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ON BALANCED 3-EDGE PRODUCT CORDIAL GRAPHS

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Abstract. A k-edge product cordial labelling is a variant of the well-known cordial labelling. In this paper, a balanced k-edge product cordial labelling is suggested and some sufficient conditions for balanced 3-edge product cordial graphs are proved. Moreover, a construction of graphs admitting a balanced 3-edge product cordial labelling is presented.

Keywords: 3-edge product cordial graphs; balanced 3-edge product cordial graphs.

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1. Introduction

We consider finite undirected graphs without loops, multiple edges and isolated vertices. If G is a graph, then $V(G)$ and $E(G)$ stand for the vertex set and the edge set of G, respectively. Cardinalities of these sets are called the *order* and the *size* of G. The set of vertices of G adjacent to a vertex $v \in V(G)$ is denoted by $N_G(v)$. For integers p, q, we denote by $[p, q]$ the set of all integers z satisfying $p \le z \le q$.

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Let $k \geq 2$ be an integer. For a graph G, an edge mapping $f : E(G) \to [0,k-1]$ induces a vertex mapping $f^* : V(G) \to [0,k-1]$ defined by

$$f^*(v) \equiv \prod_{u \in N_G(v)} f(vu) \pmod{k}.$$

We denote by $e_f(i)$ the number of edges of G having label i under f and $v_f(i)$ the number of vertices of G having label i under f^* for each $i \in [0,k-1]$. A mapping $f : E(G) \to [0,k-1]$ is called a *k-edge product cordial* (for short k-EPC) *labelling* of G if

$$|e_f(i) - e_f(j)| \leq 1 \quad \text{and} \quad |v_f(i) - v_f(j)| \leq 1 \quad \text{for all} \quad i, j \in [0,k-1].$$

A graph G is called *k-edge product cordial* (k-EPC) if it admits a k-edge product cordial labelling.

The unicyclic graph is a connected graph with exactly one cycle. The *crown* $C_n \bigodot K_1$ is the graph obtained by joining a pendant edge to each vertex of a cycle C_n. The *armed crown* AC_n is the graph obtained by attaching a path P_2 to each vertex of a cycle C_n. The *wheel* W_n is the graph obtained by connecting a vertex to each vertex of a cycle C_{n-1}. All vertices of C_{n-1} called *rim vertices* join to one vertex called an *apex vertex*. The *helm* H_n is the graph obtained by attaching a pendant edge to each rim vertex of a wheel W_n. Herein, let us recall some results on 2-edge product cordial graphs in [4] that will be referred in the next as follows.

Theorem 1.1. [4] *The cycle C_n is a 2-edge product cordial graph for odd n and not a 2-edge product cordial graph for even n.*

Theorem 1.2. [4] *The tree with order greater than 2 is a 2-edge product cordial graph.*

Corollary 1.3. [4] *The unicyclic graph of odd order is 2-edge product cordial.*

Theorem 1.4. [4] *The crown $C_n \bigodot K_1$ is a 2-edge product cordial graph.*

Theorem 1.5. [4] *The armed crown AC_n is a 2-edge product cordial graph.*

Theorem 1.6. [4] *The Helm H_n is a 2-edge product cordial graph.*

2-edge product cordial graphs were introduced by Vaidya and Barasara and they investigated several results on this concept in [4]. After, k-edge product cordial graphs were put forward by Azaizeh et al. in [1]. Moreover, the graphs admitting a 2-edge product cordial labelling are characterized and the 2-edge product cordiality of broad classes of graphs was studied by Ivančo in [3]. Currently, a balanced 2-edge product cordial labelling was recommended and some sufficient conditions for graphs admitting a balanced 2-edge product cordial labelling were investigated by Inpoonjai in [2]. Moreover, a construction of balanced 2-edge product cordial graphs was also shown in [2].

In this paper, a balanced k-edge product cordial labelling is suggested and some sufficient conditions for graphs admitting a balanced 3-edge product cordial labelling are investigated. Moreover, balanced 3-edge product cordial graphs are constructed.

2. 3-Edge Product Cordial Graphs

Now, we start with recalling an assertion on a 2-edge product cordial labelling of a graph G presented by Ivančo in [3] and then we apply this result for a k-edge product cordial labelling of G as follows.

Observation 2.1. *For an integer $k \geq 2$, let G be a graph with n vertices and m edges. Then a mapping $f : E(G) \to [0, k-1]$ is a k-edge product cordial labelling of G if and only if $e_f(i) \in \{\lfloor \frac{m}{k} \rfloor, \lceil \frac{m}{k} \rceil\}$ and $v_f(i) \in \{\lfloor \frac{n}{k} \rfloor, \lceil \frac{n}{k} \rceil\}$ for all $i \in [0, k-1]$.*

Then, we can find a sufficient condition for a graph constructed from a 2-edge product cordial graph to be 3-edge product cordial.

Theorem 2.2. *Let f be a 2-edge product cordial labelling of a graph G with n vertices and m edges and let u be a vertex of G such that $f^*(u) = 0$. If $|m - n| \leq 1$, then the graph H obtained by joining $\lfloor \frac{m}{2} \rfloor$ pendant edges to a vertex u of G is 3-edge product cordial.*

Proof. Let e_i be a pendant edge incident with a vertex u and let v_i be a pendant vertex incident with e_i for all $i \in [1, \lfloor \frac{m}{2} \rfloor]$. We consider a mapping $g : E(H) \to [0, 2]$ defined by

$$g(e)=\begin{cases} f(e) & : \quad e\in E(G),\\ 2 & : \quad e=e_i, i\in[1,\lfloor\frac{m}{2}\rfloor].\end{cases}$$

Clearly, $g(e_i)=2$ and $g^*(v_i)=2$ for all $i\in[1,\lfloor\frac{m}{2}\rfloor]$. Thus, $e_g(2)=\lfloor\frac{m}{2}\rfloor$ and $v_g(2)=\lfloor\frac{m}{2}\rfloor$. Also, $e_g(0)=e_f(0), e_g(1)=e_f(1), v_g(0)=v_f(0)$ and $v_g(1)=v_f(1)$. Applying Observation 2.1, we obtain that $e_g(0),e_g(1)\in\{\lfloor\frac{m}{2}\rfloor,\lceil\frac{m}{2}\rceil\}$ and $v_g(0),v_g(1)\in\{\lfloor\frac{n}{2}\rfloor,\lceil\frac{n}{2}\rceil\}$. Evidently, $|e_g(i)-e_g(j)|\leq 1$ for all $i,j\in[0,2]$ and $|v_g(0)-v_g(1)|\leq 1$. Since $|m-n|\leq 1$, $m=n, m=n-1$ or $m=n+1$. For $v_g(0)=\lfloor\frac{n}{2}\rfloor$, we consider 3 cases as below.

(i) If $m=n$, then

$$|v_g(0)-v_g(2)|=|\lfloor\frac{n}{2}\rfloor-\lfloor\frac{m}{2}\rfloor|=|\lfloor\frac{n}{2}\rfloor-\lfloor\frac{n}{2}\rfloor|=0.$$

(ii) If $m=n-1$, then

$$|v_g(0)-v_g(2)|=|\lfloor\frac{n}{2}\rfloor-\lfloor\frac{m}{2}\rfloor|=|\lfloor\frac{n}{2}\rfloor-\lfloor\frac{n-1}{2}\rfloor|\leq 1.$$

(iii) If $m=n+1$, then

$$|v_g(0)-v_g(2)|=|\lfloor\frac{n}{2}\rfloor-\lfloor\frac{m}{2}\rfloor|=|\lfloor\frac{n}{2}\rfloor-\lfloor\frac{n+1}{2}\rfloor|\leq 1.$$

For $v_g(0)=\lceil\frac{n}{2}\rceil$, we consider 3 cases as follows.

(i) If $m=n$, then

$$|v_g(0)-v_g(2)|=|\lceil\frac{n}{2}\rceil-\lfloor\frac{m}{2}\rfloor|=|\lceil\frac{n}{2}\rceil-\lfloor\frac{n}{2}\rfloor|\leq 1.$$

(ii) If $m=n-1$, then

$$|v_g(0)-v_g(2)|=|\lceil\frac{n}{2}\rceil-\lfloor\frac{m}{2}\rfloor|=|\lceil\frac{n}{2}\rceil-\lfloor\frac{n-1}{2}\rfloor|\leq 1.$$

(iii) If $m=n+1$, then

$$|v_g(0)-v_g(2)|=|\lceil\frac{n}{2}\rceil-\lfloor\frac{m}{2}\rfloor|=|\lceil\frac{n}{2}\rceil-\lfloor\frac{n+1}{2}\rfloor|=0.$$

These show that $|v_g(0)-v_g(2)| \leq 1$. Similarly, for $v_g(1)=\lfloor \frac{n}{2} \rfloor$ and $v_g(1)=\lceil \frac{n}{2} \rceil$, we can prove that $|v_g(1)-v_g(2)| \leq 1$. This means that g is a 3-edge product cordial labelling of H. Therefore, H is a required graph. □

Next, we immediately have the following results.

Corollary 2.3. *The graph G obtained by joining $\lfloor \frac{n}{2} \rfloor$ pendant edges to a vertex of a cycle C_n for odd n is 3-edge product cordial.*

Proof. Let u be a vertex of C_n incident with $\lfloor \frac{n}{2} \rfloor$ pendant edges. Since C_n of odd order is a 2-edge product cordial graph by Theorem 1.1, there is a 2-edge product cordial labelling f of C_n such that $f^*(u)=0$. Moreover, C_n has n vertices and n edges. Therefore, G is a 3-edge product cordial graph by Theorem 2.2. □

Corollary 2.4. *The graph G obtained by joining $\lfloor \frac{n-1}{2} \rfloor$ pendant edges to a vertex of a tree with order $n>2$ is 3-edge product cordial.*

Proof. Let u be a vertex of a tree with order $n>2$ incident with $\lfloor \frac{n-1}{2} \rfloor$ pendant edges. As the tree is a 2-edge product cordial graph by Theorem 1.2, there exists a 2-edge product cordial labelling f of the tree such that $f^*(u)=0$. Furthermore, the tree has n vertices and $n-1$ edges. Thus, by Theorem 2.2, G is a 3-edge product cordial graph. □

Corollary 2.5. *The graph G obtained by joining $\lfloor \frac{n}{2} \rfloor$ pendant edges to a vertex of a unicyclic graph of odd order n is 3-edge product cordial.*

Proof. Let u be a vertex of a unicyclic graph of odd order n incident with $\lfloor \frac{n}{2} \rfloor$ pendant edges. Since the unicyclic graph is 2-edge product cordial by Corollary 1.3, there is a 2-edge product cordial labelling f of the unicyclic graph such that $f^*(u)=0$. Besides, the unicyclic graph has n vertices and n edges. Hence, by Theorem 2.2, G is a 3-edge product cordial graph. □

Corollary 2.6. *The graph G obtained by joining n pendant edges to a vertex of a cycle C_n of the crown $C_n \bigodot K_1$ is 3-edge product cordial.*

Proof. Let u be a vertex of a cycle C_n of the crown $C_n \bigodot K_1$ incident with n pendant edges. As $C_n \bigodot K_1$ is a 2-edge product cordial graph by Theorem 1.4, there exists a 2-edge product cordial labelling f of $C_n \bigodot K_1$ such that $f^*(u) = 0$. Moreover, $C_n \bigodot K_1$ has $2n$ vertices and $2n$ edges. Therefore, by Theorem 2.2, G is a 3-edge product cordial graph. □

Corollary 2.7. *The graph G obtained by joining $\lfloor \frac{3n}{2} \rfloor$ pendant edges to a vertex of a cycle C_n of the armed crown AC_n is* 3*-edge product cordial.*

Proof. Let u be a vertex of a cycle C_n of the armed crown AC_n incident with $\lfloor \frac{3n}{2} \rfloor$ pendant edges. Since AC_n is a 2-edge product cordial graph by Theorem 1.5, there is a 2-edge product cordial labelling f of AC_n such that $f^*(u) = 0$. Furthermore, AC_n has $3n$ vertices and $3n$ edges. Thus, G admits a 3-edge product cordial labelling by Theorem 2.2. □

3. Balanced 3-Edge Product Cordial Graphs

Here, we add more definition of a k-edge product cordial labelling. A k-edge product cordial labelling $f : E(G) \to [0, k-1]$ of a graph G is called *balanced* if

$$e_f(i) = e_f(j) \quad \text{and} \quad v_f(i) = v_f(j) \quad \text{for all} \quad i, j \in [0, k-1].$$

A graph G is called *balanced k-edge product cordial* (balanced k-EPC) if it admits a balanced k-edge product cordial labelling.

After, we are able to prove the following characterization.

Theorem 3.1. [2] *The graph G is balanced* 2*-edge product cordial if and only if it is* 2*-edge product cordial having both even order and even size.*

Proof. Let f be a balanced 2-edge product cordial labelling of G. Then, $e_f(0) = e_f(1)$ and $v_f(0) = v_f(1)$. Obviously, it is a 2-edge product cordial labelling. Since $|E(G)| = e_f(0) + e_f(1) = 2e_f(0)$ and $|V(G)| = v_f(0) + v_f(1) = 2v_f(0)$, G has both even size and even order.

On the other hand, let G be a graph of even order and even size and let f be a 2-edge product cordial labelling of G. Suppose that $|e_f(0) - e_f(1)| = 1$, then $e_f(0) = e_f(1) + 1$ or $e_f(0) = e_f(1) - 1$. As $|E(G)| = e_f(0) + e_f(1) = e_f(1) + 1 + e_f(1) = 2e_f(1) + 1$ or $|E(G)| = e_f(0) + e_f(1) = e_f(1) - 1 + e_f(1) = 2e_f(1) - 1$, the size is odd, a contradiction. Similarly, suppose that $|v_f(0) - v_f(1)| = 1$, then $v_f(0) = v_f(1) + 1$ or $v_f(0) = v_f(1) - 1$.

Since $|V(G)| = v_f(0)+v_f(1) = v_f(1)+1+v_f(1) = 2v_f(1)+1$ or $|V(G)| = v_f(0)+v_f(1) = v_f(1)-1+v_f(1) = 2v_f(1)-1$, the order is odd, a contradiction. This shows that $e_f(0) = e_f(1)$ and $v_f(0) = v_f(1)$. Therefore, f is a balanced 2-edge product cordial labelling of G. □

Next, using the known findings on 2-edge product cordial graphs in [4] and applying Theorem 3.1, we suddenly have the following assertions.

Corollary 3.2. [2] *The crown $C_n \odot K_1$ is a balanced 2-edge product cordial graph.*

Proof. Since the crown $C_n \odot K_1$ has $2n$ vertices and $2n$ edges, by Theorem 1.4 and Theorem 3.1, it is a desired graph. □

Corollary 3.3. [2] *The armed crown AC_n of even n is a balanced 2-edge product cordial graph.*

Proof. As the order and the size of the armed crown AC_n are equal to $3n$ and $3n$ is an even number for even n, by Theorem 1.5 and Theorem 3.1, AC_n is a required graph. □

Then, we can find some sufficient conditions for some graphs constructed by a 2-edge product cordial graph of both odd order and odd size to be balanced 2-edge product cordial.

Theorem 3.4. [2] *Let f be a 2-edge product cordial labelling of a graph G having both odd order and odd size and let u be a vertex of G such that $f^*(u) = 0$. If $e_f(0) < e_f(1)$ and $v_f(0) < v_f(1)$, then the graph H obtained by joining a pendant edge to a vertex u of G is balanced 2-edge product cordial.*

Proof. Let e_1 be a pendant edge joining a vertex u of G and let w be a pendant vertex incident with e_1. Consider a mapping $g : E(H) \to \{0,1\}$ defined by

$$g(e) = \begin{cases} f(e) & : \quad e \in E(G), \\ 0 & : \quad e = e_1. \end{cases}$$

Clearly, $g(e) = f(e)$ for all $e \in E(G)$, $g(e_1) = 0$, $g^*(v) = f^*(v)$ for all $v \in V(G)$ and $g^*(w) = 0$. Thus, $e_g(0) = e_f(0)+1 = e_f(1) = e_g(1)$ and $v_g(0) = v_f(0)+1 = v_f(1) = v_g(1)$. This means that g is a balanced 2-edge product cordial labelling of H. Therefore, H is an expected graph. □

Notice that we can create a balanced 2-edge product cordial graph from the armed crown AC_n with odd n as the following finding.

Corollary 3.5. [2] *The graph G obtained by joining a pendant edge to a vertex of a cycle C_n of the armed crown AC_n with odd n is balanced 2-edge product cordial.*

Proof. For odd n, it is clear that the armed crown AC_n has $3n$ vertices and $3n$ edges such that $3n$ is also an odd number. Let v_i be a vertex of C_n of AC_n, let u_i be a vertex of AC_n adjacent to v_i and let w_i be a pendant vertex of AC_n adjacent to u_i for all $i \in [1,n]$. Consider a mapping $f : E(AC_n) \to \{0,1\}$ defined by

$$f(e) = \begin{cases} 0 & : \quad e \in E(C_n), \\ 0 & : \quad e = v_iu_i, i \in [1, \lfloor \frac{n}{2} \rfloor], \\ 1 & : \quad e = v_iu_i, i \in [\lfloor \frac{n}{2} \rfloor + 1, n], \\ 1 & : \quad e = u_iw_i, i \in [1,n]. \end{cases}$$

Evidently, $e_f(0) = n + \lfloor \frac{n}{2} \rfloor < n + \lfloor \frac{n}{2} \rfloor + 1 = e_f(1)$. Moreover, $f^*(v_i) = 0$ for all $i \in [1,n]$, $f^*(u_i) = 0$ for all $i \in [1, \lfloor \frac{n}{2} \rfloor]$, $f^*(u_i) = 1$ for all $i \in [\lfloor \frac{n}{2} \rfloor + 1, n]$ and $f^*(w_i) = 1$ for all $i \in [1,n]$. Thus, $v_f(0) = n + \lfloor \frac{n}{2} \rfloor < n + \lfloor \frac{n}{2} \rfloor + 1 = v_f(1)$. Since $|e_f(0) - e_f(1)| = 1$ and $|v_f(0) - v_f(1)| = 1$, f is a 2-edge product cordial labelling of AC_n. By applying Theorem 3.4, G is a balanced 2-edge product cordial graph. □

Theorem 3.6. [2] *Let f be a 2-edge product cordial labelling of a graph G having both odd order and odd size. If $e_f(0) > e_f(1)$ and $v_f(0) > v_f(1)$, then the graph H obtained by joining a pendant edge to a vertex of G is balanced 2-edge product cordial.*

Proof. Let e_1 be a pendant edge joining a vertex of G and let u be a pendant vertex incident with e_1. Consider a mapping $g : E(H) \to \{0,1\}$ defined by

$$g(e) = \begin{cases} f(e) & : \quad e \in E(G), \\ 1 & : \quad e = e_1. \end{cases}$$

Obviously, $g(e) = f(e)$ for all $e \in E(G)$, $g(e_1) = 1$, $g^*(v) = f^*(v)$ for all $v \in V(G)$ and $g^*(u) = 1$. Hence, $e_g(0) = e_f(0) = e_f(1) + 1 = e_g(1)$ and $v_g(0) = v_f(0) = v_f(1) + 1 = v_g(1)$. That is, g is a balanced 2-edge product cordial labelling of H. Thus, H is a desired graph. □

We can see that the Helm H_n is a 2-edge product cordial graph by Theorem 1.6, but it is not balanced 2-edge product cordial for both even n and odd n. However, a balanced 2-edge product cordial graph is able to construct from the helm H_n with even n as the following assertion.

Corollary 3.7. [2] *The graph G obtained by joining a pendant edge to a vertex of the helm H_n with even n is balanced 2-edge product cordial.*

Proof. For even n, it is obvious that the helm H_n has odd order $2n-1$ and odd size $3n-3$. Let x be an apex vertex of W_n of H_n, let v_i be a rim vertex of W_n of H_n and let u_i be a pendant vertex of H_n adjacent to v_i for all $i \in [1, n-1]$. Consider a mapping $f : E(H_n) \to \{0, 1\}$ defined by

$$f(e) = \begin{cases} 0 & : \quad e \in E(C_{n-1}), \\ 0 & : \quad e = xv_i, i \in [1, \frac{n}{2}], \\ 1 & : \quad e = xv_i, i \in [\frac{n}{2}+1, n-1], \\ 1 & : \quad e = v_iu_i, i \in [1, n-1]. \end{cases}$$

Evidently, $e_f(0) = \frac{3n}{2} - 1 > \frac{3n}{2} - 2 = e_f(1)$. Moreover, $f^*(x) = 0, f^*(v_i) = 0$ for all $i \in [1, n-1]$ and $f^*(u_i) = 1$ for all $i \in [1, n-1]$. Hence, $v_f(0) = n > n-1 = v_f(1)$. As $|e_f(0) - e_f(1)| = 1$ and $|v_f(0) - v_f(1)| = 1$, f is a 2-edge product cordial labelling of H_n. By applying Theorem 3.6, G is a balanced 2-edge product cordial graph. □

Now, the following result for a balanced k-edge product cordial graph is obvious.

Observation 3.8. *For an integer $k \geq 2$, let G be a graph with kn vertices and km edges. Then a mapping $f : E(G) \to [0, k-1]$ is a balanced k-edge product cordial labelling of G if and only if $e_f(i) = m$ and $v_f(i) = n$ for all $i \in [0, k-1]$.*

Corollary 3.9. *The armed crown AC_n is a balanced 3-edge product cordial graph.*

Proof. Let v_i be a vertex of C_n of AC_n, let u_i be a vertex of AC_n adjacent to v_i and let w_i be a pendant vertex of AC_n adjacent to u_i for all $i \in [1,n]$. Consider a mapping $f : E(AC_n) \to [0,2]$ defined by

$$f(e) = \begin{cases} 0 & : \quad e \in E(C_n), \\ 2 & : \quad e = v_iu_i, i \in [1,n], \\ 1 & : \quad e = u_iw_i, i \in [1,n]. \end{cases}$$

Evidently, $e_f(0) = e_f(1) = e_f(2) = n$. Moreover, $f^*(v_i) = 0, f^*(u_i) = 2$ and $f^*(w_i) = 1$ for all $i \in [1,n]$. Hence, $v_f(0) = v_f(1) = v_f(2) = n$. By applying Observation 3.8, AC_n is a balanced 3-edge product cordial graph. □

Next, we are able to find a sufficient condition for a balanced 3-edge product cordial graph constructed from a balanced 2-edge product cordial graph, which its order is the same as its size, as below.

Theorem 3.10. *Let f be a balanced 2-edge product cordial labelling of a graph G with $2n$ vertices and $2n$ edges and let u be a vertex of G such that $f^*(u) = 0$. Then the graph H obtained by joining n pendant edges to a vertex u of G is balanced 3-edge product cordial.*

Proof. Let e_i be a pendant edge incident with a vertex u and let v_i be a pendant vertex incident with e_i for all $i \in [1,n]$. We consider a mapping $g : E(H) \to [0,2]$ defined by

$$g(e) = \begin{cases} f(e) & : \quad e \in E(G), \\ 2 & : \quad e = e_i, i \in [1,n]. \end{cases}$$

Clearly, $g(e_i) = 2$ and $g^*(v_i) = 2$ for all $i \in [1,n]$. Thus, $e_g(2) = n$ and $v_g(2) = n$. Also, $e_g(0) = e_f(0) = e_f(1) = e_g(1)$ and $v_g(0) = v_f(0) = v_f(1) = v_g(1)$. By Observation 3.8, we obtain that $e_g(0) = e_g(1) = n$ and $v_g(0) = v_g(1) = n$. This means that g is a balanced 3-edge product cordial labelling of H. Therefore, H is a required graph. □

Corollary 3.11. *The graph G obtained by joining n pendant edges to a vertex of a cycle C_n of the crown $C_n \odot K_1$ is balanced 3-edge product cordial.*

Proof. Let u be a vertex of a cycle C_n of the crown $C_n \bigodot K_1$ incident with n pendant edges. As $C_n \bigodot K_1$ is a balanced 2-edge product cordial graph by Corollary 3.2, there exists a balanced 2-edge product cordial labelling f of $C_n \bigodot K_1$ such that $f^*(u) = 0$. Moreover, $C_n \bigodot K_1$ has $2n$ vertices and $2n$ edges. Therefore, by Theorem 3.10, G is a balanced 3-edge product cordial graph. □

Corollary 3.12. *The graph G obtained by joining $\frac{3n}{2}$ pendant edges to a vertex of a cycle C_n of the armed crown AC_n with even n is balanced 3-edge product cordial.*

Proof. For even n, let u be a vertex of a cycle C_n of the crown AC_n incident with $\frac{3n}{2}$ pendant edges. Since AC_n is a balanced 2-edge product cordial graph by Corollary 3.3, there is a balanced 2-edge product cordial labelling f of AC_n such that $f^*(u) = 0$. Furthermore, $C_n \bigodot K_1$ has $3n$ vertices and $3n$ edges. Therefore, by Theorem 3.10, G is a balanced 3-edge product cordial graph. □

After, we are able to obtain some sufficient conditions for some graphs constructed by a 2-edge product cordial graph, which its odd order is similar to its odd size, to be balanced 3-edge product cordial.

Theorem 3.13. *Let f be a 2-edge product cordial labelling of a graph G with $2n-1$ vertices and $2n-1$ edges and let u be a vertex of G such that $f^*(u) = 0$. If $e_f(0) < e_f(1)$ and $v_f(0) < v_f(1)$, then the graph H obtained by joining $n+1$ pendant edges to a vertex u of G is balanced 3-edge product cordial.*

Proof. Let e_i be pendant edges joining a vertex u of G and let w_i be a pendant vertex incident with e_i for all $i \in [1, n+1]$. Consider a mapping $g : E(H) \to [0,2]$ defined by

$$g(e) = \begin{cases} f(e) & : \quad e \in E(G), \\ 0 & : \quad e = e_1, \\ 2 & : \quad e = e_i, i \in [2, n+1]. \end{cases}$$

Clearly, $g(e) = f(e)$ for all $e \in E(G)$, $g(e_1) = 0, g(e_i) = 2$ for all $i \in [2, n+1]$, $g^*(v) = f^*(v)$ for all $v \in V(G), g^*(w_1) = 0$ and $g^*(w_i) = 2$ for all $i \in [2, n+1]$. Thus, $e_g(0) = e_f(0) + 1 = e_f(1) =$

$e_g(1) = e_g(2) = n$ and $v_g(0) = v_f(0)+1 = v_f(1) = v_g(1) = v_g(2) = n$. This means that g is a balanced 3-edge product cordial labelling of H. Therefore, H is an expected graph. □

Corollary 3.14. *The graph G obtained by joining $\frac{3n+1}{2}+1$ pendant edges to a vertex of a cycle C_n of the armed crown AC_n with odd n is balanced 3-edge product cordial.*

Proof. For odd n, it is clear that the armed crown AC_n has $3n$ vertices and $3n$ edges such that $3n$ is also an odd number. Let v_i be a vertex of C_n of AC_n, let u_i be a vertex of AC_n adjacent to v_i and let w_i be a pendant vertex of AC_n adjacent to u_i for all $i \in [1,n]$. Consider a mapping $f : E(AC_n) \to \{0,1\}$ defined by

$$f(e) = \begin{cases} 0 & : \quad e \in E(C_n), \\ 0 & : \quad e = v_iu_i, i \in [1, \lfloor \frac{n}{2} \rfloor], \\ 1 & : \quad e = v_iu_i, i \in [\lfloor \frac{n}{2} \rfloor + 1, n], \\ 1 & : \quad e = u_iw_i, i \in [1,n]. \end{cases}$$

Evidently, $e_f(0) = n + \lfloor \frac{n}{2} \rfloor < n + \lfloor \frac{n}{2} \rfloor + 1 = e_f(1)$. Moreover, $f^*(v_i) = 0$ for all $i \in [1,n]$, $f^*(u_i) = 0$ for all $i \in [1, \lfloor \frac{n}{2} \rfloor]$, $f^*(u_i) = 1$ for all $i \in [\lfloor \frac{n}{2} \rfloor + 1, n]$ and $f^*(w_i) = 1$ for all $i \in [1,n]$. Thus, $v_f(0) = n + \lfloor \frac{n}{2} \rfloor < n + \lfloor \frac{n}{2} \rfloor + 1 = v_f(1)$. Since $|e_f(0) - e_f(1)| = 1$ and $|v_f(0) - v_f(1)| = 1$, f is a 2-edge product cordial labelling of AC_n. By applying Theorem 3.13, G is a balanced 3-edge product cordial graph. □

Theorem 3.15. *Let f be a 2-edge product cordial labelling of a graph G with $2n-1$ vertices and $2n-1$ edges and let u be a vertex of G such that $f^*(u) = 0$. If $e_f(0) > e_f(1)$ and $v_f(0) > v_f(1)$, then the graph H obtained by joining $n+1$ pendant edges to a vertex u of G is balanced 3-edge product cordial.*

Proof. Let e_i be pendant edges joining a vertex u of G and let w_i be a pendant vertex incident with e_i for all $i \in [1, n+1]$. Consider a mapping $g : E(H) \to [0,2]$ defined by

$$g(e) = \begin{cases} f(e) & : \quad e \in E(G), \\ 1 & : \quad e = e_1, \\ 2 & : \quad e = e_i, i \in [2, n+1]. \end{cases}$$

Clearly, $g(e) = f(e)$ for all $e \in E(G)$, $g(e_1) = 1, g(e_i) = 2$ for all $i \in [2, n+1]$, $g^*(v) = f^*(v)$ for all $v \in V(G), g^*(w_1) = 1$ and $g^*(w_i) = 2$ for all $i \in [2, n+1]$. Hence, $e_g(0) = e_f(0) = e_f(1) + 1 = e_g(1) = e_g(2) = n$ and $v_g(0) = v_f(0) = v_f(1) + 1 = v_g(1) = v_g(2) = n$. This means that g is a balanced 3-edge product cordial labelling of H. Therefore, H is a desired graph. □

Corollary 3.16. *Let G be a graph obtained by joining 2 pendant edges to each vertex of a cycle C_n with odd n. Then the graph H obtained by joining $\frac{3n+1}{2} + 1$ pendant edges to a vertex of C_n of G is balanced 3-edge product cordial.*

Proof. For odd n, it is obvious that G has $3n$ vertices and $3n$ edges such that $3n$ is also an odd number. Let v_i be a vertex of C_n of G for all $i \in [1, n]$ and let u_i, w_i be two vertices of G adjacent to v_i for all $i \in [1, n]$. Consider a mapping $f : E(G) \to \{0, 1\}$ defined by

$$f(e) = \begin{cases} 0 & : \quad e \in E(C_n), \\ 0 & : \quad e = v_i u_i, i \in [1, \lceil \frac{n}{2} \rceil], \\ 1 & : \quad e = v_i u_i, i \in [\lceil \frac{n}{2} \rceil + 1, n], \\ 1 & : \quad e = v_i w_i, i \in [1, n]. \end{cases}$$

Clearly, $e_f(0) = n + \lceil \frac{n}{2} \rceil > n + \lceil \frac{n}{2} \rceil - 1 = e_f(1)$. Besides, $f^*(v_i) = 0$ for all $i \in [1, n]$, $f^*(u_i) = 0$ for all $i \in [1, \lceil \frac{n}{2} \rceil]$, $f^*(u_i) = 1$ for all $i \in [\lceil \frac{n}{2} \rceil + 1, n]$ and $f^*(w_i) = 1$ for all $i \in [1, n]$. Thus, $v_f(0) = n + \lceil \frac{n}{2} \rceil > n + \lceil \frac{n}{2} \rceil - 1 = v_f(1)$. As $|e_f(0) - e_f(1)| = 1$ and $|v_f(0) - v_f(1)| = 1$, f is a 2-edge product cordial labelling of G. By applying Theorem 3.15, H is a balanced 3-edge product cordial graph. □

Then, we can see that the following characterization for a balanced 3-edge product cordial graph is evident.

Theorem 3.17. *The graph G is balanced 3-edge product cordial if and only if it is 3-edge product cordial such that 3 is a divisor of both $|V(G)|$ and $|E(G)|$.*

Proof. Let f be a balanced 3-edge product cordial labelling of G. Then, $e_f(0) = e_f(1) = e_f(2)$ and $v_f(0) = v_f(1) = v_f(2)$. Obviously, it is a 3-edge product cordial labelling. Since $|E(G)| = e_f(0) + e_f(1) + e_f(2) = 3e_f(0)$ and $|V(G)| = v_f(0) + v_f(1) + v_f(2) = 3v_f(0)$, 3 is a divisor of both $|V(G)|$ and $|E(G)|$.
On the other hand, let f be a 3-edge product cordial labelling of G such that 3 is a divisor of $|V(G)|$ and $|E(G)|$. Suppose that $|e_f(0) - e_f(1)| = 1$, then $e_f(0) = e_f(1) + 1$ or $e_f(0) = e_f(1) - 1$. Since $|E(G)| = e_f(0) + e_f(1) + e_f(2) = e_f(0) + e_f(0) - 1 + e_f(0) = 3e_f(0) - 1$, $|E(G)| = e_f(0) + e_f(1) + e_f(2) = e_f(1) + 1 + e_f(1) + e_f(1) = 3e_f(1) + 1$, $|E(G)| = e_f(0) + e_f(1) + e_f(2) = e_f(0) + e_f(0) + 1 + e_f(0) = 3e_f(0) + 1$ or $|E(G)| = e_f(0) + e_f(1) + e_f(2) = e_f(1) - 1 + e_f(1) + e_f(1) = 3e_f(1) - 1$, 3 is not a divisor of the size, a contradiction. By the same way, we can check that $|e_f(i) - e_f(j)| \neq 1$ for all $i, j \in [0,2]$. Similarly, Suppose that $|v_f(0) - v_f(1)| = 1$, then $v_f(0) = v_f(1) + 1$ or $v_f(0) = v_f(1) - 1$. Since $|V(G)| = v_f(0) + v_f(1) + v_f(2) = v_f(0) + v_f(0) - 1 + v_f(0) = 3v_f(0) - 1$, $|V(G)| = v_f(0) + v_f(1) + v_f(2) = v_f(1) + 1 + v_f(1) + v_f(1) = 3v_f(1) + 1$, $|V(G)| = v_f(0) + v_f(1) + v_f(2) = v_f(0) + v_f(0) + 1 + v_f(0) = 3v_f(0) + 1$ or $|V(G)| = v_f(0) + v_f(1) + v_f(2) = v_f(1) - 1 + v_f(1) + v_f(1) = 3v_f(1) - 1$, 3 is not a divisor of the order, a contradiction. Likewise, we are able to prove that $|v_f(i) - v_f(j)| \neq 1$ for all $i, j \in [0,2]$. This shows that $e_f(0) = e_f(1) = e_f(2)$ and $v_f(0) = v_f(1) = v_f(2)$. Therefore, f is a balanced 3-edge product cordial labelling of G. □

For the last result, a construction of graphs admitting a balanced 3-edge product cordial labelling is presented.

Theorem 3.18. *For a connected graph G of order $n \geq 3$ and size m, there is a balanced 3-edge product cordial graph constructed from G.*

Proof. Let v_i be a vertex of G for all $i \in [1,n]$. Since G is a connected graph, $m \geq n - 1$. Thus, we consider 3 cases as follows.
(i) If $m = n - 1$, then G is a tree. Thus, there exist at least two pendant vertices v_j, v_k of G for some $j, k \in [1,n]$. Let H be a graph obtained by joining two pendant edges $e_i, e_i^{'}$ to each vertex v_i

of G for all $i \in [1,n]$ and adding an edge e_1 incident with vertices v_j, v_k. Let $u_i, u_i^{'}$ be two pendant vertices incident with $e_i, e_i^{'}$ for all $i \in [1,n]$, respectively. Consider a mapping $f : E(H) \rightarrow [0,2]$ defined by

$$f(e) = \begin{cases} 0 & : \quad e \in E(G), \\ 0 & : \quad e = e_1 = v_j v_k, \\ 1 & : \quad e = e_i, i \in [1,n], \\ 2 & : \quad e = e_i^{'}, i \in [1,n]. \end{cases}$$

Clearly, $f^*(v_i) = 0$, $f^*(u_i) = 1$ and $f^*(u_i^{'}) = 2$ for all $i \in [1,n]$. Since $e_f(0) = m+1 = e_f(1) = e_f(2)$ and $v_f(0) = n = v_f(1) = v_f(2)$, H is a balanced 3-edge product cordial graph.

(ii) If $m = n$, then let H be a graph obtained by joining two pendant edges $e_i, e_i^{'}$ to each vertex v_i of G for all $i \in [1,n]$. Let $u_i, u_i^{'}$ be two pendant vertices incident with $e_i, e_i^{'}$ for all $i \in [1,n]$, respectively. Consider a mapping $f : E(H) \rightarrow [0,2]$ defined by

$$f(e) = \begin{cases} 0 & : \quad e \in E(G), \\ 1 & : \quad e = e_i, i \in [1,n], \\ 2 & : \quad e = e_i^{'}, i \in [1,n]. \end{cases}$$

Evidently, $f^*(v_i) = 0$, $f^*(u_i) = 1$ and $f^*(u_i^{'}) = 2$ for all $i \in [1,n]$. As $e_f(0) = m = e_f(1) = e_f(2)$ and $v_f(0) = n = v_f(1) = v_f(2)$, H admits a balanced 3-edge product cordial labelling.

(iii) If $m > n$, then let G_1 be a graph obtained by joining two pendant edges $e_i, e_i^{'}$ to each vertex v_i of G for all $i \in [1,n]$. Let $u_i, u_i^{'}$ be two pendant vertices incident with $e_i, e_i^{'}$ of G_1 for all $i \in [1,n]$, respectively. Now, a mapping $f : E(G_1) \rightarrow [0,2]$ is defined by

$$f(e) = \begin{cases} 0 & : \quad e \in E(G), \\ 1 & : \quad e = e_i, i \in [1,n], \\ 2 & : \quad e = e_i^{'}, i \in [1,n]. \end{cases}$$

Clearly, $f^*(v_i) = 0$, $f^*(u_i) = 1$ and $f^*(u_i^{'}) = 2$ for all $i \in [1,n]$. Hence, $e_f(0) = m, e_f(1) = e_f(2) = n$ and $v_f(0) = n = v_f(1) = v_f(2)$.

After, we construct H_1 by attaching two edges $e_{h1}, e_{h1}^{'}$ incident with different two vertices of G_1 having labels 1 and 2. Consider a mapping $g_1 : E(H_1) \to [0,2]$ defined by

$$g_1(e) = \begin{cases} f(e) & : \quad e \in E(G_1), \\ 1 & : \quad e = e_{h1}, \\ 2 & : \quad e = e_{h1}^{'}. \end{cases}$$

It is easy to see that $e_{g_1}(0) = m, e_{g_1}(1) = e_{g_1}(2) = n+1$ and $v_{g_1}(0) = n = v_{g_1}(1) = v_{g_1}(2)$.
We create H_2 by adding two edges $e_{h2}, e_{h2}^{'}$ incident with different two vertices of H_1 having labels 1 and 2. Consider a mapping $g_2 : E(H_2) \to [0,2]$ defined by

$$g_2(e) = \begin{cases} g_1(e) & : \quad e \in E(H_1), \\ 1 & : \quad e = e_{h2}, \\ 2 & : \quad e = e_{h2}^{'}. \end{cases}$$

We can see that $e_{g_2}(0) = m, e_{g_2}(1) = e_{g_2}(2) = n+2$ and $v_{g_2}(0) = n = v_{g_2}(1) = v_{g_2}(2)$.
By the same way, we can construct the graphs $H_3, H_4, ..., H_{m-n}$. Consider a mapping $g_{m-n} : E(H_{m-n}) \to [0,2]$ defined by

$$g_{m-n}(e) = \begin{cases} g_{m-n-1}(e) & : \quad e \in E(H_{m-n-1}), \\ 1 & : \quad e = e_{h(m-n)}, \\ 2 & : \quad e = e_{h(m-n)}^{'}. \end{cases}$$

Obviously, $e_{g_{m-n}}(0) = m = e_{g_{m-n}}(1) = e_{g_{m-n}}(2)$ and $v_{g_{m-n}}(0) = n = v_{g_{m-n}}(1) = v_{g_{m-n}}(2)$. Thus, H_{m-n} is a balanced 3-edge product cordial graph. □

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CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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