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EXPRESSION FOR PRIMITIVE IDEMPOTENTS OF LENGTH $8p^n$ AND CORRESPONDING CODES

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Abstract. The group algebra FG of the group G of order $8p^n$ over the field F of prime power order q, where p is an odd prime $n \geq 1$, q is of the form $8k+1$ and q is primitive root modulo p^n, have $8(n+1)$ primitive idempotents. The explicit expressions for these idempotents are obtained. Generating polynomials, minimum distances and dimensions for the corresponding minimal cyclic codes are also obtained.

Keywords: group algebra; cyclotomic cosets; primitive idempotents; generating polynomials; minimum distance.

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1. Introduction

Let F be a finite field of prime power order q and G be a cyclic group of order m such that $g.c.d.(m,q) = 1$. Then FG , the group algebra of the cyclic group G over F, is semi-simple and has only a finite number of primitive idempotents which equals the number of cyclotomic cosets modulo m. Let t be the multiplicative order of q modulo m. If $t = \phi(m)$ and $m = 2, 4, p^n, 2p^n$, the complete sets of primitive idempotents were calculated by Pruthi and Arora [1,9]. The minimal quadratic residue cyclic codes of length p^n were obtained by Batra and Arora [3]. The minimal cyclic codes of length p^nq were discussed by Bakshi and Raka [2]. Explicit expression of

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primitive idempotents in FC_{4p^n} and corresponding codes were obtained by Singh and Ahlawat $[10,11]$. The minimal cyclic codes of length $8p^n$ were discussed by Singh and Arora [12, 13]. The equivalence relation $a \sim b$ whenever $a \equiv bq^i(\mod 8p^n)$ partitions the set $S = \{1,2,.....,8p^n\}$ into

$\Omega_{sp^n} = \{sp^n\}$ for $0 \leq s \leq 7$

and for $0 \leq j \leq n-1$,

Ω_{tp^j}=$\{tp^j, tp^jq, tp^jq^2, ..., tp^jq^{\phi(p^{n-j})-1}\}$

where $t = 1,2,4,8, \lambda = 1+2p^n, \mu = 2(1+2p^n), \nu = 1+4p^n, \chi = 1+6p^n$.

In the following lemmas some results for cyclotomic cosets are obtained:

Lemma 1.1. *If* $p^n \equiv 1(mod\ 4)$*, then* $(1+6p^n)q^{\frac{\phi(p^n)}{2}} \equiv -1(mod\ 8p^n)$ *and if* $p^n \equiv 3(mod\ 4)$*, then* $(1+2p^n)q^{\frac{\phi(p^n)}{2}} \equiv -1(mod\ 8p^n)$.

Proof. For an odd prime p, $\frac{\phi(p^n)}{2}$ is odd iff $p \equiv 3(mod\ 4)$.
Now, $q^{\frac{\phi(p^n)}{2}} \equiv -1(mod\ p^n)$ and $(1+6p^n) \equiv 1(mod\ p^n)$, so $(1+6p^n)q^{\frac{\phi(p^n)}{2}} \equiv -1(mod\ p^n)$.
Further, $p^n = 4t+1$, thus $(1+6p^n)q^{\frac{\phi(p^n)}{2}} \equiv -1(mod\ 8p^n)$.
When $p^n = 4t+3$, then $(1+2p^n)q^{\frac{\phi(p^n)}{2}} \equiv -1(mod\ 8)$ and hence $(1+2p^n)q^{\frac{\phi(p^n)}{2}} \equiv -1(mod\ 8p^n)$.

□

Remark 1.2. *By above lemma, we obtained the following "If* $p \equiv 3(mod\ 4)$ *and n is even or* $p \equiv 1(mod\ 4)$ *then* $-\Omega_1 = \Omega_\chi$ *and if* $p \equiv 3(mod\ 4)$ *and n is odd, then* $-\Omega_1 = \Omega_\lambda$*."*

Lemma 1.3. *For cyclotomic cosets* $\Omega_{p^j}, 0 \leq j \leq n-1$.
(i) $\lambda^2\Omega_{p^j} = \Omega p^j = \lambda\Omega_{\lambda p^j}$.
(ii) $\nu^2\Omega_{p^j} = \Omega p^j = \nu\Omega_{\nu p^j}$.
(iii) $\chi^2\Omega_{p^j} = \Omega p^j = \chi\Omega_{\chi p^j}$.
(iv) $\mu^2\Omega_{p^j} = 4\Omega p^j = \mu\Omega_{\mu p^j} = 2\Omega_{2p^j} = \Omega_{4p^j}$.

Proof. Since $\lambda^2 = (1+2p^n)^2 = 1+4p^{2n}+4p^n = 1+4p^n(p^n+1)$.
As p is odd, so $p^n+1 \equiv 0(mod\ 2)$ and hence $\lambda^2 \equiv 1(mod\ 8p^n)$.
Therefore, $\lambda^2\Omega_{p^i} = \Omega_{p^i}$. Also,

$\lambda^2\Omega_{p^j}=\{\lambda^2 p^j, \lambda^2 p^j q, ..., \lambda^2 p^j q^{\phi(p^{n-j})-1}\} = \lambda\{\lambda p^j, \lambda p^j q, ..., \lambda p^j q^{\phi(p^{n-j})-1}\} = \lambda\Omega_{\lambda p^i}$.

Thus, $\lambda^2\Omega_{p^j} = \Omega p^j = \lambda\Omega_{\lambda p^j}$. □

Proof of remaining parts will go on similar lines.

Lemma 1.4. *For cyclotomic cosets* $\Omega_{p^n}, \Omega_{2p^n}, \Omega_{3p^n}, \Omega_{4p^n}, \Omega_{5p^n}, \Omega_{6p^n}$ *and* Ω_{7p^n} :

(i) $-\Omega_{p^n} = \Omega_{7p^n}$ *and so* $-\Omega_{7p^n} = \Omega_{p^n}$

(ii) $-\Omega_{2p^n} = \Omega_{6p^n}$ *and so* $-\Omega_{6p^n} = \Omega_{2p^n}$

(iii) $-\Omega_{3p^n} = \Omega_{5p^n}$ *and so* $-\Omega_{5p^n} = \Omega_{3p^n}$

(iv) $-\Omega_{4p^n} = \Omega_{4p^n}$

Proof. Proof is trivial. □

Lemma 1.5. *If* $r = 1, 2, 4, 8, \lambda, \mu, \nu$ *or* χ *then for* $0 \leq j \leq n-1$.

$r\Omega_{8p^j} = \Omega_{8p^j} = 8\Omega_{rp^j} = 4\Omega_{\mu p^j}$.

Proof. Since $8rp^j \not\equiv lp^jq^k(mod\ 8p^n)$ for $l = 1, 2, 4, 8, \lambda, \mu, \nu, \chi$. So, $8rp^j \equiv 8p^jq^k(mod\ 8p^n)$. Hence the required result holds. □

Lemma 1.6. *For* $0 \leq j \leq n-1$

(i) $-\Omega_{2p^j} = \Omega_{\mu p^j}$.

(ii) $-\Omega_{4p^j} = \Omega_{4p^j}$ *and* $-\Omega_{8p^j} = \Omega_{8p^j}$,

(iii) *If* $p \equiv 3(mod\ 4)$ *and n is even or* $p \equiv 1(mod\ 4)$, *then* $-\Omega_{\lambda p^j} = \Omega_{\nu p^j}$.

(iv) *If* $p \equiv 3(mod\ 4)$ *and n is odd, then* $-\Omega_{\chi p^j} = \Omega_{\nu p^j}$

Proof of these can be obtained by using lemma 1.1.

The $8(n+1)$ primitive idempotents are obtained in Theorem 2.8. In Section 3, generating polynomials and dimensions for the corresponding minimal cyclic codes of length $8p^n$ are discussed. The minimum distances or the bounds for these codes are obtained in Section 4. Minimal cyclic codes of length 24 for $q = 17$ are calculated in Section 5.

Notation 1.7. *Let* α *be a fixed primitive* $8p^n$*th root of unity in some extension field of* F. *For* $0 \leq j \leq n-1$, *we define:*

$T_j = p^j \sum_{s\in\Omega_{p^j}} \alpha^s$, $S_j = p^j \sum_{s\in\Omega_{\nu p^j}} \alpha^s$, $Q_j = p^j \sum_{s\in\Omega_{\lambda p^j}} \alpha^s$, $P_j = p^j \sum_{s\in\Omega_{\chi p^j}} \alpha^s$, *and* $R_j = p^j \sum_{s\in\Omega_{2p^j}} \alpha^s$.

Then, T_j,S_j,Q_j,P_j *and* $R_j \in F$.

2. PRIMITIVE IDEMPOTENTS

Throughout this paper, α is assumed to be $8p^n$ th root of unity in some extension field of F, M_s the minimal ideal in $R_{8p^n} = \frac{F[x]}{<x^{8p^n}-1>} \cong FG$; generated by $\frac{(x^{8p^n}-1)}{m_s x}$ and $\theta_s(x)$, the primitive idempotent in R_{8p^n}, corresponding to the minimal ideal M_s, given by $\theta_s(x) = \frac{1}{8p^n}\sum_{i=0}^{8p^n-1} \varepsilon_i^s x^i$ where $\varepsilon_i^s = \sum_{j\in\Omega_s} \alpha^{-ij}$[3]. Also, denote $\overline{C}_s = \sum_{s\in\Omega_s} x^s$.

Lemma 2.1. *For any odd prime p and a positive integer k, if δ is a primitive p^kth root of unity, ζ is a primitive $2p^k$th root of unity in some extension field of F and q a primitive root modulo p^k,*

$$\sum_{s=0}^{\phi(p^k)-1} \delta^{q^s} = \begin{cases} -1, k=1 \\ 0, k\geq 2 \end{cases} \quad and \quad \sum_{s=0}^{\phi(2p^k)-1} \delta^{q^s} = \begin{cases} 1, k=1 \\ 0, k\geq 2 \end{cases}$$

Proof. Since $\{1,q,...,q^{\phi(p^k)-1}\}$ and $\{1,q,...,q^{\phi(2p^k)-1}\}$ are reduced residue system modulo p^k and $2p^k$ respectively and q is a primitive root modulo p^k. Thus,

$$\sum_{s=0}^{\phi(p^k)-1} \delta^{q^s} = \sum_{s=1}^{p^k} \delta^s - \sum_{s=1}^{p^k-1} \delta^{ps}$$

and $\sum_{s=0}^{\phi(2p^k)-1} \zeta^{q^s} = \sum_{s=1}^{2p^k} \zeta^s - \sum_{s=1}^{p^k} \zeta^{2s} - \sum_{s=1}^{2p^{k-1}} \zeta^{ps} + \sum_{s=1}^{p^{k-1}} \zeta^{2ps}$.

Now, the result can be obtained using the properties of δ and ζ. □

Lemma 2.2. *For $0 \leq j \leq n-1$, $T_j + S_j = 0$, $P_j + Q_j = 0$ and $P_j + \beta^{p^j} T_j = 0$, where $\beta = \alpha^{2p^n}$ and is 4th root of unity in some extension field of F.*

Proof. Since $\alpha^{\nu p^j} = \alpha^{(1+4p^n)p^j} = \alpha^{p^j}\alpha^{4p^n p^j} = -\alpha^{p^j}$ and $\alpha^{\chi p^j} = \alpha^{(1+6p^n)p^j} = -\beta^{p^j}\alpha^{p^j}$.

Using these the result follows. □

Lemma 2.3. *For $0 \leq i, j \leq n-1$,*

$$\sum_{s\in\Omega_{p^j}} \alpha^{p^i s} = \sum_{s\in\Omega_{\nu p^j}} \alpha^{\nu p^i s} = \sum_{s\in\Omega_{\lambda p^j}} \alpha^{\lambda p^i s} = \sum_{s\in\Omega_{\chi p^j}} \alpha^{\chi p^i s} = -\sum_{s\in\Omega_{\lambda p^j}} \alpha^{\chi p^i s} = -\sum_{s\in\Omega_{p^j}} \alpha^{\nu p^i s}$$

$$= -\sum_{s\in\Omega_{\nu p^j}} \alpha^{p^i s} = \begin{cases} \phi(p^{n-j})\alpha^{p^{i+j}}, if\ i+j \geq n, \\ \frac{1}{p^j} T_{i+j}, if\ i+j \leq n-1. \end{cases}$$

Proof. Let $\delta = \alpha^{p^{i+j}}$. Then $\sum_{s\in\Omega_{p^j}} \alpha^{p^i s} = \sum_{t=0}^{\phi(p^{n-j})-1} \alpha^{p^{i+j} q^t} = \sum_{t=0}^{\phi(p^{n-j})-1} \delta^{q^t}$.

Consider the following cases:

Case:1. If $i+j \geq n$, then $\delta = \alpha^{p^n}$ is $8th$ root of unity and

$\delta^{q^l} \equiv \delta^{q^r}$ iff $q^l \equiv q^r (mod\ 8)$ iff $l \equiv r(mod\ \phi(8))$.

Hence $\sum_{s\varepsilon\Omega_{p^j}} \alpha^{p^i s} = \sum_{t=0}^{\phi(p^{n-j})-1} \delta^{q^t} = \frac{\phi(p^{n-j})}{\phi(8)} \sum_{t=0}^{\phi(8)-1} \delta^{q^t} = \phi(p^{n-j})\alpha^{p^{i+j}}$.

Case 2. If $i+j \leq n-1$, then δ is $8p^{n-i-j}th$ root of unity and

$\delta^{q^l} \equiv \delta^{q^r}$ iff $q^l \equiv q^r (mod\ 8p^{n-i-j})$ iff $l \equiv r(mod\ \phi(p^{n-i-j}))$

Therefore, $\sum_{s\varepsilon\Omega_{p^j}} \alpha^{p^i s} = \frac{\phi(p^{n-j})}{\phi(p^{n-i-j})} \sum_{t=0}^{\phi(p^{n-i-j})-1} \delta^{q^t} = \frac{1}{p^j} p^{i+j} \sum_{t=0}^{\phi(p^{n-i-j})-1} \alpha^{p^{i+j}q^t}$

$= \frac{1}{p^j} p^{i+j} \sum_{s\varepsilon\Omega_{p^{i+j}}} \alpha^s$ Since $p^{i+j} \sum_{s\varepsilon\Omega_{p^{i+j}}} \alpha^s = T_{i+j}$, therefore, $\sum_{s\varepsilon\Omega_{p^j}} \alpha^s = \frac{1}{p^j} T_{i+j}$. □

The proof of the following lemmas will go on similar lines as of the above lemma.

Lemma 2.4. *For* $0 \leq i, j \leq n$,

$$\sum_{s\in\Omega_{\lambda p^j}} \alpha^{p^i s} = \sum_{s\in\Omega_{p^j}} \alpha^{\lambda p^i s} = -\sum_{s\in\Omega_{\lambda p^j}} \alpha^{\nu p^i s} = \sum_{s\in\Omega_{\nu p^j}} \alpha^{\chi p^i s} = -\sum_{s\in\Omega_{\nu p^j}} \alpha^{\lambda p^i s} = \sum_{s\in\Omega_{\chi p^j}} \alpha^{\nu p^i s}$$

$$= -\sum_{s\in\Omega_{p^j}} \alpha^{\chi p^i s} = \begin{cases} \phi(p^{n-j})\alpha^{p^{i+j}}\beta^{p^{i+j}}, if\ i+j \geq n, \\ -\frac{1}{p^j} P_{i+j}, if\ i+j \leq n-1. \end{cases}$$

Lemma 2.5. *For* $0 \leq i \leq n$*;* $0 \leq j \leq n-1$

$$\sum_{s\in\Omega_{p^j}} \alpha^{4p^i s} = \sum_{s\in\Omega_{2p^j}} \alpha^{2p^i s} = \sum_{s\in\Omega_{\lambda p^j}} \alpha^{4p^i s} = \sum_{s\in\Omega_{4p^j}} \alpha^{p^i s} = \sum_{s\in\Omega_{4p^j}} \alpha^{\lambda p^i s} = \sum_{s\in\Omega_{\nu p^j}} \alpha^{4p^i s} = \sum_{s\in\Omega_{4p^j}} \alpha^{\nu p^i s}$$

$$= \sum_{s\in\Omega_{\chi p^j}} \alpha^{4p^i s} = \sum_{s\in\Omega_{4p^j}} \alpha^{\chi p^i s} = \sum_{s\in\Omega_{\mu p^j}} \alpha^{2p^i s} = \sum_{s\in\Omega_{2p^j}} \alpha^{\mu p^i s} = \sum_{s\in\Omega_{\mu p^j}} \alpha^{\mu p^i s}$$

$$= \begin{cases} -\phi(p^{n-j}), if\ i+j \geq n \\ p^{n-j-1}, if\ i+j = n-1 \\ 0, if i+j < n-1. \end{cases}$$

Lemma 2.6. *For* $0 \leq i \leq n$*;* $0 \leq j \leq n-1$

$$\sum_{s\in\Omega_{p^j}} \alpha^{8p^i s} = \sum_{s\in\Omega_{4p^j}} \alpha^{2p^i s} = \sum_{s\in\Omega_{2p^j}} \alpha^{4p^i s} = \sum_{s\in\Omega_{\lambda p^j}} \alpha^{8p^i s} = \sum_{s\in\Omega_{8p^j}} \alpha^{\lambda p^i s} = \sum_{s\in\Omega_{\nu p^j}} \alpha^{8p^i s}$$

$$= \sum_{s\in\Omega_{8p^j}} \alpha^{\nu p^i s} = \sum_{s\in\Omega_{\chi p^j}} \alpha^{8p^i s} = \sum_{s\in\Omega_{8p^j}} \alpha^{\chi p^i s} = \sum_{s\in\Omega_{\mu p^j}} \alpha^{4p^i s} = \sum_{s\in\Omega_{4p^j}} \alpha^{\mu p^i s} = \sum_{s\in\Omega_{\mu p^j}} \alpha^{8p^i s}$$

$$= \sum_{s\in\Omega_{8p^j}} \alpha^{\mu p^i s} = \sum_{s\in\Omega_{p^j}} \alpha^{16p^i s} = \sum_{s\in\Omega_{\mu p^j}} \alpha^{16p^i s} = \sum_{s\in\Omega_{p^j}} \alpha^{32p^i s} = \sum_{s\in\Omega_{p^j}} \alpha^{64p^i s}$$

$$= \begin{cases} \phi(p^{n-j}), if\ i+j \geq n, j=n \\ -p^{n-j-1}, if\ i+j = n-1 \\ 0, if i+j < n-1. \end{cases}$$

Lemma 2.7. *For* $0 \leq i, j \leq n-1$,

$$\sum_{s\in\Omega_{p^j}} \alpha^{2p^i s} = \sum_{s\in\Omega_{2p^j}} \alpha^{p^i s} = -\sum_{s\in\Omega_{\mu p^j}} \alpha^{2p^i s} = -\sum_{s\in\Omega_{\lambda p^j}} \alpha^{2p^i s} = \sum_{s\in\Omega_{\nu p^j}} \alpha^{2p^i s} = \sum_{s\in\Omega_{\chi p^j}} \alpha^{2p^i s}$$

$$= \begin{cases} \phi(p^{n-j})\alpha^{2p^{i+j}}, if\ i+j \geq n \\ \frac{1}{p^j} R_{i+j}, if\ i+j \leq n-1. \end{cases}$$

Theorem 2.8. *The explicit expressions for the* $8(n+1)$ *primitive idempotents in* R_{8p^n} *are given by*

$\theta_0(x) = \frac{1}{8p^n}[\overline{C}_0 + \overline{C}_{p^n} + \overline{C}_{2p^n} + \overline{C}_{3p^n} + \overline{C}_{4p^n} + \overline{C}_{5p^n} + \overline{C}_{6p^n} + \overline{C}_{7p^n} + \sum_{i=0}^{n-1}\{\overline{C}_{p^i} + \overline{C}_{2p^i} + \overline{C}_{4p^i} + \overline{C}_{8p^i} + \overline{C}_{\lambda p^i} + \overline{C}_{\mu p^i} + \overline{C}_{\nu p^i} + \overline{C}_{\chi p^i}\}]$

$\theta_{p^n}(x) = \frac{1}{8p^n}[\overline{C}_0 - \beta^{p^n}\alpha^{p^{2n}}\overline{C}_{p^n} - \beta^{p^n}\overline{C}_{2p^n} - \alpha^{p^{2n}}\overline{C}_{3p^n} - \overline{C}_{4p^n} + \beta^{p^n}\alpha^{p^{2n}}\overline{C}_{5p^n} + \beta^{p^n}\overline{C}_{6p^n} + \alpha^{p^{2n}}\overline{C}_{7p^n} - \sum_{i=0}^{n-1}\{\beta^{p^i}\alpha^{p^{n+i}}\overline{C}_{p^i} + \beta^{p^i}\overline{C}_{2p^i} + \overline{C}_{4p^i} - \overline{C}_{8p^i} - \beta^{p^i(1+p^n)}\alpha^{p^{n+i}}\overline{C}_{\lambda p^i} - \beta^{p^i}\overline{C}_{\mu p^i} - \beta^{p^i}\alpha^{p^{n+i}}\overline{C}_{\nu p^i} - \beta^{p^i(1+p^n)}\alpha^{p^{n+i}}\overline{C}_{\chi p^i}\}]$

$\theta_{2p^n}(x) = \frac{1}{8p^n}[\overline{C}_0 - \beta^{p^n}\overline{C}_{p^n} - \overline{C}_{2p^n} + \beta^{p^n}\overline{C}_{3p^n} + \overline{C}_{4p^n} - \beta^{p^n}\overline{C}_{5p^n} - \overline{C}_{6p^n} + \beta^{p^n}\overline{C}_{7p^n} - \sum_{i=0}^{n-1}\{\beta^{p^i}\overline{C}_{p^i} + \overline{C}_{2p^i} - \overline{C}_{4p^i} - \overline{C}_{8p^i} - \beta^{p^i}\overline{C}_{\lambda p^i} + \overline{C}_{\mu p^i} + \beta^{p^i}\overline{C}_{\nu p^i} - \beta^{p^i}\overline{C}_{\chi p^i}\}]$

$\theta_{3p^n}(x) = \frac{1}{8p^n}[\overline{C}_0 - \alpha^{p^{2n}}\overline{C}_{p^n} + \beta^{p^n}\overline{C}_{2p^n} - \beta^{p^n}\alpha^{p^{2n}}\overline{C}_{3p^n} - \overline{C}_{4p^n} + \alpha^{p^{2n}}\overline{C}_{5p^n} - \beta^{p^n}\overline{C}_{6p^n} + \beta^{p^n}\alpha^{p^{2n}}\overline{C}_{7p^n} - \sum_{i=0}^{n-1}\{\alpha^{p^{n+i}}\overline{C}_{p^i} - \beta^{p^i}\overline{C}_{2p^i} + \overline{C}_{4p^i} - \overline{C}_{8p^i} + \beta^{p^{n+i}}\alpha^{p^{n+i}}\overline{C}_{\lambda p^i} + \beta^{p^i}\overline{C}_{\mu p^i} - \alpha^{p^{n+i}}\overline{C}_{\nu p^i} - \beta^{p^{n+i}}\alpha^{p^{n+i}}\overline{C}_{\chi p^i}\}]$

$\theta_{4p^n}(x) = \frac{1}{8p^n}[\overline{C}_0 - \overline{C}_{p^n} + \overline{C}_{2p^n} - \overline{C}_{3p^n} + \overline{C}_{4p^n} - \overline{C}_{5p^n} + \overline{C}_{6p^n} - \overline{C}_{7p^n} - \sum_{i=0}^{n-1}\{\overline{C}_{p^i} - \overline{C}_{2p^i} - \overline{C}_{4p^i} - \overline{C}_{8p^i} + \overline{C}_{\lambda p^i} - \overline{C}_{\mu p^i} + \overline{C}_{\nu p^i} + \overline{C}_{\chi p^i}\}]$

$\theta_{5p^n}(x) = \frac{1}{8p^n}[\overline{C}_0 + \beta^{p^n}\alpha^{p^{2n}}\overline{C}_{p^n} - \beta^{p^n}\overline{C}_{2p^n} + \alpha^{p^{2n}}\overline{C}_{3p^n} - \overline{C}_{4p^n} - \beta^{p^n}\alpha^{p^{2n}}\overline{C}_{5p^n} + \beta^{p^n}\overline{C}_{6p^n} - \alpha^{p^{2n}}\overline{C}_{7p^n} + \sum_{i=0}^{n-1}\{\beta^{p^i}\alpha^{p^{n+i}}\overline{C}_{p^i} - \beta^{p^i}\overline{C}_{2p^i} - \overline{C}_{4p^i} + \overline{C}_{8p^i} - \beta^{p^i(1+p^n)}\alpha^{p^{n+i}}\overline{C}_{\lambda p^i} + \beta^{p^i}\overline{C}_{\mu p^i} - \beta^{p^i}\alpha^{p^{n+i}}\overline{C}_{\nu p^i} + \beta^{p^i(1+p^n)}\alpha^{p^{n+i}}\overline{C}_{\chi p^i}\}]$

$\theta_{6p^n}(x) = \frac{1}{8p^n}[\overline{C}_0 + \beta^{p^n}\overline{C}_{p^n} - \overline{C}_{2p^n} - \beta^{p^n}\overline{C}_{3p^n} + \overline{C}_{4p^n} + \beta^{p^n}\overline{C}_{5p^n} - \overline{C}_{6p^n} - \beta^{p^n}\overline{C}_{7p^n} + \sum_{i=0}^{n-1}\{\beta^{p^i}\overline{C}_{p^i} - \overline{C}_{2p^i} + \overline{C}_{4p^i} + \overline{C}_{8p^i} - \beta^{p^i}\overline{C}_{\lambda p^i} - \overline{C}_{\mu p^i} + \beta^{p^i}\overline{C}_{\nu p^i} - \beta^{p^i}\overline{C}_{\chi p^i}\}]$

$\theta_{7p^n}(x) = \frac{1}{8p^n}[\overline{C}_0 + \alpha^{p^{2n}}\overline{C}_{p^n} + \beta^{p^n}\overline{C}_{2p^n} + \beta^{p^n}\alpha^{p^{2n}}\overline{C}_{3p^n} - \overline{C}_{4p^n} - \alpha^{p^{2n}}\overline{C}_{5p^n} - \beta^{p^n}\overline{C}_{6p^n} - \beta^{p^n}\alpha^{p^{2n}}\overline{C}_{7p^n} + \sum_{i=0}^{n-1}\{\alpha^{p^{n+i}}\overline{C}_{p^i} + \beta^{p^i}\overline{C}_{2p^i} - \overline{C}_{4p^i} + \overline{C}_{8p^i} + \beta^{p^{n+i}}\alpha^{p^{n+i}}\overline{C}_{\lambda p^i} - \beta^{p^i}\overline{C}_{\mu p^i} - \alpha^{p^{n+i}}\overline{C}_{\nu p^i} - \beta^{p^{n+i}}\alpha^{p^{n+i}}\overline{C}_{\chi p^i}\}]$

and for $0 \le j \le n-1$,

$\theta_{2p^j}(x) = \frac{1}{8p^n}[\phi(p^{n-j})\{\overline{C}_0 - \beta^{p^j}\overline{C}_{p^n} - \overline{C}_{2p^n} + \beta^{p^j}\overline{C}_{3p^n} + \overline{C}_{4p^n} - \beta^{p^j}\overline{C}_{5p^n} - \overline{C}_{6p^n} + \beta^{p^j}\overline{C}_{7p^n}\} + p^{n-j-1}\{\overline{C}_{2p^{n-j-1}} - \overline{C}_{4p^{n-j-1}} - \overline{C}_{8p^{n-j-1}} + \overline{C}_{\mu p^{n-j-1}}\} - \frac{1}{p^j}\sum_{i=0}^{n-j-1}\{R_{i+j}\overline{C}_{p^i} - R_{i+j}\overline{C}_{\lambda p^i} + R_{i+j}\overline{C}_{\nu p^i} - R_{i+j}\overline{C}_{\chi p^i}\} - \phi(p^{n-j})\sum_{i=n-j}^{n-1}\alpha^{2p^{i+j}}\overline{C}_{p^i} + \overline{C}_{2p^i} - \overline{C}_{4p^i} - \overline{C}_{8p^i} - \alpha^{2p^{i+j}}\overline{C}_{\lambda p^i} + \overline{C}_{\mu p^i} + \alpha^{2p^{i+j}}\overline{C}_{\nu p^i} - \alpha^{2p^{i+j}}\overline{C}_{\chi p^i}\}]$

$\theta_{4p^j}(x) = \frac{1}{8p^n}[\phi(p^{n-j})\{\overline{C}_0 - \overline{C}_{p^n} + \overline{C}_{2p^n} - \overline{C}_{3p^n} + \overline{C}_{4p^n} - \overline{C}_{5p^n} + \overline{C}_{6p^n} - \overline{C}_{7p^n}\} + p^{n-j-1}\{\overline{C}_{p^{n-j-1}} - \overline{C}_{2p^{n-j-1}} - \overline{C}_{4p^{n-j-1}} - \overline{C}_{8p^{n-j-1}} + \overline{C}_{\lambda p^{n-j-1}} - \overline{C}_{\mu p^{n-j-1}} + \overline{C}_{\nu p^{n-j-1}} + \overline{C}_{\chi p^{n-j-1}}\} - \phi(p^{n-j})\sum_{i=n-j}^{n-1}\{\overline{C}_{p^i} - \overline{C}_{2p^i} - \overline{C}_{4p^i} - \overline{C}_{8p^i} + \overline{C}_{\lambda p^i} - \overline{C}_{\mu p^i} + \overline{C}_{\nu p^i} + \overline{C}_{\chi p^i}\}]$

$\theta_{8p^j}(x) = \frac{1}{8p^n}[\phi(p^{n-j})\{\overline{C}_0 + \overline{C}_{p^n} + \overline{C}_{2p^n} - \overline{C}_{3p^n} + \overline{C}_{4p^n} - \overline{C}_{5p^n} + \overline{C}_{6p^n} - \overline{C}_{7p^n}\} + p^{n-j-1}\{\overline{C}_{p^{n-j-1}} - \overline{C}_{2p^{n-j-1}} - \overline{C}_{4p^{n-j-1}} - \overline{C}_{8p^{n-j-1}} + \overline{C}_{\lambda p^{n-j-1}} - \overline{C}_{\mu p^{n-j-1}} + \overline{C}_{\nu p^{n-j-1}} + \overline{C}_{\chi p^{n-j-1}}\} - \phi(p^{n-j})\sum_{i=n-j}^{n-1}\{\overline{C}_{p^i} - \overline{C}_{2p^i} - \overline{C}_{4p^i} - \overline{C}_{8p^i} + \overline{C}_{\lambda p^i} - \overline{C}_{\mu p^i} + \overline{C}_{\nu p^i} + \overline{C}_{\chi p^i}\}]$

$\theta_{\mu p^j}(x) = \frac{1}{8p^n}[\phi(p^{n-j})\{\overline{C}_0 + \beta^{p^j}\overline{C}_{p^n} - \overline{C}_{2p^n} - \beta^{p^j}\overline{C}_{3p^n} + \overline{C}_{4p^n} + \beta^{p^j}\overline{C}_{5p^n} - \overline{C}_{6p^n} - \beta^{p^j}\overline{C}_{7p^n}\} + p^{n-j-1}\{\overline{C}_{2p^{n-j-1}} - \overline{C}_{4p^{n-j-1}} - \overline{C}_{8p^{n-j-1}} + \overline{C}_{\mu p^{n-j-1}}\} + \frac{1}{p^j}\sum_{i=0}^{n-j-1}\{R_{i+j}\overline{C}_{p^i} - R_{i+j}\overline{C}_{\lambda p^i} + R_{i+j}\overline{C}_{\nu p^i} - R_{i+j}\overline{C}_{\chi p^i}\} + \phi(p^{n-j})\sum_{i=n-j}^{n-1}\alpha^{2p^{i+j}}\overline{C}_{p^i} - \overline{C}_{2p^i} + \overline{C}_{4p^i} + \overline{C}_{8p^i} - \alpha^{2p^{i+j}}\overline{C}_{\lambda p^i} - \overline{C}_{\mu p^i} + \alpha^{2p^{i+j}}\overline{C}_{\nu p^i} - \alpha^{2p^{i+j}}\overline{C}_{\chi p^i}\}]$

If $p^n \equiv 3 (mod\ 4)$

$\theta_{p^j}(x) = \frac{1}{8p^n}[\phi(p^{n-j})\{\overline{C}_0 + \alpha^{p^{n+j}}\beta^{p^{n+j}}\overline{C}_{p^n} - \beta^{p^j}\overline{C}_{2p^n} - \alpha^{p^{n+j}}\overline{C}_{3p^n} - \overline{C}_{4p^n} - \alpha^{p^{n+j}}\beta^{p^{n+j}}\overline{C}_{5p^n} +$

$\beta^{p^{n+j}}\overline{C}_{6p^n} + \alpha^{p^{n+j}}\overline{C}_{7p^n}\}+p^{n-j-1}\{\overline{C}_{4p^{n-j-1}}-\overline{C}_{8p^{n-j-1}}\}$-$\frac{1}{p^j}\sum_{i=0}^{n-j-1}\{P_{i+j}\overline{C}_{p^i} + R_{i+j}\overline{C}_{2p^i} - T_{i+j}\overline{C}_{\lambda p^i} - R_{i+j}\overline{C}_{\mu p^i} - P_{i+j}\overline{C}_{\nu p^i} + T_{i+j}\overline{C}_{\chi p^i}\}+ \phi(p^{n-j})\sum_{i=n-j}^{n-1}\{\alpha^{p^{i+j}}\beta^{p^{i+j}}\overline{C}_{p^i} - \alpha^{2p^{i+j}}\overline{C}_{2p^i} - \overline{C}_{4p^i}+\overline{C}_{8p^i}+\alpha^{p^{i+j}}\overline{C}_{\lambda p^i}+\alpha^{2p^{i+j}}\overline{C}_{\mu p^i}-\alpha^{p^{i+j}}\beta^{p^{i+j}}\overline{C}_{\nu p^i}-\alpha^{p^{i+j}}\overline{C}_{\chi p^i}\}]$

$\theta_{\lambda p^j}(x)=\frac{1}{8p^n}[\phi(p^{n-j})\{\overline{C}_0+\alpha^{p^{n+j}}\overline{C}_{p^n}+\beta^{p^j}\overline{C}_{2p^n}+\alpha^{p^{n+j}}\beta^{p^j}\overline{C}_{3p^n}-\overline{C}_{4p^n}-\alpha^{p^{n+j}}\overline{C}_{5p^n}- \beta^{p^j}\overline{C}_{6p^n} - \alpha^{p^{n+j}}\beta^{p^j}\overline{C}_{7p^n}\}+p^{n-j-1}\{\overline{C}_{4p^{n-j-1}}-\overline{C}_{8p^{n-j-1}}\}+\frac{1}{p^j}\sum_{i=0}^{n-j-1}\{T_{i+j}\overline{C}_{p^i} + R_{i+j}\overline{C}_{2p^i} - P_{i+j}\overline{C}_{\lambda p^i}-R_{i+j}\overline{C}_{\mu p^i}-T_{i+j}\overline{C}_{\nu p^i}+P_{i+j}\overline{C}_{\chi p^i}\}+ \phi(p^{n-j})\sum_{i=n-j}^{n-1}\{\alpha^{p^{i+j}}\overline{C}_{p^i}+\alpha^{2p^{i+j}}\overline{C}_{2p^i}-\overline{C}_{4p^i}+ \overline{C}_{8p^i}+\alpha^{p^{i+j}}\beta^{p^{i+j}}\overline{C}_{\lambda p^i}-\alpha^{2p^{i+j}}\overline{C}_{\mu p^i}-\alpha^{p^{i+j}}\overline{C}_{\nu p^i}+\alpha^{p^{i+j}}\beta^{p^{i+j}}\overline{C}_{\chi p^i}\}]$

$\theta_{\nu p^j}(x)=\frac{1}{8p^n}[\phi(p^{n-j})\{\overline{C}_0-\alpha^{p^{n+j}}\beta^{p^{n+j}}\overline{C}_{p^n}-\beta^{p^j}\overline{C}_{2p^n}+\alpha^{p^{n+j}}\overline{C}_{3p^n}-\overline{C}_{4p^n}+\alpha^{p^{n+j}}\beta^{p^{n+j}}\overline{C}_{5p^n}+ \beta^{p^{n+j}}\overline{C}_{6p^n} - \alpha^{p^{n+j}}\overline{C}_{7p^n}\}+p^{n-j-1}\{\overline{C}_{4p^{n-j-1}}-\overline{C}_{8p^{n-j-1}}\}+\frac{1}{p^j}\sum_{i=0}^{n-j-1}\{P_{i+j}\overline{C}_{p^i} - R_{i+j}\overline{C}_{2p^i} - T_{i+j}\overline{C}_{\lambda p^i} + R_{i+j}\overline{C}_{\mu p^i} - P_{i+j}\overline{C}_{\nu p^i} + T_{i+j}\overline{C}_{\chi p^i}\}$- $\phi(p^{n-j})\sum_{i=n-j}^{n-1}\{\alpha^{p^{i+j}}\beta^{p^{i+j}}\overline{C}_{p^i} + \alpha^{2p^{i+j}}\overline{C}_{2p^i} + \overline{C}_{4p^i}-\overline{C}_{8p^i}+\alpha^{p^{i+j}}\overline{C}_{\lambda p^i}-\alpha^{2p^{i+j}}\overline{C}_{\mu p^i}-\alpha^{p^{i+j}}\beta^{p^{i+j}}\overline{C}_{\nu p^i}-\alpha^{p^{i+j}}\overline{C}_{\chi p^i}\}]$

$\theta_{\chi p^j}(x)=\frac{1}{8p^n}[\phi(p^{n-j})\{\overline{C}_0-\alpha^{p^{n+j}}\overline{C}_{p^n}+\beta^{p^j}\overline{C}_{2p^n}-\alpha^{p^{n+j}}\beta^{p^j}\overline{C}_{3p^n}-\overline{C}_{4p^n}+\alpha^{p^{n+j}}\overline{C}_{5p^n}- \beta^{p^j}\overline{C}_{6p^n} + \alpha^{p^{n+j}}\beta^{p^j}\overline{C}_{7p^n}\}+p^{n-j-1}\{\overline{C}_{4p^{n-j-1}}-\overline{C}_{8p^{n-j-1}}\}$-$\frac{1}{p^j}\sum_{i=0}^{n-j-1}\{T_{i+j}\overline{C}_{p^i} - R_{i+j}\overline{C}_{2p^i} - P_{i+j}\overline{C}_{\lambda p^i}+R_{i+j}\overline{C}_{\mu p^i}-T_{i+j}\overline{C}_{\nu p^i}+P_{i+j}\overline{C}_{\chi p^i}\}$- $\phi(p^{n-j})\sum_{i=n-j}^{n-1}\{\alpha^{p^{i+j}}\overline{C}_{p^i}-\alpha^{2p^{i+j}}\overline{C}_{2p^i}+\overline{C}_{4p^i}- \overline{C}_{8p^i}+\alpha^{p^{i+j}}\beta^{p^{i+j}}\overline{C}_{\lambda p^i}+\alpha^{2p^{i+j}}\overline{C}_{\mu p^i}-\alpha^{p^{i+j}}\overline{C}_{\nu p^i}-\alpha^{p^{i+j}}\beta^{p^{i+j}}\overline{C}_{\chi p^i}\}]$

Further, when $p^n \equiv 1(mod\ 4)$, then the expressions for $\theta_{\chi p^j}$ and $\theta_{\lambda p^j}$ in case when $p^n \equiv 3(mod\ 4)$, are the expressions for $\theta_{\lambda p^j}$ and $\theta_{\chi p^j}$ respectively. Further, the expressions for θ_{p^j} and $\theta_{\nu p^j}$ can be obtained from $\theta_{\nu p^j}$ and θ_{p^j} by replacing $\overline{C}_{3p^n}$ and $\overline{C}_{7p^n}$ with $-\overline{C}_{3p^n}$ and $-\overline{C}_{7p^n}$ respectively.

Here β is a 4th root of unity in F and $P_j = -\beta^{p^j}T_j$, $R_{n-1} = \sqrt{-1}p^{n-1}$, for all $j \leq n-2$, $R_j = 0$. If $p \equiv 3(mod\ 4)$, then $T_{n-1} = \sqrt{-\beta}p^{n-1}$, for $j \leq n-2$, $T_j = 0$ and if $p \equiv 1(mod\ 4)$, then $T_{n-1} = \sqrt{\beta}p^{n-1}$, for $j \leq n-2$, $T_j = 0$

3. Dimension and Generating Polynomial

If α is primitive $8p^nth$ root of unity, then $m_s(x) = \prod_{s\varepsilon\Omega_s}(x-\alpha^s)$ denote the minimal polynomial for α^s and so the generating polynomial for cyclic code of length $8p^n$ corresponding to the cyclotomic coset Ω_s is $\frac{x^{8p^n}-1}{m_s(x)}$ and the dimension of minimal cyclic code M_s is equal to the cardinality of the class Ω_s. Thus, the dimensions of the codes M_0, M_{p^n}, M_{2p^n}, M_{3p^n}, M_{4p^n}, M_{5p^n}, M_{6p^n}, M_{7p^n}, M_{4p^j} and M_{8p^j} are 1, 1, 1, 1, 1, 1, 1, 1, $\phi(p^{n-j})$ and $\phi(p^{n-j})$ respectively.

Theorem 3.1. ***(i)*** *The generating polynomials for the codes $M_0, M_{p^n}, M_{2p^n}, M_{3p^n}, M_{4p^n}, M_{5p^n}, M_{6p^n}$ and M_{7p^n} are $(1+x+x^2+\ldots+x^{8p^n-1}), (x^4-1)(x^2+\beta)(x+\delta)(1+x^8+\ldots+x^{8(p^n-1)})$, $(x^6-x^4+x^2-1)(x+\beta)(1+x^8+\ldots+x^{8(p^n-1)})$, $\frac{x^{8p^n}-1}{x-\delta_1}$,*
$(x^7-x^6+x^5-x^4+x^3-x^2+x-1)(1+x^8+\ldots+x^{8(p^n-1)})$,
$(x^4-1)(x^2+\beta)(x-\delta)(1+x^8+\ldots+x^{8(p^n-1)})$,
$(x^4-1)(x^2+\beta)(x-\beta)(1+x^8+\ldots+x^{8(p^n-1)})$ and $\frac{x^{8p^n}-1}{x+\delta_1}$ respectively.
(ii)*The generating polynomials for the codes M_{4p^j} and M_{8p^j} are*
$(x^{p^{n-j-1}}+1)(x^{p^{n-j}}-1)(x^{2p^{n-j}}+1)(x^{4p^{n-j}}+1)(1+x^8+\ldots+x^{8p^{n-j}(p^j-1)})$,
$(x^{p^{n-j-1}}-1)(x^{p^{n-j}}+1)(x^{2p^{n-j}}+1)(x^{4p^{n-j}}+1)(1+x^8+\ldots+x^{8p^{n-j}(p^j-1)})$ respectively.
(iii) *The generating polynomials for $M_{p^j}\oplus M_{2p^j}\oplus M_{\lambda p^j}\oplus M_{\mu p^j}\oplus M_{\nu p^j}\oplus M_{\chi p^j}$ is*
$(x^{2p^{n-j-1}}+1)(x^{4p^{n-j-1}}+1)(x^{2p^{n-j}}-1)(1+x^8+\ldots+x^{8p^{n-j}(p^j-1)})$.

Proof. **(i)** The minimal polynomials for $\alpha^0, \alpha^{p^n}, \alpha^{2p^n}, \alpha^{3p^n}, \alpha^{4p^n}, \alpha^{5p^n}, \alpha^{6p^n}$ and α^{7p^n} are $(x-1), (x-\delta), (x-\beta), (x-\delta_1), (x+1), (x+\delta), (x+\beta)$ and $(x+\delta_1)$ respectively, where $\delta = \alpha^{p^n}$ and $\delta_1 = \alpha^{3p^n}$. $(1+x+x^2+\ldots+x^{8p^n-1}), (x^4-1)(x^2+\beta)(x+\delta)(1+x^8+\ldots+x^{8(p^n-1)})$,
$(x^6-x^4+x^2-1)(x+\beta)(1+x^8+\ldots+x^{8(p^n-1)})$, $\frac{x^{8p^n}-1}{x-\delta_1}$,
$(x^7-x^6+x^5-x^4+x^3-x^2+x-1)(1+x^8+\ldots+x^{8(p^n-1)})$,
$(x^4-1)(x^2+\beta)(x-\delta)(1+x^8+\ldots+x^{8(p^n-1)})$,
$(x^4-1)(x^2+\beta)(x-\beta)(1+x^8+\ldots+x^{8(p^n-1)})$ and $\frac{x^{8p^n}-1}{x+\delta_1}$ respectively.
(ii) The minimal polynomials for α^{4p^j} and α^{8p^j} are $\frac{(x^{p^{n-j}}+1)}{(x^{p^{n-j-1}}+1)}$ and $\frac{(x^{p^{n-j}}-1)}{(x^{p^{n-j-1}}-1)}$ respectively. Then, the corresponding generating polynomials are
$(x^{p^{n-j-1}}+1)(x^{p^{n-j}}-1)(x^{2p^{n-j}}+1)(x^{4p^{n-j}}+1)(1+x^8+\ldots+x^{8p^{n-j}(p^j-1)})$
and $(x^{p^{n-j-1}}-1)(x^{p^{n-j}}+1)(x^{2p^{n-j}}+1)(x^{4p^{n-j}}+1)(1+x^8+\ldots+x^{8p^{n-j}(p^j-1)})$.

(iii) The product of minimal polynomial satisfied by $\alpha^{p^j}, \alpha^{2p^j}, \alpha^{\lambda p^j}, \alpha^{\mu p^j}, \alpha^{\nu p^j}$ and $\alpha^{\chi p^j}$ is $\frac{(x^{2p^{n-j}}+1)(x^{4p^{n-j}}+1)}{(x^{2p^{n-j-1}}+1)(x^{4p^{n-j-1}}+1)}$. Therefore, the corresponding generating polynomial for $M_{p^j} \oplus M_{2p^j} \oplus M_{\lambda p^j} \oplus M_{\mu p^j} \oplus M_{\nu p^j} \oplus M_{\chi p^j}$ is $(x^{2p^{n-j-1}}+1)(x^{4p^{n-j-1}}+1)(x^{2p^{n-j}}-1)(1+x^8+...+x^{8p^{n-j}(p^j-1)})$. □

4. Minimum Distance

If l is a cyclic code of length m generated by $g(x)$ and its minimum distance is d, then the code $\bar{l}$ of length mk generated by $g(x)(1+x^m+x^{2m}+...+x^{(k-1)m})$ is a repetition code of l repeated k times and its minimum distance is dk.

Theorem 4.1. *The minimum distance of the codes $M_0, M_{p^n}, M_{2p^n}, M_{3p^n}, M_{4p^n}, M_{5p^n}, M_{6p^n}$ and M_{7p^n} are $8p^n$.*

Proof. Since the generating polynomial for the code M_0 is $(1+x+x^2+...+x^{8p^n-1})$, which is itself a polynomial of length $8p^n$, hence its minimum distance is $8p^n$.
Also, the generating polynomial for the code M_{p^n} is $(x^4-1)(x^2+\beta)(x+\delta)(1+x^8+...+x^{8(p^n-1)})$. If we take a cyclic code of length 8 generated by $(x^4-1)(x^2+\beta)(x+\delta)$, then the minimum distance of this code is 8. Since the cyclic code of length $8p^n$ with generating polynomial $(x^4-1)(x^2+\beta)(x+\delta)(1+x^8+...+x^{8(p^n-1)})$, is a repetition code of length 8 with generating polynomial $(x^4-1)(x^2+\beta)(x+\delta)$, repeated p^n times, therefore its minimum distance is $8p^n$.
Similarly, the minimum distance of each of the cyclic codes $M_{2p^n}, M_{3p^n}, M_{4p^n}, M_{5p^n}, M_{6p^n}$ and M_{7p^n} is $8p^n$. □

Proofs for theorem 4.2 and theorem 4.3 are similar to that of theorem 4.1.

Theorem 4.2. *For $0 \le j \le n-1$,the minimum distance of the cyclic codes M_{4p^j} and M_{8p^j} are $16p^j$.*

Theorem 4.3. *For $0 \le j \le n-1$,the minimum distance of the codes $M_{p^j}, M_{2p^j}, M_{\lambda p^j}, M_{\mu p^j}, M_{\nu p^j}$ and $M_{\chi p^j}$ are greater than equal to $8p^j$.*

5. EXAMPLE

Example 5.1. *cyclic code of length* 24.

Take $p=3, n=1, q=17$. *Then, the q-cyclotomic cosets modulo* 24 *are*

$\Omega_0=\{0\}$, $\Omega_1=\{1,17\}$, $\Omega_2=\{2,10\}$, $\Omega_3=\{3\}$, $\Omega_4=\{4,20\}$,

$\Omega_6=\{6\}$, $\Omega_7=\{7,23\}$, $\Omega_8=\{8,16\}$, $\Omega_9=\{9\}$, $\Omega_{12}=\{12\}$, $\Omega_{13}=\{5,13\}$,

$\Omega_{14}=\{14,22\}$, $\Omega_{15}=\{15\}$, $\Omega_{18}=\{18\}$, $\Omega_{19}=\{11,19\}$, $\Omega_{21}=\{21\}$,

Also, $\beta=4$, $T_0=8$, $R_0=4$, $P_0=2$, $\alpha^3=2$ *and the corresponding primitive idempotents in* $\frac{GF(17)[x]}{<x^{24}-1>}$ *are*

$\Omega_0(x)=\frac{1}{24}[\overline{C}_0+\overline{C}_3+\overline{C}_6+\overline{C}_9+\overline{C}_{12}+\overline{C}_{15}+\overline{C}_{18}+\overline{C}_{21}+\overline{C}_1+\overline{C}_2+\overline{C}_4+\overline{C}_8+\overline{C}_7+\overline{C}_{14}+\overline{C}_{13}+\overline{C}_{19}]$

$\Omega_1(x)=\frac{1}{24}[2\overline{C}_0+\overline{C}_3-8\overline{C}_6-4\overline{C}_9-2\overline{C}_{12}-\overline{C}_{15}+8\overline{C}_{18}+4\overline{C}_{21}-2\overline{C}_1-4\overline{C}_2+\overline{C}_4-\overline{C}_8+8\overline{C}_7+4\overline{C}_{14}+2\overline{C}_{13}-8\overline{C}_{19}]$

$\Omega_2(x)=\frac{1}{24}[2\overline{C}_0-8\overline{C}_3-2\overline{C}_6+8\overline{C}_9+2\overline{C}_{12}-8\overline{C}_{15}-2\overline{C}_{18}+8\overline{C}_{21}-4\overline{C}_1+\overline{C}_2-\overline{C}_4-\overline{C}_8+4\overline{C}_7+\overline{C}_{14}-4\overline{C}_{13}+4\overline{C}_{19}]$

$\Omega_3(x)=\frac{1}{24}[\overline{C}_0-2\overline{C}_3+4\overline{C}_6-8\overline{C}_9-\overline{C}_{12}+2\overline{C}_{15}-4\overline{C}_{18}+8\overline{C}_{21}-8\overline{C}_1-4\overline{C}_2-\overline{C}_4+\overline{C}_8+2\overline{C}_7+4\overline{C}_{14}+8\overline{C}_{13}-2\overline{C}_{19}]$

$\Omega_4(x)=\frac{1}{24}[2\overline{C}_0-2\overline{C}_3+2\overline{C}_6-2\overline{C}_9+2\overline{C}_{12}-2\overline{C}_{15}+2\overline{C}_{18}-2\overline{C}_{21}+\overline{C}_1-\overline{C}_2-\overline{C}_4-\overline{C}_8+\overline{C}_7-\overline{C}_{14}+\overline{C}_{13}+\overline{C}_{19}]$

$\Omega_6(x)=\frac{1}{24}[\overline{C}_0+4\overline{C}_3-\overline{C}_6-4\overline{C}_9+\overline{C}_{12}+4\overline{C}_{15}-\overline{C}_{18}-4\overline{C}_{21}-4\overline{C}_1-\overline{C}_2+\overline{C}_4+\overline{C}_8+4\overline{C}_7-\overline{C}_{14}-4\overline{C}_{13}+4\overline{C}_{19}]$

$\Omega_7(x)=\frac{1}{24}[2\overline{C}_0+4\overline{C}_3+8\overline{C}_6-\overline{C}_9-2\overline{C}_{12}-4\overline{C}_{15}-8\overline{C}_{18}+\overline{C}_{21}+8\overline{C}_1+4\overline{C}_2+\overline{C}_4-\overline{C}_8-2\overline{C}_7-4\overline{C}_{14}-8\overline{C}_{13}+2\overline{C}_{19}]$

$\Omega_8(x)=\frac{1}{24}[2\overline{C}_0+2\overline{C}_3+2\overline{C}_6+2\overline{C}_9+2\overline{C}_{12}+2\overline{C}_{15}+2\overline{C}_{18}+2\overline{C}_{21}-\overline{C}_1-\overline{C}_2-\overline{C}_4-\overline{C}_8-\overline{C}_7-\overline{C}_{14}-\overline{C}_{13}-\overline{C}_{19}]$

$\Omega_9(x)=\frac{1}{24}[\overline{C}_0-8\overline{C}_3-4\overline{C}_6-2\overline{C}_9-\overline{C}_{12}+8\overline{C}_{15}+4\overline{C}_{18}+2\overline{C}_{21}-2\overline{C}_1+4\overline{C}_2-\overline{C}_4+\overline{C}_8-2\overline{C}_7-4\overline{C}_{14}+2\overline{C}_{13}+2\overline{C}_{19}]$

$\Omega_{12}(x)=\frac{1}{24}[\overline{C}_0-\overline{C}_3+\overline{C}_6-\overline{C}_9+\overline{C}_{12}-\overline{C}_{15}+\overline{C}_{18}-\overline{C}_{21}-\overline{C}_1+\overline{C}_2+\overline{C}_4+\overline{C}_8-\overline{C}_7+\overline{C}_{14}-\overline{C}_{13}-\overline{C}_{19}]$

$\Omega_{13}(x)=\frac{1}{24}[2\overline{C}_0-\overline{C}_3-8\overline{C}_6+4\overline{C}_9-2\overline{C}_{12}+\overline{C}_{15}+8\overline{C}_{18}-4\overline{C}_{21}+2\overline{C}_1-4\overline{C}_2+\overline{C}_4-\overline{C}_8-8\overline{C}_7+$

$4\overline{C}_{14}-2\overline{C}_{13}+8\overline{C}_{19}]$

$\Omega_{14}(x)=\frac{1}{24}[2\overline{C}_0+8\overline{C}_3-2\overline{C}_6-8\overline{C}_9+2\overline{C}_{12}+8\overline{C}_{15}-2\overline{C}_{18}-8\overline{C}_{21}+4\overline{C}_1+\overline{C}_2-\overline{C}_4-\overline{C}_8-4\overline{C}_7+\overline{C}_{14}+4\overline{C}_{13}-4\overline{C}_{19}]$

$\Omega_{15}(x)=\frac{1}{24}[\overline{C}_0+2\overline{C}_3+4\overline{C}_6+8\overline{C}_9-\overline{C}_{12}-2\overline{C}_{15}-4\overline{C}_{18}-8\overline{C}_{21}+8\overline{C}_1-4\overline{C}_2-\overline{C}_4+\overline{C}_8-2\overline{C}_7+4\overline{C}_{14}-8\overline{C}_{13}+2\overline{C}_{19}]$

$\Omega_{18}(x)=\frac{1}{24}[\overline{C}_0-4\overline{C}_3-\overline{C}_6+4\overline{C}_9+\overline{C}_{12}-4\overline{C}_{15}-\overline{C}_{18}+4\overline{C}_{21}+4\overline{C}_1-\overline{C}_2+\overline{C}_4+\overline{C}_8-4\overline{C}_7-\overline{C}_{14}+4\overline{C}_{13}-4\overline{C}_{19}]$

$\Omega_{19}(x)=\frac{1}{24}[2\overline{C}_0-4\overline{C}_3+8\overline{C}_6+\overline{C}_9-2\overline{C}_{12}+4\overline{C}_{15}-8\overline{C}_{18}-\overline{C}_{21}-8\overline{C}_1+4\overline{C}_2+\overline{C}_4-\overline{C}_8+2\overline{C}_7-4\overline{C}_{14}+8\overline{C}_{13}-2\overline{C}_{19}]$

$\Omega_{21}(x)=\frac{1}{24}[\overline{C}_0+8\overline{C}_3-4\overline{C}_6+2\overline{C}_9-\overline{C}_{12}-8\overline{C}_{15}+4\overline{C}_{18}-2\overline{C}_{21}+2\overline{C}_1+4\overline{C}_2-\overline{C}_4+\overline{C}_8-8\overline{C}_7-4\overline{C}_{14}-2\overline{C}_{13}+8\overline{C}_{19}]$

Minimal polynomials of $\alpha^0,\alpha^1,\alpha^2,\alpha^3,\alpha^4,\alpha^6,\alpha^7,\alpha^8,\alpha^9,\alpha^{12},\alpha^{13},\alpha^{14},\alpha^{15},\alpha^{18},\alpha^{19}$ *and* α^{21} *are*

$x-1,\ x^2-9x+13,\ x^2-4x-1,\ x-2,\ x^2-x+1,\ x-4,\ x^2-2x+4,\ x^2+x+1,\ x-8,\ x+1,\ x^2-8x+13,\ x^2+4x-1,\ x+2,\ x+4,\ x^2+2x+4$ *and* $x+8$ *respectively.*

The minimal codes $M_0,M_1,M_2,M_3,M_4,M_6,M_7,M_8,M_9,M_{12},M_{13},M_{14},M_{15},M_{18},M_{19}$, *and* M_{21} *of length* 24 *are as follows:*

Code	*Dimension*	*Minimum Distance*	*Generating Polynomial*
M_0	1	24	$1+x+x^2+x^3+x^4+x^5+x^6+x^7+x^8+x^9+x^{10}+x^{11}+x^{12}+x^{13}+x^{14}+x^{15}+x^{16}+x^{17}+x^{18}+x^{19}+x^{20}+x^{21}+x^{22}+x^{23}$
M_1	2	$8\leq d\leq 16$	$13+9x+15x^3+13x^4+16x^6+15x^7+8x^9+16x^{10}+4x^{12}+8x^{13}+2x^{15}+4x^{16}+x^{18}+2x^{19}+9x^{21}+x^{22}$
M_2	2	$8\leq d\leq 16$	$1+13x+13x^3+16x^4+16x^6+4x^7+4x^9+x^{10}+x^{12}+13x^{13}+13x^{15}+16x^{16}+16x^{18}+4x^{19}+4x^{21}+x^{22}$

M_3	*1*	24	$9+13x+15x^2+16x^3+8x^4+4x^5+2x^6+x^7+9x^8+13x^9+15x^{10}+16x^{11}+8x^{12}+4x^{13}+2x^{14}+x^{15}+9x^{16}+13x^{17}+15x^{18}+16x^{19}+8x^{20}+4x^{21}+2x^{22}+x^{23}$
M_4	2	*16*	$-1-x+x^3+x^4-x^6-x^7+x^9+x^{10}-x^{12}-x^{13}+x^{15}+x^{16}-x^{18}-x^{19}+x^{21}+x^{22}$
M_6	*1*	*24*	$13+16x+4x^2+x^3+13x^4+16x^5+4x^6+x^7+13x^8+16x^9+4x^{10}+x^{11}+13x^{12}+16x^{13}+4x^{14}+x^{15}+13x^{16}+16x^{17}+4x^{18}+x^{19}+13x^{20}+16x^{21}+4x^{22}+x^{23}$
M_7	2	$8 \leq d \leq 16$	$4-15x+8x^3+4x^4-x^6-9x^7+15x^9+16x^{10}-4x^{12}-2x^{13}+9x^{15}+13x^{16}-16x^{18}-8x^{19}+2x^{21}+x^{22}$
M_8	2	*16*	$-1+x-x^3+x^4-x^6+x^7-x^9+x^{10}-x^{12}+x^{13}-x^{15}+x^{16}-x^{18}+x^{19}-x^{21}+x^{22}$
M_9	*1*	*24*	$15+4x+9x^2+16x^3+2x^4+13x^5+8x^6+x^7+15x^8+4x^9+9x^{10}+16x^{11}+2x^{12}+13x^{13}+8x^{14}+x^{15}=15x^{16}+4x^{17}+9x^{18}+16x^{19}+2x^{20}+13x^{21}+8x^{22}+x^{23}$
M_{12}	*1*	*24*	$-1+x-x^2+x^3-x^4+x^5-x^6+x^7-x^8+x^9-x^{10}+x^{11}-x^{12}+x^{13}-x^{14}+x^{15}-x^{16}+x^{17}-x^{18}+x^{19}-x^{20}+x^{21}-x^{22}+x^{23}$
M_{13}	2	$8 \leq d \leq 16$	$13+8x+2x^3+13x^4+16x^6+2x^7+9x^9+16x^{10}+4x^{12}+9x^{13}+15x^{15}+4x^{16}+x^{18}+15x^{19}+8x^{21}+x^{22}$
M_{14}	2	$8 \leq d \leq 16$	$1-13x-13x^3+16x^4+16x^6-4x^7-4x^9+x^{10}+x^{12}-13x^{13}-13x^{15}+16x^{16}+16x^{18}-4x^{19}-4x^{21}+x^{22}$

M_{15}	*2*	24	$-9+13x-15x^2+16x^3-8x^4+4x^5-2x^6+x^7-9x^8+13x^9-15x^{10}+16x^{11}-8x^{12}+4x^{13}-2x^{14}+x^{15}-9x^{16}+13x^{17}-15x^{18}+16x^{19}-8x^{20}+4x^{21}-2x^{22}+x^{23}$
M_{18}	*2*	24	$-13+16x-4x^2+x^3-13x^4+16x^5-4x^6+x^7-13x^8+16x^9-4x^{10}+x^{11}-13x^{12}+16x^{13}-4x^{14}+x^{15}-13x^{16}+16x^{17}-4x^{18}+x^{19}-13x^{20}+16x^{21}-4x^{22}+x^{23}$
M_{19}	*2*	$8 \le d \le 16$	$-13+15x-8x^3+4x^4-x^6+9x^7-15x^9+16x^{10}-4x^{12}+2x^{13}-9x^{15}+13x^{16}-16x^{18}+8x^{19}-2x^{21}+x^{22}$
M_{21}	*1*	24	$-15+4x-9x^2+16x^3-2x^4+13x^5-8x^6+x^7-15x^8+4x^9-9x^{10}+16x^{11}-2x^{12}+13x^{13}-8x^{14}+x^{15}-15x^{16}+4x^{17}-9x^{18}+16x^{19}-2x^{20}+13x^{21}-8x^{22}+x^{23}$

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

REFERENCES

[1] S.K. Arora, M. Pruthi, Minimal cyclic codes of length $2p^n$, Finite Fields Appl. 5 (1999), 177-187.

[2] G.K. Bakshi, M. Raka, Minimal cyclic codes of length p^nq, Finite Fields Appl. 9 (2003), 432-448.

[3] S. Batra, S.K. Arora, Minimal quadratic residue cyclic codes of length p^n(p odd prime), J. Appl. Math. Comput. 8 (2001), 531-547.

[4] D.M. Burton, Elementary number theory, Wm. C. Brown Publisher (1989). (Reprinted, New Delhi: University Book Stall, 1990).

[5] V. Pless, Introduction of the theory of error correcting codes, Wiley, New York, (1981).

[6] M. Pruthi, S.K. Arora, Minimal codes of prime-power length, Finite Fields Appl. 3 (1997), 99-113.

[7] J. Singh, S. Ahlawat, Explicit expression of primitive idempotents in FC_{4p^n} and corresponding codes, Far East J. Math. Sci. 103 (2018), 1115-1150.

[8] J. Singh, S. Ahlawat, Some cyclic codes of length $4p^n$ and their minimum distance bounds, Int. J. Appl. Eng. Res. 13 (2018), 16708-16718.

[9] J. Singh, S.K. Arora, Minimal cyclic codes of length $8p^n$ over $GF(q)$, where q is prime power of the form $8k+5$, J. Appl. Math. Comput. 48 (2015), 55-69.

[10] J. Singh, S.K. Arora, S. Chawla, Some cyclic codes of length $8p^n$, Int. J. Pure Appl. Math. 116 (2017), 217-241.