u,v,w) &\leq s(G(u,u_n,u_n)+G(u_n,v,w)) \\ &\leq s\left(\limsup_{n\to+\infty} G(u,u_n,u_n)+\limsup_{n\to+\infty} G(u_n,v,w)\right) \\ &\leq s\left(G(u,u,u)+\limsup_{n\to+\infty} G(u_n,v,w)\right) \\ &= s\left(\limsup_{n\to+\infty} G(u_n,v,w)+G(u,u,u)\right), \end{aligned}$$

that is, $(Gg3)$ holds for $\alpha = s$. Thus, (U,G) is a G^*-metric space.

Remark 2.3. Every GP-metric space is a G^*-metric space. Consider a GP-metric space (U,G). Then $(Gg2)$ is obvious and $(Gg3)$ is easy to check. For $(Gg1)$, let $(u,v,w) \in U \times U \times U$ such that $G(u,v,w) = 0$. Suppose that $v \neq w$, then by (G_p2'), we have

$$G(u,u,u) \leq G(u,u,v) \leq G(u,v,w) = 0.$$

Also, then by (G_p4), we have

$$G(v,v,w) \leq G(v,u,u)+G(u,v,w)-G(u,u,u) = 0+0-0 = 0.$$

So, again by (G_p2'), we have

$$G(v,v,v) \leq G(v,v,w) = 0.$$

Similarly, we can prove that $G(w,w,w) = 0$. Thus, $G(u,v,w) = G(u,u,u) = G(v,v,v) = G(w,w,w) = 0$, therefore, by (G_p1), we have $u = v = w$. Thus, (U,G) is a G^*-metric space.

Remark 2.4. We see the following implication diagram.

G-metric space $\Rightarrow$ GP-metric space $\Rightarrow$ G^*-metric space

$\Downarrow$

generalized b-metric space

$\Downarrow$

generalized G_b-metric space $\Rightarrow$ G^*-metric space

Now, some examples are presented here, which assures that G^*-metric space generalizes the generalized G_b-metric space and GP-metric space.

Example 2.5. Let $U = [0,1]$ and $G : U \times U \times U \to [0,+\infty]$ be a mapping defined by

$$G(u,v,w) = \begin{cases} +\infty, & \text{if atleast one of } u,v,w \text{ is } 1. \\ 2, & \text{if } u,v,w \in (\frac{1}{2},1), \\ |u-v|+|v-w|+|w-u|, & \text{otherwise.} \end{cases}$$

Then $(Gg1)$ and $(Gg2)$ are obvious. For $(Gg3)$, let $(u,v,w) \in U \times U \times U$ and $\{u_n\} \in C_U(G,u)$. Then

$$\lim_{n,m\to+\infty} G(u_n,u_m,u) = G(u,u,u) < +\infty. \tag{2}$$

Clearly $u \neq 1$. Consider the following cases:

Case I: If $v = 1$ or $w = 1$, then $(Gg3)$ holds obviously.

Case II: If $v \neq 1$ and $w \neq 1$, then consider further two subcases.

Subcase 1: If $u \in [0,\frac{1}{2}]$, then by (2), $\lim\limits_{n\to+\infty} G(u_n,u_n,u) = 0$, i.e., $\lim\limits_{n\to+\infty} |u_n - u| = 0$, therefore, $\{u_n\}$ is any sequence in U such that $u_n \to u$ in usual sense.

Also, $G(u,v,w) = |u-v|+|v-w|+|w-u|$ and

$$G(u_n,v,w) = \begin{cases} +\infty, & \text{if } u_n = 1. \\ 2, & \text{if } u_n \in (\frac{1}{2},1), \\ |u_n-v|+|v-w|+|w-u_n|, & \text{if } u_n \in [0,\frac{1}{2}]. \end{cases}$$

Thus $G(u,v,w) \leq \alpha \left(\limsup\limits_{n\to+\infty} G(u_n,v,w) + G(u,u,u) \right)$ for $\alpha = 1$.

Subcase 2: If $u \in (\frac{1}{2},1)$, then by (2), $\lim\limits_{n\to+\infty} G(u_n,u_n,u) = 2$, therefore, $\{u_n\}$ is any sequence in U such that $u_n \in (\frac{1}{2},1)$ for all $n \geq n_0$, for some natural n_0. Now

$$G(u,v,w) = \begin{cases} 2, & \text{if } v,w \in (\frac{1}{2},1), \\ |u-v|+|v-w|+|w-u|, & \text{otherwise.} \end{cases}$$

Thus $G(u,v,w) \leq \alpha \left(\limsup\limits_{n\to+\infty} G(u_n,v,w) + G(u,u,u) \right)$ for $\alpha = 1$.

Hence, (U,G) is a G^*-metric space. But (U,G) is not a generalized G_b-metric space as $G(0.9,0.9,0.9) = 2 \neq 0$. Also, (U,G) is not a GP-metric space as for $u = 0.6,\ v = 0.7,\ w = 0.8$, $G(u,v,w) = G(u,u,u) = G(v,v,v) = G(w,w,w) = 2$, but $u \neq v \neq w$, i.e., (G_p1) does not hold.

Example 2.6. Let $V = \left\{\frac{1}{n} \mid n \in \mathbb{N}\right\}$ and $U = V \cup \{0\}$. Let $G : U \times U \times U \to [0,+\infty]$ be a mapping defined such that G satisfies $(Gg2)$ and

$$G(u,v,w) = \begin{cases} u+v+w, & \text{if atleast one of } u,v,w \text{ is } 0; \ or \\ & \text{if } u = \frac{1}{n}, v = \frac{1}{n+m}, w = \frac{1}{n+l}, \ where \ n,m,l \geq 5; \\ 5, & \text{otherwise.} \end{cases}$$

Then $(Gg1)$ is obvious. For $(Gg3)$, let $(u,v,w) \in U \times U \times U$ and $\{u_n\} \in C_U(G,u)$. Then

$$\lim_{n,m\to+\infty} G(u_n,u_m,u) = G(u,u,u) < +\infty. \tag{3}$$

Case I: If u=0, then by (3), $\lim\limits_{n\to+\infty} G(u_n,u_n,u) = 0$, i.e., $\lim\limits_{n\to+\infty} u_n = 0$ (in usual sense). Thus,

$$\begin{aligned} G(u,v,w) &= v+w \\ &\leq \lim_{n\to+\infty}(u_n+v+w) \ or \ 5 \\ &= \limsup_{n\to+\infty} G(u_n,v,w) \\ &= \limsup_{n\to+\infty} G(u_n,v,w) + G(u,u,u). \end{aligned}$$

Case II: If $u = \frac{1}{k}$, for some $k \in \mathbb{N}$, then by (3), $\lim\limits_{n\to+\infty} G(u_n,u_n,u) = G(u,u,u) = 5$.
Thus, $u_n \in \left\{\frac{1}{k+j} \mid j = -4,-3,-2,-1,0,1,2,3,4\right\}$ for all $n \in \mathbb{N}$ with $\frac{1}{u_m} - \frac{1}{u_l} \leq 4$ for all $m,l \in \mathbb{N}$. Also, clearly $G(u,v,w) \leq \alpha\left(\limsup\limits_{n\to+\infty} G(u_n,v,w) + G(u,u,u)\right)$ for $\alpha = 1$.
Then, (U,G) is a G^*-metric space. But (U,G) is not a generalized G_b-metric space as $G(0.5,0.5,0.5) = 5 \neq 0$. Also, (U,G) is not a GP-metric space as for $u = \frac{1}{10}$ and $v = \frac{1}{5}$, $G(u,u,u) = 5 \nleq \frac{2}{5} = G(u,u,v)$, i.e., (G_p2') does not hold.

Remark 2.7. In Example 2.6, (U,G) is a nonsymmetric G^*-metric space, as $G(\frac{1}{5},\frac{1}{10},\frac{1}{10}) = \frac{2}{5}$ and $G(\frac{1}{5},\frac{1}{5},\frac{1}{10}) = 5$.

2.1. Some Basic Concepts.

Some basic concepts in the context of G^*-metric space have been presented in this section.

Definition 2.8. Let (U,G) be a G^*-metric space. Let $\{u_n\}$ be a sequence in U. If there exists $u \in U$ such that $\{u_n\} \in C_U(G,u)$, that is, $\lim\limits_{n,m\to+\infty} G(u_n,u_m,u) = G(u,u,u) < +\infty$, then we say that sequence $\{u_n\}$ is G^*-convergent and G^*-converges to u. Denote this by $\lim_{n\to+\infty} u_n = u$ or $u_n \to u$. Also, in this case, we say u is a limit of sequence $\{u_n\}$.

Proposition 2.9. *Let (U,G) be a G^*-metric space and $\{u_n\}$ be a sequence in U such that $\{u_n\} \in C_U(G,u)$ for some $u \in U$, then*
(I) $\lim_{n\to+\infty} G(u_n,u_n,u) = G(u,u,u)$;
(II) $\limsup_{n\to+\infty} G(u_n,u,u) \le 2\alpha G(u,u,u)$.

Proof. (I) is obvious. Now for (II), for each $n \in \mathbb{N}$ using $(Gg3)$, we have

$$G(u,u_n,u) \le \alpha \limsup_{m\to+\infty} G(u_m,u_n,u) + \alpha G(u,u,u).$$

Taking limit supremum $n \to +\infty$ on both sides, we have

$$\begin{aligned}
\limsup_{n\to+\infty} G(u,u_n,u) &\le \alpha \limsup_{n\to+\infty}\left(\limsup_{m\to+\infty} G(u_m,u_n,u)\right) + \alpha G(u,u,u) \\
&= \alpha \lim_{n,m\to+\infty} G(u_m,u_n,u) + \alpha G(u,u,u) \\
&= \alpha G(u,u,u) + \alpha G(u,u,u) \\
&= 2\alpha G(u,u,u).
\end{aligned}$$

Hence (II) holds. □

Proposition 2.10. *Let (U,G) be a G^*-metric space and $\{u_n\}$ be a sequence in U such that $u_n \to u$ implies $G(u,u,u) = 0$. Then sequence $\{u_n\}$ have a unique limit in U.*

Proof. Suppose $u_n \to v$. Since $v,u \in U$, therefore, by $(Gg3)$,

$$\begin{aligned}
G(v,u,u) &\le \alpha \limsup_{n\to+\infty} G(u_n,u,u) + \alpha G(v,v,v) \\
&\le 2\alpha^2 G(u,u,u) + \alpha G(v,v,v) \\
&= 0 + 0 = 0.
\end{aligned}$$

Thus $G(v,u,u) = 0$, i.e., $v = u$. □

Now, some new concepts in G^*-metric spaces are:

Definition 2.11. Let (U,G) be a G^*-metric space. Let $\{u_n\}$ be a sequence in U. Then we say that $\{u_n\}$ is:

(i) G^*-*Cauchy sequence if there exists a real r such that for each* $\varepsilon > 0$ *there exists* $n_0 \in \mathbb{N}$ *such that* $|G(u_n,u_m,u_l) - r| < \varepsilon$ *for all* $n,m,l > n_0$.

Denote this by $\lim_{n,m,l\to+\infty} G(u_n,u_m,u_l) = r$.

(ii) G^*-*Cauchy*-1 *sequence if there exists a real r such that for each* $\varepsilon > 0$ *there exists* $n_0 \in \mathbb{N}$ *such that* $|G(u_n,u_{n+m},u_{n+l}) - r| < \varepsilon$ *for all* $n,m,l > n_0$.

Denote this by $\lim_{n,m,l\to+\infty} G(u_n,u_{n+m},u_{n+l}) = r$.

(iii) G^*-*Cauchy*-2 *sequence if there exists a real r such that for each* $\varepsilon > 0$ *there exists* $n_0 \in \mathbb{N}$ *such that* $|G(u_n,u_m,u_{m+l}) - r| < \varepsilon$ *for all* $n,m,l > n_0$ *with* $m \geq n$.

Denote this by $\lim_{n,m(\geq n),l\to+\infty} G(u_n,u_m,u_{m+l}) = r$.

(iv) G^*-*Cauchy*-3 *sequence if there exists a real r such that for each* $\varepsilon > 0$ *there exists* $n_0 \in \mathbb{N}$ *such that* $|G(u_n,u_{n+m},u_{n+m+l}) - r| < \varepsilon$ *for all* $n,m,l > n_0$.

Denote this by $\lim_{n,m,l\to+\infty} G(u_n,u_{n+m},u_{n+m+l}) = r$.

From the above definition, the next proposition directly follows.

Proposition 2.12. *In a* G^**-metric space,*

(I) *Every* G^**-Cauchy sequence is a* G^**-Cauchy*-1 *sequence.*

(II) *Every* G^**-Cauchy sequence is a* G^**-Cauchy*-2 *sequence.*

(III) *Every* G^**-Cauchy*-1 *sequence is a* G^**-Cauchy*-3 *sequence.*

(IV) *Every* G^**-Cauchy*-2 *sequence is a* G^**-Cauchy*-3 *sequence.*

(V) *Every* G^**-Cauchy sequence is a* G^**-Cauchy*-3 *sequence.*

Also, it is noted that the reverse implication in all (I) to (V) of the above proposition doesn't hold good in general. Consider the following examples.

Example 2.13. Consider G^*-metric space as in Example in 2.6. Sequence $\{u_n\}$, $u_n = \frac{1}{n}$, $n \in \mathbb{N}$, is a G^*-*Cauchy*-1 *sequence*, but not a G^*-*Cauchy sequence*.

Example 2.14. Let $V = \left\{\frac{1}{n} \mid n \in \mathbb{N}\right\}$ and $U = V \cup \{0\}$. Let $G : U \times U \times U \to [0,+\infty]$ be a mapping defined such that G satisfies $(Gg2)$ and

$$G(u,v,w) = \begin{cases} u+v+w, & \text{if atleast one of } u,v,w \text{ is } 0; \textit{ or} \\ & \text{if } u = \frac{1}{n}, v = \frac{1}{m}, w = \frac{1}{m+l}, \textit{ where } n,m,l \geq 5 \textit{ with } m \geq n; \\ 5, & \text{otherwise.} \end{cases}$$

Then (U,G) will be a G^*-metric space. Here, the sequence $\{u_n\}$, $u_n = \frac{1}{n}$, $n \in \mathbb{N}$, is a G^*-*Cauchy*-2 *sequence*, but not a G^*-*Cauchy sequence*.

Example 2.15. Let $V = \left\{\frac{1}{n} \mid n \in \mathbb{N}\right\}$ and $U = V \cup \{0\}$. Let $G : U \times U \times U \to [0,+\infty]$ be a mapping defined such that G satisfies $(Gg2)$ and

$$G(u,v,w) = \begin{cases} u+v+w, & \text{if atleast one of } u,v,w \text{ is } 0; \textit{ or} \\ & \text{if } u = \frac{1}{n}, v = \frac{1}{n+m}, w = \frac{1}{n+m+l}, \textit{ where } n,m,l \geq 5; \\ 5, & \text{otherwise.} \end{cases}$$

Then (U,G) will be a G^*-metric space. Here, the sequence $\{u_n\}$, $u_n = \frac{1}{n}$, $n \in \mathbb{N}$, is a G^*-*Cauchy*-3 *sequence*, but not a G^*-*Cauchy*-1 sequence and not a G^*-*Cauchy*-2 *sequence*.

We also notice another interesting fact in G^*-metric space as in the following remark.

Remark 2.16. In a G^*-metric space, a G^*-convergent sequence need not be a G^*-*Cauchy sequence*. For example, in Example 2.6, sequence $\{u_n\}$, $u_n = \frac{1}{n}$, $n \in \mathbb{N}$, is a G^*-convergent sequence and G^*-converges to 0 as $\lim_{n,m \to +\infty} G\left(\frac{1}{n}, \frac{1}{m}, 0\right) = \lim_{n,m \to +\infty} \left(\frac{1}{n} + \frac{1}{m} + 0\right) = 0 = G(0,0,0)$, but $\{u_n\}$ is a not G^*-*Cauchy sequence* as

$$G(u_n,u_m,u_l) = \begin{cases} \frac{1}{n} + \frac{1}{m} + \frac{1}{l}, & if\ n \geq 5, m \geq n+5, l \geq n+5; \\ 5, & \text{otherwise;} \end{cases}$$

implies that $\lim_{n,m,l \to +\infty} G(u_n,u_m,u_l)$ does not exist.

Now, another important remark as follows.

Remark 2.17. From elementary concepts in generalized G_b-metric space, one can prove that concepts G^*-*Cauchy sequence*, G^*-*Cauchy*-1 *sequence*, G^*-*Cauchy*-2 *sequence* and G^*-*Cauchy*-3 *sequence* are equivalent in generalized G_b-metric space.

Now, we define some more new concepts as.

Definition 2.18. Let (U,G) be a G^*-metric space. Then we say that (U,G) is:

(i) G^*-*complete if every* G^*-*Cauchy sequence* $\{u_n\}$ *in* U *is* G^*-*convergent to some* $u \in U$ *and* $\lim_{n,m,l\to+\infty} G(u_n,u_m,u_l) = G(u,u,u) = \lim_{n,m\to+\infty} G(u_n,u_m,u)$.

(ii) G^*-*complete*-1 *if every* G^*-*Cauchy*-1 *sequence* $\{u_n\}$ *in* U *is* G^*-*convergent to some* $u \in U$ *and* $\lim_{n,m,l\to+\infty} G(u_n,u_{n+m},u_{n+l}) = G(u,u,u) = \lim_{n,m\to+\infty} G(u_n,u_m,u)$.

(iii) G^*-*complete*-2 *if every* G^*-*Cauchy*-2 *sequence* $\{u_n\}$ *in* U *is* G^*-*convergent to some* $u \in U$ *and* $\lim_{n,m(\geq n),l\to+\infty} G(u_n,u_m,u_{m+l}) = G(u,u,u) = \lim_{n,m\to+\infty} G(u_n,u_m,u)$.

(iv) G^*-*complete*-3 *if every* G^*-*Cauchy*-3 *sequence* $\{u_n\}$ *in* U *is* G^*-*convergent to some* $u \in U$ *and* $\lim_{n,m,l\to+\infty} G(u_n,u_{n+m},u_{n+m+l}) = G(u,u,u) = \lim_{n,m\to+\infty} G(u_n,u_m,u)$.

Definition 2.19. Let (U,G) be a G^*-metric space, $M: U \to U$ a given mapping and $k \in (0,1)$. Then M is a k-contraction in (U,G) if

$$G(Mu,Mv,Mw) \leq kG(u,v,w) \text{ for every } (u,v,w) \in U \times U \times U.$$

Definition 2.20. Let (U,G) be a G^*-metric space and $M: U \to U$ a given mapping, then for each $u \in U$, we define

$$\delta(G,M,u) = \sup\{G(M^p(u),M^q(u),M^r(u)) \ : \ p,q,r \in \mathbb{N}\}.$$

3. Fixed Point Results in G^*-Metric Space

In this section, some fixed point theorems in G^*-metric spaces have been proved. Now, the first result of this section is as follows.

Theorem 3.1. *Let* (U,G) *be a* G^*-*complete metric space and* $M: U \to U$ *a* k-*contraction mapping for some* $k \in (0,1)$. *Also, let there exists* $u_0 \in U$ *such that* $\delta(G,M,u_0) < +\infty$, *then* M *has a fixed point (say* θ*) and if* $M(\theta') = \theta'$ *with* $G(\theta,\theta,\theta') < +\infty$, *then* $\theta' = \theta$.

Proof. Let $n \in \mathbb{N}$. Since M is a k-contraction, therefore, for all $p,q,r \in \mathbb{N}$,

$$G(M^{n+p}(u_0),M^{n+q}(u_0),M^{n+r}(u_0)) \ \leq kG(M^{n-1+p}(u_0),M^{n-1+q}(u_0),M^{n-1+r}(u_0)),$$

taking supremum over all $p,q,r \in \mathbb{N}$ on both sides, we have

$$\delta(G,M,M^n(u_0)) \leq k\delta(G,M,M^{n-1}(u_0)).$$

and thus a simple induction gives that

$$\delta(G,M,M^n(u_0)) \leq k^n\delta(G,M,u_0).$$

Now for all $n,m,l \in \mathbb{N}$, with $n \leq m \leq l$, we have

$$G(M^{n+1}(u_0),M^{m+1}(u_0),M^{l+1}(u_0)) \leq \delta(G,M,M^n(u_0)) \leq k^n\delta(G,M,u_0).$$

Since $\delta(G,M,u_0) < +\infty$ and $k \in (0,1)$, therefore,

$$\lim_{n,m,l\to+\infty} G(M^{n+1}(u_0),M^{m+1}(u_0),M^{l+1}(u_0)) = 0.$$

Hence, $\{M^n(u_0)\}$ is a G^*-Cauchy sequence in (U,G), but (U,G) is G^*-complete metric space, therefore there exists $\theta \in U$ such that $M^n(u_0) \to \theta$ and

$$\lim_{n,m\to+\infty} G(M^n(u_0),M^m(u_0),\theta) = G(\theta,\theta,\theta) = \lim_{n,m,l\to+\infty} G(M^n(u_0),M^m(u_0),M^l(u_0)) = 0. \tag{4}$$

Now as M is a k-contraction, therefore,

$$G(M(\theta),M(\theta),M(\theta)) \leq kG(\theta,\theta,\theta) = 0. \tag{5}$$

Again as M is a k-contraction, therefore, for all $n,m,l \in \mathbb{N}$, we have

$$G(M^{n+1}(u_0),M^{m+1}(u_0),M(\theta)) \leq kG(M^n(u_0),M^m(u_0),\theta).$$

Taking limit $n,m \to +\infty$ and using (4) and (5), we get

$$\lim_{n,m\to+\infty} G(M^{n+1}(u_0),M^{m+1}(u_0),M(\theta)) = 0 = G(M(\theta),M(\theta),M(\theta)),$$

which implies that $M^n(u_0) \to M(\theta)$. Thus in view of Proposition 2.10, $M(\theta) = \theta$, that is, θ is a fixed point of M.

Now, suppose that $\theta' \in U$ such that $M(\theta') = \theta'$ and $G(\theta,\theta,\theta') < +\infty$, then

$$G(\theta,\theta,\theta') = G(M\theta,M\theta,M\theta') \leq kG(\theta,\theta,\theta'),$$

and since $G(\theta,\theta,\theta') < +\infty$ and $k \in (0,1)$, therefore, $G(\theta,\theta,\theta') = 0$ which implies that $\theta = \theta'$. $\square$

Now, we present the next fixed point result.

Theorem 3.2. *Let (U,G) be a G^*-complete metric space with constant α and $M : U \to U$ be a mapping such that*

$$(T1)\quad G(Mu,Mv,Mw) \leq \beta \max\{G(u,v,w),\ G(u,Mu,Mu),\ G(v,Mv,Mv),\ G(w,Mw,Mw),\ G(u,Mv,Mv),\ G(v,Mw,Mw),\ G(w,Mu,Mu)\}$$

$$\textit{for every } (u,v,w) \in U \times U \times U, \textit{ and for some } \beta \in [0,1) \textit{ with } \alpha\beta < 1;$$

$(T2)$ *and there exists $u_0 \in U$ such that $\delta(G,M,u_0) < +\infty$;*

then sequence $\{M^n(u_0)\}$ G^*-converges to some $\theta \in U$. Also, if

$$\limsup_{n\to+\infty} G(M^n(u_0),M(\theta),M(\theta)) < +\infty, \tag{6}$$

then θ is a fixed point of M. Further, if θ' be another fixed point of M with $G(\theta,\theta,\theta') < +\infty$, $G(\theta,\theta',\theta') < +\infty$, and $G(\theta',\theta',\theta') < +\infty$, then $\theta = \theta'$.

Proof. Let $n \in \mathbb{N}$. By $(T1)$, for all $p,q,r \in \mathbb{N}$,

$$\begin{aligned}
&G(M^{n+p}(u_0),M^{n+q}(u_0),M^{n+r}(u_0))\\
&\leq\ \beta \max\{G(M^{n-1+p}(u_0),M^{n-1+q}(u_0),M^{n-1+r}(u_0)),\ G(M^{n-1+p}(u_0),M^{n+p}(u_0),M^{n+p}(u_0)),\\
&\ G(M^{n-1+q}(u_0),M^{n+q}(u_0),M^{n+q}(u_0)),\ G(M^{n-1+r}(u_0),M^{n+r}(u_0),M^{n+r}(u_0))\\
&\ G(M^{n-1+p}(u_0),M^{n+q}(u_0),M^{n+q}(u_0)),\ G(M^{n-1+q}(u_0),M^{n+r}(u_0),M^{n+r}(u_0)),\\
&\ G(M^{n-1+r}(u_0),M^{n+p}(u_0),M^{n+p}(u_0))\},
\end{aligned}$$

which implies that

$$\delta(G,M,M^n(u_0)) \leq \beta\delta(G,M,M^{n-1}(u_0)).$$

Thus a simple induction gives that

$$\delta(G,M,M^n(u_0)) \leq \beta^n\delta(G,M,u_0).$$

Now for all $n,m,l \in \mathbb{N}$, with $n \leq m \leq l$, we have

$$G(M^{n+1}(u_0),M^{m+1}(u_0),M^{l+1}(u_0)) \leq \delta(G,M,M^n(u_0)) \leq \beta^n \delta(G,M,u_0).$$

Since $\delta(G,M,u_0) < +\infty$ and $\beta \in [0,1)$, therefore,

$$\lim_{n,m,l\to+\infty} G(M^{n+1}(u_0),M^{m+1}(u_0),M^{l+1}(u_0)) = 0.$$

Hence, $\{M^n(u_0)\}$ is a G^*-Cauchy sequence in (U,G), but (U,G) is a G^*-complete metric space, therefore there exists $\theta \in U$ such that $M^n(u_0) \to \theta$ and

$$\lim_{n,m\to+\infty} G(M^n(u_0),M^m(u_0),\theta) = G(\theta,\theta,\theta) = \lim_{n,m,l\to+\infty} G(M^n(u_0),M^m(u_0),M^l(u_0)) = 0. \tag{7}$$

Also, by Proposition 2.9,

$$\limsup_{n\to+\infty} G(M^n(u_0),\theta,\theta) \leq 2\alpha G(\theta,\theta,\theta) = 0. \tag{8}$$

Now, using $(T1)$, we have

$$\begin{aligned} G(M^n(u_0),M(\theta),M(\theta)) \ \leq \ \beta \max\{&G(M^{n-1}(u_0),\theta,\theta),\ G(M^{n-1}(u_0),M^n(u_0),M^n(u_0)),\\ &G(\theta,M(\theta),M(\theta)),\ G(M^{n-1}(u_0),M(\theta),M(\theta)),\ G(\theta,M^n(u_0),M^n(u_0))\}. \end{aligned}$$

Taking limit supremum over all n on both sides and using (8), (7) and (6), we have

$$\limsup_{n\to+\infty} G(M^n(u_0),M(\theta),M(\theta)) \ \leq \beta G(\theta,M(\theta),M(\theta)). \tag{9}$$

Now by $(Gg3)$, (9) and (7), we have

$$\begin{aligned} G(\theta,M(\theta),M(\theta)) \ &\leq \ \alpha \limsup_{n\to+\infty} G(M^n(u_0),M(\theta),M(\theta)) + \alpha G(\theta,\theta,\theta) \\ &\leq \ \alpha\beta G(\theta,M(\theta),M(\theta)). \end{aligned} \tag{10}$$

Since $\alpha\beta < 1$ and $G(\theta,M(\theta),M(\theta)) < +\infty$ (in view of (10) and (6)), therefore, $G(\theta,M(\theta),M(\theta)) = 0$, which gives $M(\theta) = \theta$.

Let θ' be another fixed point of M with $G(\theta,\theta,\theta') < +\infty$, $G(\theta,\theta',\theta') < +\infty$, and

$G(\theta',\theta',\theta') < +\infty$. Now,

$$\begin{aligned} G(\theta',\theta',\theta') &= G(M(\theta'),M(\theta'),M(\theta')) \\ &\leq \beta\max\{G(\theta',\theta',\theta'),\ G(\theta',M(\theta'),M(\theta')), G(\theta',M(\theta'),M(\theta'))\} \\ &= \beta G(\theta',\theta',\theta'), \end{aligned}$$

as $\beta\in[0,1)$ and $G(\theta',\theta',\theta') < \infty$, therefore, $G(\theta',\theta',\theta') = 0$. Now,

$$\begin{aligned} &G(\theta,\theta,\theta') \\ &= G(M(\theta),M(\theta),M(\theta')) \\ &\leq \beta\max\{G(\theta,\theta,\theta'),\ G(\theta,M(\theta),M(\theta)),\ G(\theta',M(\theta'),M(\theta')), \\ &\quad G(\theta,M(\theta'),M(\theta')),\ G(\theta',M(\theta),M(\theta))\} \\ &= \beta\max\{G(\theta,\theta,\theta'),\ G(\theta,\theta,\theta),\ G(\theta',\theta',\theta'),\ G(\theta,\theta',\theta'),\ G(\theta',\theta,\theta)\} \\ &= \beta\max\{G(\theta,\theta,\theta'),\ G(\theta,\theta',\theta')\}. \end{aligned}$$

Similarly, $G(\theta,\theta',\theta') \leq \beta\max\{G(\theta,\theta,\theta'),\ G(\theta,\theta',\theta')\}$, therefore,

$$\max\{G(\theta,\theta,\theta'),\ G(\theta,\theta',\theta')\} \leq \beta\max\{G(\theta,\theta,\theta'),\ G(\theta,\theta',\theta')\}.$$

Now, as $\beta\in[0,1)$, $G(\theta,\theta,\theta') < +\infty$, and $G(\theta,\theta',\theta') < +\infty$, therefore, $\max\{G(\theta,\theta,\theta'),\ G(\theta,\theta',\theta')\} = 0$, which gives that $\theta = \theta'$. □

Now, by omitting the conditions $\limsup_{n\to+\infty} G(M^n(u_0),M(\theta),M(\theta)) < +\infty$ and $G(\theta,\theta',\theta') < +\infty$, in the previous result and with a similar proof, we have the following result.

Theorem 3.3. *Let (U,G) be a complete G^*-metric space with constant α and $M: U\to U$ be a mapping such that*

$(T3)$ $G(Mu,Mv,Mw)$

$\leq \beta\max\{G(u,v,w),\ G(u,Mu,Mu),\ G(v,Mv,Mv),\ G(w,Mw,Mw)\}$

for every $(u,v,w)\in U\times U\times U$, and for some $\beta\in[0,1)$ with $\alpha\beta<1$;

$(T4)$ *and there exists $u_0\in U$ such that $\delta(G,M,u_0) < +\infty$;*

then sequence $\{M^n(u_0)\}$ G^*-converges to some $\theta \in U$. If $G(\theta, M(\theta), M(\theta)) < +\infty$, then θ is a fixed point of M. Further, if θ' be another fixed point of M with $G(\theta, \theta, \theta') < +\infty$ and $G(\theta', \theta', \theta') < +\infty$, then $\theta = \theta'$.

4. Consequences in Generalized G_b-Metric Space

4.1. Lemmas. Before we deduce the consequences (in generalized G_b-metric space) of results of previous section, we state and prove the following results in generalized G_b-metric space.

Lemma 4.1. *Let (U, G) be a generalized G_b-metric space and $M : U \to U$ be a k-contraction in (U, G), then for each $u \in U$, $\delta(G, M, u) < +\infty$.*

Proof. Define a sequence $\{u_n\}$ in U by $u_n = M^n(u)$ for each $n \in \mathbb{N}$. Now, as M is a k-contraction, therefore,

$$G(u_n, u_{n+1}, u_{n+1}) \leq kG(u_{n-1}, u_n, u_n).$$

Thus, using the inequality at the end of the proof of Lemma 2 in [49] , we have for every $q, r \in \mathbb{N}$, with $q \geq r$,

$$G(M^r(u), M^q(u), M^q(u)) \leq \frac{G(u, M(u), M(u))k^{r-1}\mu}{1-k} \leq \frac{G(u, M(u), M(u))\mu}{1-k}, \tag{11}$$

where $\mu = \sum_{n=1}^{\infty} s^{2n} k^{2^{n-1}} < +\infty$. Also, for every $p, q \in \mathbb{N}$, with $p \geq q$,

$$G(M^q(u), M^q(u), M^p(u)) \leq 2sG(M^q(u), M^p(u), M^p(u)) \leq \frac{2sG(u, M(u), M(u))\mu}{1-k}. \tag{12}$$

Now, using (11) and (12), for $p, q, r \in \mathbb{N}$, with $p \geq q \geq r$,

$$\begin{aligned} G(M^p(u), M^q(u), M^r(u)) &\leq sG(M^p(u), M^q(u), M^q(u)) + sG(M^q(u), M^q(u), M^r(u)) \\ &\leq \frac{2s^2 G(u, M(u), M(u))\mu}{1-k} + \frac{sG(u, M(u), M(u))\mu}{1-k}, \end{aligned}$$

that is,

$$G(M^p(u), M^q(u), M^r(u)) \leq \frac{(2s^2 + s)G(u, M(u), M(u))\mu}{1-k}. \tag{13}$$

Thus, we get

$$\begin{aligned}\delta(G,M,u) &= \sup\{G(M^p(u),M^q(u),M^r(u)) \;:\; p,q,r\in\mathbb{N}\}\\ &= \sup\{G(M^p(u),M^q(u),M^r(u)) \;:\; p\geq q\geq r\}\\ &\leq \frac{(2s^2+s)G(u,Mu,Mu)\mu}{1-k}<+\infty.\end{aligned}$$

□

Lemma 4.2. *Let (U,G) be a generalized G_b-metric space and $M:U\to U$ be a mapping such that there exists some $k\in[0,1)$ and for every $(u,v,w)\in U\times U\times U$,*

$$G(Mu,Mv,Mw)\leq k\max\{G(u,v,w),\ G(u,Mu,Mu),\ G(v,Mv,Mv),\ G(w,Mw,Mw)\}. \tag{14}$$

Then for each $u\in U$, $\delta(G,M,u)<+\infty$.

Proof. Define a sequence $\{u_n\}$ in U by $u_n=M^n(u)$. Then, for each $n\in\mathbb{N}$,

$$G(u_n,u_{n+1},u_{n+1})\leq k\max\{G(u_{n-1},u_n,u_n),G(u_n,u_{n+1},u_{n+1})\}=kG(u_{n-1},u_n,u_n).$$

Now, rest of the proof is similar as the proof in Lemma 4.1. □

Lemma 4.3. *Let (U,G) be a generalized G_b-metric space and $M:U\to U$ be a mapping such that there exists some $k\in[0,\frac{1}{2s})$ and for every $(u,v,w)\in U\times U\times U$,*

$$\begin{aligned}G(Mu,Mv,Mw) &\leq k\max\{G(u,v,w),\ G(u,Mu,Mu),\ G(v,Mv,Mv),\ G(w,Mw,Mw),\\ &\qquad G(u,Mv,Mv),\ G(v,Mw,Mw),\ G(w,Mu,Mu)\}.\end{aligned}$$

Then for each $u\in U$, $\delta(G,M,u)<+\infty$.

Proof. Define a sequence $\{u_n\}$ in U by $u_n=M^n(u)$. Then for each $n\in\mathbb{N}$,

$$\begin{aligned}G(u_n,u_{n+1},u_{n+1}) &\leq k\max\{G(u_{n-1},u_n,u_n),G(u_n,u_{n+1},u_{n+1}),G(u_{n-1},u_{n+1},u_{n+1})\}.\\ &= k\max\{G(u_{n-1},u_n,u_n),G(u_{n-1},u_{n+1},u_{n+1})\}.\end{aligned}$$

If $\max\{G(u_{n-1},u_n,u_n),G(u_{n-1},u_{n+1},u_{n+1})\}=G(u_{n-1},u_n,u_n)$, then

$$G(u_n,u_{n+1},u_{n+1})\leq kG(u_{n-1},u_n,u_n).$$

If $\max\{G(u_{n-1},u_n,u_n),G(u_{n-1},u_{n+1},u_{n+1})\} = G(u_{n-1},u_{n+1},u_{n+1})$, then

$$G(u_n,u_{n+1},u_{n+1}) \leq kG(u_{n-1},u_{n+1},u_{n+1}) \leq ks(G(u_{n-1},u_n,u_n)+G(u_n,u_{n+1},u_{n+1}))$$

which implies that

$$G(u_n,u_{n+1},u_{n+1}) \leq \frac{ks}{1-ks}G(u_{n-1},u_n,u_n).$$

Now, as $k \in [0,\frac{1}{2s})$, so rest of the proof is similar as proof in Lemma 4.1. □

4.2. Consequences.

Now, due to Lemma 4.1, the following result is a consequence of Theorem 3.1.

Corollary 4.4. *(Theorem 3, [49]) Let (U,G) be a complete generalized G_b-metric space with $s \geq 1$ and $M : U \to U$ a k-contraction mapping for some $k \in (0,1)$. Then M has a unique fixed point.*

Next, we provide an example for which the hypothesis of Theorem 3.1 holds, but not of Corollary 4.4.

Example 4.5. Consider a G^*-metric space as in Example 2.5, which is a G^*-complete metric space. Now, define a mapping $M : U \to U$ by

$$M(u) = \begin{cases} 1, & \text{if } u = 1, \\ \frac{u}{2}, & \text{if } u \in [0,1). \end{cases}$$

Then M is a k-contraction for any $k \in (\frac{1}{2},1)$. Also for $u_0 \in [0,1)$, $\delta(G,M,u_0) < +\infty$, therefore, the hypothesis of Theorem 3.1 are satisfied. Further, we see that sequence $M^n(u_0) = \frac{u_0}{2^n} \to 0$, and 0 is a fixed point of M. Also, M has another fixed point namely 1, but then $G(0,0,1) = +\infty$.

Now, in view of Lemma 4.3, the following result is a consequence of Theorem 3.2.

Corollary 4.6. *Let (U,G) be a complete generalized G_b-metric space with $s \geq 1$ and $M : U \to U$ be a mapping such that there exists some $\beta \in [0,\frac{1}{2s})$ and for every $(u,v,w) \in U \times U \times U$,*

$$\begin{aligned} G(Mu,Mv,Mw) \quad \leq \quad & \beta \max\{G(u,v,w),\ G(u,Mu,Mu),\ G(v,Mv,Mv),\ G(w,Mw,Mw), \\ & G(u,Mv,Mv),\ G(v,Mw,Mw),\ G(w,Mu,Mu)\}. \end{aligned}$$

Then M has a unique fixed point.

The following example is such that the hypothesis of Theorem 3.2 are satisfied, but not of Theorem 3.1 and not of Corollary 4.6.

Example 4.7. Consider a G^*-metric space as in Example 2.5, which is a G^*-complete metric space with constant $\alpha = 1$. Now, define a mapping $M : U \to U$ by

$$M(u) = \begin{cases} 1, & \text{if } u = 1 \text{ or } \frac{1}{2}, \\ \frac{u}{3}, & \text{otherwise.} \end{cases}$$

Then M is not a k-contraction as for $u = v = w = \frac{1}{2}$, $G(Mu, Mv, Mw) = +\infty \nleq 0 = kG(u, v, w)$ for any $k \in [0, 1)$. Thus, hypothesis of Theorem 3.1 are not satisfied. But we can check that $(T1)$ is satisfied for any $\beta \in (\frac{1}{3}, \frac{1}{\alpha})$. Also, for $u_0 \in [0, 1) - \{\frac{1}{2}\}$, $\delta(G, M, u_0) < +\infty$, therefore, hypothesis of Theorem 3.2 are satisfied. Further, we see that sequence $M^n(u_0) = \frac{u_0}{3^n} \to 0$, and 0 is a fixed point of M. Also, M has another fixed point namely 1, but then $G(0, 0, 1) = +\infty$ and $G(1, 1, 1) = +\infty$.

Also, the following result (in generalized G_b-metric space) is a consequence of Theorem 3.3 by Lemma 4.2.

Corollary 4.8. *Let (U, G) be a complete generalized G_b-metric space with $s \geq 1$ and $M : U \to U$ be a mapping such that there exist some $\beta \in [0, \frac{1}{s})$ and for every $(u, v, w) \in U \times U \times U$,*

$$G(Mu, Mv, Mw) \leq \beta \max\{G(u, v, w),\ G(u, Mu, Mu),\ G(v, Mv, Mv),\ G(w, Mw, Mw)\}.$$

Then M has a unique fixed point.

Conclusion

In the present article, a generalization of generalized G_b-metric space and GP-metric space has been investigated and named as G^*-metric space. Some basic concepts are extended in newly defined space. Also, some new types of Cauchy's sequence are defined. But, in the context of generalized G_b-metric space, all types coincide with usual Cauchy's sequence. New concepts can be of interest to many researchers in this field. We have proved some fixed point results in G^*-metric space which generalized the various results of generalized G_b-metric space in the literature.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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