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EXISTENCE AND EXTREMAL SOLUTIONS IN BANACH ALGEBRAS FOR A FRACTIONAL ORDER DIFFERENTIAL EQUATION

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Abstract. The existence of a solution in Banach algebras for a fractional order nonlinear quadratic differential equation with initial value condition is investigated in this paper. Furthermore, we demonstrate that the solutions to this equation are appealing locally. The primary conclusion is established using basic hybrid fixed point theory methods for three operators. Under certain monotonicity criteria, existence theorems for extremal solutions can also be established. Finally, we provide a specific example to demonstrate our findings.

Keywords: fractional order quadratic differential equation; fixed point theorem; locally attractivitty and extremal solutions; Banach Space.

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1. Introduction

Fractional differential equations (FDEs) appear in a variety of scientific and engineering disciplines because the mathematical analysis of systems and processes in the fields of physics, aerodynamics of complex media, and others requires fractional order derivatives of the order [1,5,9]. Many writers have recently examined fractional Order differential equations (FODE) from two perspectives: the theoretical characteristics of solution existence and uniqueness, and

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the analytic and numerical methods for solving them. FDEs are very useful for describing the hereditary features of different materials and processes. As a result, the subject of fractional differential equations is becoming increasingly popular. See $[3,7,10]$ and the references therein for some recent developments on the subject. In the area of nonlinear analysis and associated applications, fractional order nonlinear differential equations play an essential role. Dealing with nonlinear differential equations can be done in a variety of ways.

Fixed point theory is a crucial aspect of nonlinear analysis. For the existence of a solution to a fractional order nonlinear differential equation, I employed the fixed point approach. This approach has been demonstrated to handle a wide range of nonlinear problems successfully, quickly, and precisely. Several fixed point theorems are now used in nonlinear differential and integral equations applications. The fixed point theorem is chosen based on the data that is provided. With the given initial conditions, we investigate the FNQDE.

$$D^{\alpha}\left[\frac{x(t)}{f(t,x(t),x(\delta(t)))}\right]+\lambda\left[\frac{x(t)}{f(t,x(t),x(\delta(t)))}\right]=g(t,x(t),x(\delta(t))),t\in\mathbb{R} \tag{1}$$

$$x(t_0)=x_0,x(\delta(t))=z_0\in\mathbb{R} \tag{2}$$

$$f(t_0,x(t_0),x(\delta(t_0)))=f(t_0,x_0,z_0)\in\mathbb{R}. \tag{3}$$

for $\lambda>0\in\mathbb{R},\alpha\in(0,1)$

Where, $\phi:\mathbb{R}_+\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}-\{0\}$ and $\psi:\mathbb{R}_+\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ are continuous functions.

By a solution of Fractional Order Nonlinear Quadratic Differential Equation (1) we mean a function $x\in\mathscr{C}(\mathbb{R}_+,\mathbb{R})$ such that:

(i) The function $t\to\left[\frac{x(t)}{f(t,x(t),x(\delta(t)))}\right]$ is continuous for each $x\in\mathbb{R}$.

(ii)x satisfies $(1),(2)$ and (3)

2. Preliminaries

We provide definitions, notation, hypothesis, and preliminary tools in this area, which will be utilised in the next section.

Let $\mathbb{X} = \mathscr{C}(\mathbb{R}_+, \mathbb{R})$ be the space of continuous real valued function on $\mathbb{R}_+$ and Ω be a subset of $\mathbb{X}$. Let a mapping $\mathbb{A}\colon \mathbb{X} \to \mathbb{X}$ be an operator and consider the following operator equation in $\mathbb{X}$, namely,

$$x(t) = (\mathbb{A}x)(t),\ \forall\, t\ \in \mathbb{R}_+ \tag{4}$$

Below we give some different characterization of the solutions for operator equation(4) on $\mathbb{R}_+$. We need the following definitions.

Definition 2.1. [4] We say that solution of the equation (4) are locally attractive if there exists a closed ball $\overline{B_r(0)}$ in the space $\mathscr{AC}(\mathbb{R}_+, \mathbb{R})$ and for some real number $r > 0$ such that for arbitrary solution $x = x(t)$ and $y = y(t)$ of equation (4) belonging to $\overline{B_r(0)} \cap \Omega$ we have that $\lim_{t\to\infty}(x(t) - y(t)) = 0$.

Definition 2.2.[2] Let $\mathbb{X}$ be a Banach space. A mapping $\mathbb{A} : \mathbb{X} \to \mathbb{X}$ is called Lipschitz if there is a constant $\alpha > 0$ such that, $\|\mathbb{A}x - \mathbb{A}y\| \le \alpha\|x - y\|$ for all $x, y \in \mathbb{X}$. If $\alpha < 1$, then $\mathbb{A}$ is called a contraction on $\mathbb{X}$ with the contraction constant α.

Definition 2.3.[8] An operator $\mathbb{Q}$ on a Banach space $\mathbb{X}$ into itself is called compact if for any bounded subset S of $\mathbb{X}, \mathbb{Q}(S)$ is relatively compact subset of $\mathbb{X}$. If $\mathbb{Q}$ is continuous and compact, then it is called completely continuous on $\mathbb{X}$.

Definition 2.4.[2](Dugunji and Granas) Let $\mathbb{X}$ be a Banach space with the norm $\|\cdot\|$ and let $\mathbb{Q} : \mathbb{X} \to \mathbb{X}$, be an operator (in general nonlinear). Then $\mathbb{Q}$ is called

i. Compact if $\mathbb{Q}(X)$ is relatively compact subset of $\mathbb{X}$.

ii. Totally compact if $\mathbb{Q}(S)$ is totally bounded subset of $\mathbb{X}$ for any bounded subset S of $\mathbb{X}$.

iii. Completely continuous if it is continuous and totally bounded operator on $\mathbb{X}$.

It is clear that every compact operator is totally bounded but the converse need not be true. We recall the basic definitions of fractional calculus which are useful in what follows.

Definition 2.5.[1] The Riemann - Liouville fractional derivative of order $\alpha > 0, n-1 < \alpha < n, n \in \mathscr{N}$ with lower limit zero for a function f is defined as

$$D^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t \frac{f(s)}{(t-s)^\alpha} ds \quad ,t > 0$$

Such that $D^{-\alpha}f(t)=I^{\alpha}f(t)=\frac{1}{\Gamma(\alpha)}\int_0^t\frac{f(s)}{(t-s)^{1-\alpha}}ds$ respectively.

Definition 2.6. [9] The Riemann-Liouville fractional integral of order $\xi>0, n-1<\xi<n, n\in\mathscr{N}$ with lower limit zero for a function f is defined by the formula:

$$I^{\xi}f(t)=\frac{1}{\Gamma(\xi)}\int_0^t\frac{f(s)}{(t-s)^{1-\xi}}ds,\quad t>0$$

,

where $\Gamma(\xi)$ denote the Euler gamma function. The Riemann-Liouville fractional derivative operator of order ξ defined by $D^{\alpha}=\frac{d^{\xi}}{dt}=\frac{d}{dt}\circ I^{1-\xi}$.

Theorem 2.7.[2] (A rzela-Ascoli Theorem) If every uniformly bounded and equicontinuous sequence $\{f_n\}$ of functions in $\mathscr{C}(\mathbb{R}_+,\mathbb{R})$, then it has a convergent subsequence.

Theorem 2.8.[2] A metric space X is compact iff every sequence in X has a convergent subsequence.

We employ a new hybrid fixed pint theorem proved by Dhage [2] which is the main tool in the existence theorem of solutions of FNQDE.

Theorem 2.9.[2] : Let S be a non-empty, bounded and closed-convex subset of the Banach space $\mathbb{X}$ and let A: $\mathbb{X}\to\mathbb{X}$ and $\mathbb{B}:S\to\mathbb{X}$ are two operators satisfying:

a) $\mathbb{A}$ is Lipschitz with a lipschitz constant α

b) $\mathbb{B}$ is completely continuous, and

c) $\mathbb{A}x\mathbb{B}x\in S$ for all $x\in S$, and

d) $\alpha M<1$ where $M=\|\mathbb{B}(S)\|:\sup\{\|\mathbb{B}x\|:x\in S\}$.Then the operator equation $\mathbb{A}x\mathbb{B}x=x$ has a solution in S.

Existence Theory:

We seek the solution of (4) in the space $\mathscr{C}(\mathbb{R}_+,\mathbb{R},\mathbb{R})$ of continuous and real-valued function defined on $\mathbb{R}_+$. Define a standard norm $\|\cdot\|$ and a multiplication "$\cdot$" in $\mathscr{C}(\mathbb{R}_+,\mathbb{R})$ by,

$\|x\|=\sup\{|x(t)|:t\in\mathbb{R}_+\},(xy)(t)=x(t)y(t),t\in\mathbb{R}_+$

Clearly, $\mathscr{C}(\mathbb{R}_+,\mathbb{R})$ becomes a Banach space with respect to the above norm and the multiplication in it.

Definition 2.10. [2] : A mapping $g:\mathbb{R}_+\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is Caratheodory if:

i) $t \to g(t,x,y)$ is measurable for each $x, y \in \mathbb{R}$ and

ii) $(x) \to g(t,x,y)$is continuous almost everywhere for $t \in \mathbb{R}_+$.

Furthermore a Caratheodory function g is $\mathscr{L}^1$ -Caratheodory if:

iii) For each real number $r > 0$ there exists a function $h_r \in \mathscr{L}^1(\mathbb{R}_+, \mathbb{R})$ such that $|g(t,x,y)| \leq h_r(t)$ a.e. $t \in \mathbb{R}_+$ for all $x, y \in \mathbb{R}$ with $|x|_r \leq r, |y|_r \leq r$.

Finally a caratheodory function g is $\mathscr{L}^1_{\mathbb{X}}$ -caratheodory if:

iv) There exists a function $h \in \mathscr{L}^1(\mathbb{R}_+, \mathbb{R})$ such that $|g(t,x,y)| \leq h(t), \quad$ a.e. $t \in \mathbb{R}_+$ for all $x, y \in \mathbb{R}$.

For convenience, the function h is referred to as a bound function for g.

Lemma 2.11. Suppose that $\alpha \in (0,1)$ and the function f, g satisfying FNQDE $(1-3)$. Then x is the solution of the FNQDE $(1-3)$ if and only if it is the solution of integral equation

$$x(t) = f(t,x(t),x(\delta(t))) \left\{ \begin{array}{r} \dfrac{x_0}{f(t_0,x_0,z_0)} - \dfrac{\lambda}{\Gamma(\alpha)} \displaystyle\int_{t_0}^{t} \frac{x(s)}{f(s,x(s),x(\delta(s)))(t-s)^{1-\alpha}} ds \\ + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_{t_0}^{t} \frac{g(s,x(s),x(\delta(s)))}{(t-s)^{1-\alpha}} ds \end{array} \right\} \tag{5}$$

for all $t \in \mathbb{R}_+$

Proof : Integrating equation $(1-3)$ of fractional order α with respect to, we get,

$$\mathfrak{D}^{\xi} I^{\xi} \left[\frac{x(t)}{f(t,x(t),x(\delta(t)))} \right] + \lambda I^{\xi} \left[\frac{x(t)}{f(t,x(t),x(\delta(t)))} \right] = I^{\xi} [g(t,x(t),x(\delta(t)))]$$

$$\left[\frac{x(t)}{f(t,x(t),x(\delta(t)))} \right]_{t_0}^{t} + \lambda I^{\xi} \left[\frac{x(t)}{f(t,x(t),x(\delta(t)))} \right] = I^{\alpha} [g(t,x(t),x(\delta(t)))]$$

$$\frac{x(t)}{f(t,x(t),x(\delta(t)))} - \frac{x(t_0)}{f(t_0,x(t_0),x(\delta(t_0)))} + \frac{\lambda}{\Gamma(\alpha)} \int_{t_0}^{t} \frac{x(s)}{f(s,x(s),x(\delta(s)))} (t-s)^{\alpha-1} ds$$
$$= \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} g(s,x(s),x(\delta(s)))(t-s)^{\alpha-1} ds$$

$$x(t) = f(t,x(t),x(\delta(t))) \left\{ \begin{array}{r} \dfrac{x(t_0)}{f(t_0,x(t_0),x(\delta(t_0)))} - \dfrac{\lambda}{\Gamma(\alpha)} \displaystyle\int_{t_0}^{t} \frac{x(s)}{f(s,x(s),x(\delta(s)))} (t-s)^{\alpha-1} ds \\ + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_{t_0}^{t} \frac{g(s,x(s),x(\delta(s)))}{(t-s)^{1-\alpha}} ds \end{array} \right\}$$

Since $x(t_0) = x_0, x(\delta(t_0)) = z_0 \in \mathbb{R}$ and $f(t_0, x(t_0), x(\delta(t_0))) = f(t_0, x_0, z_0) \in \mathbb{R}$

Conversely differentiate (5) of order α with respetive to, we get,

$$\mathfrak{D}^\alpha \left[\frac{x(t)}{f(t,x(t),x(\delta(t)))}\right] = \mathfrak{D}^\alpha \left\{ \begin{array}{r} \dfrac{x(t_0)}{f(t_0,x(t_0),x(\delta(t_0)))} - \dfrac{\lambda}{\Gamma(\alpha)} \displaystyle\int_{t_0}^{t} \frac{x(s)}{f(s,x(s),x(\delta(s)))} (t-s)^{\alpha-1} ds \\ + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_{t_0}^{t} \frac{g(s,x(s))}{(t-s)^{1-\alpha}} ds \end{array} \right\}$$

$$\mathfrak{D}^\alpha \left[\frac{x(t)}{f(t,x(t),x(\delta(t)))}\right] = \mathfrak{D}^\alpha \left[\frac{x(t_0)}{f(t_0,x(t_0),x(\delta(t_0)))}\right] - \lambda \mathfrak{D}^\alpha I^\alpha \left[\frac{x(t)}{f(t,x(t),x(\delta(t)))}\right]$$

$$+ \mathfrak{D}^\alpha I^\alpha [g(t,x(t),x(\delta(t)))]$$

$$\mathfrak{D}^\alpha \left[\frac{x(t)}{f(t,x(t),x(\delta(t)))}\right] = 0 - \lambda \left[\frac{x(t)}{f(t,x(t),x(\delta(t)))}\right] + [g(t,x(t),x(\delta(t)))]$$

$$\mathfrak{D}^\alpha \left[\frac{x(t)}{f(t,x(t),x(\delta(t)))}\right] + \lambda \left[\frac{x(t)}{f(t,x(t),x(\delta(t)))}\right] = [g(t,x(t),x(\delta(t)))]$$

We need following hypothesis for existence the solution of FNQDE (1).

$(\mathscr{H}_1)$ The function $f : \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} - \{0\}$ is continuous and bounded with bound $\mathbb{F} = \sup_{(t,x(t),x(\delta(t))) \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}} |f(t,x(t),x(\delta(t)))|$. There exist a bounded function $\kappa : \mathbb{R}_+ \to \mathbb{R}$ with bound $\|\kappa\|$ satisfying:

$|f(t,x(t),x(\delta(t))) - f(t,y(t),y(\delta(t)))| \leq \kappa(t) \max\{|x(t)-y(t)|, |x(\delta(t))-y(\delta(t))|\}$ for all $x,y \in \mathbb{R}$

$(\mathscr{H}_2)$ The function $g(t,x,y) = g : \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is satisfying caratheodory condition with continuous function $h(t) : \mathbb{R}_+ \to \mathbb{R}$ such that $g(t,x,y) \leq h(t)\ \forall\ t \in \mathbb{R}_+$ and $x,y \in \mathbb{R}$

$(\mathscr{H}_3)$ The function $f : \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} - \{0\}$ is satisfying caratheodory condition with continuous function $\mathbb{P}(t) : \mathbb{R}_+ \to \mathbb{R}$ such that $\frac{x(t)}{f(t,x(t),x(\delta(t)))} \leq \mathbb{P}(t), \forall\ t \in \mathbb{R}_+$ and $x \in \mathbb{R}$

$(\mathscr{H}_4)$ The function $u,v : \mathbb{R}_+ \to \mathbb{R}$ defined by the formulas $u(t) = \int_0^t \frac{\mathbb{P}(s)}{(t-s)^{1-\alpha}} ds$ and $v(t) = \int_0^t \frac{h(s)}{(t-s)^{1-\alpha}} ds$ is bounded on $\mathbb{R}_+$ and the functions $\mathbb{P}(t), u(t)$ and$v(t)$ vanish at infinity.

Remark 3.1: Note that the $(\mathscr{H}_1)$ and $(\mathscr{H}_4)$ hold, then there exists a constant $K_1, K_2 > 0$ such that $K_1 = \sup\left\{\frac{\lambda u(t)}{\Gamma(\alpha)} : t \in \mathbb{R}_+\right\}$ and $K_2 = \sup\left\{\frac{v(t)}{\Gamma(\alpha)} : t \in \mathbb{R}_+\right\}$ for all $t \in \mathbb{R}_+$ and $\mathbb{P} + K_1 + K_2 = \mathbb{K}$ say.

3. Main Result

In this part, we'll look at the FNQDE (1). The preceding B.D. Karande [2] hybrid fixed point theorem for three operators in Banach algebras $\mathbb{X}$ will be utilised to show the existence of the solution for the following problem (1).

Theorem 3.1. Assume that conditions $(\mathscr{H}_1)-(\mathscr{H}_4)$ hold. Further if $\mathbb{F}\mathbb{K} < r$ and $\mathbb{K}K_1 < 1$, where $\mathbb{K}$ and K_1 is defined in remark (3.1). Then FNQDE (1) has a solution in the spacee $\mathscr{C}(\mathbb{R}_+,\mathbb{R})$, moreover solution of (1) are locally attractive on $\mathbb{R}_+$.

Proof: By a solution of FNQDE (1) we mean a continuous function
$\mathbb{X}:\mathbb{R}_+\to\mathbb{R}$ that satisfies FNQDE (1) on $\mathbb{R}_+$. Set $\mathbb{X}=\mathscr{C}(\mathbb{R}_+,\mathbb{R})$ and define a subset S of X as $S=\{x\in\mathbb{X}:\|x\|\leq r\}$. Wherer satisfies the inequality, $\mathbb{F}\mathbb{K}\leq r$
Let $\mathbb{X}=\mathscr{C}(\mathbb{R}_+,\mathbb{R})$ be a Banach Algebra of all continuous real-valued function on $\mathbb{R}_+$ with the norm,

$$\|x\| = \sup|x(t)|, t\in\mathbb{R}_+ \tag{6}$$

We shall obtain the solution of FNQDE (1) under some suitable conditions involved in (1) Now the FNQDE (1) is equivalent to the FNQIE (5)
Let us define the two mappings $\mathbb{A}:\mathbb{X}\to\mathbb{X}$ *and* $\mathbb{B}:S\to\mathbb{X}$ by,

$$\mathbb{A}x(t) = f(t,x(t),x(\delta(t))), t\in\mathbb{R}_+ \tag{7}$$

$$\begin{aligned}\mathbb{B}x(t) = {} & \frac{x(t_0)}{f(t_0,x(t_0),x(\delta(t_0)))} - \frac{\lambda}{\Gamma(\alpha)}\int_{t_0}^{t}\frac{x(s)}{f(s,x(s),x(\delta(s)))}(t-s)^{\alpha-1}ds \\ & + \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}\frac{g(s,x(s),x(\delta(s)))}{(t-s)^{1-\alpha}}ds, \quad t\in\mathbb{R}_+\end{aligned} \tag{8}$$

Thus from the FNQDE (1) we obtain the operator equation as follows:

$$x(t) = \mathbb{A}x(t)\mathbb{B}x(t), t\in\mathbb{R}_+ \tag{9}$$

If the operator $\mathbb{A}$ and $\mathbb{B}$ satisfy all the hypothesis of theorem (2.9), then the operator equation (9) has a solution on S.

Step I: Firstly we show that $\mathbb{A}$ is Lipschitz on S. Let $x, y \in \overline{B_r(0)}$; then by $(\mathscr{H}_1)$,

$$|\mathbb{A}x(t) - \mathbb{A}y(t)| \leq |f(t, x(t), x(\delta(t))) - f(t, y(t), y(\delta(t)))|$$

$$\leq \kappa(t)|x(t) - y(t)|$$

$\leq \kappa(t)|x(t) - y(t)|$ for all $t \in \mathbb{R}_+, x, y \in S$

Taking suprimum over t we get, $\|\mathbb{A}x - \mathbb{A}y\| \leq \|\kappa\|\|x - y\|$ for all $x, y \in S$ Thus, $\mathbb{A}$ is Lipchitz on S with Lipschitz constant $\|\kappa\|$.

Step II: To show the operator $\mathbb{B}$ is completely continuous on $\mathbb{X}$. Let $\{x_n\}$ be a sequence in S converging to a point x. Then by lebesgue dominated convergence theorem for all $t \in \mathbb{R}_+$, we obtain $\lim_{n\to\infty} \mathbb{B}x_n(t)$

$$= \lim_{n\to\infty} \left\{ \begin{array}{r} \dfrac{x_n(t_0)}{f(t_0, x_n(t_0), x(\delta(t_0)))} - \dfrac{\lambda}{\Gamma(\alpha)} \displaystyle\int_{t_0}^{t} \dfrac{x_n(s)}{f(s, x_n(s)), x_n(\delta(s))} (t-s)^{\alpha-1} ds \\ + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_{t_0}^{t} \dfrac{g(s, x_n(s), x_n(\delta(t)))}{(t-s)^{1-\alpha}} ds \end{array} \right\}$$

$$= \frac{x(t_0)}{f(t_0, x(t_0), x(\delta(t_0)))} - \frac{\lambda}{\Gamma(\alpha)} \int_{t_0}^{t} \frac{x(s)}{f(s, x(s), x(\delta(s)))} (t-s)^{\alpha-1} ds + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} \frac{g(s, x(s), x(\delta(s)))}{(t-s)^{1-\alpha}} ds$$

$$= \mathbb{B}x(t), \forall t \in \mathbb{R}_+$$

This shows that $\mathbb{B}$ is continuous on S.

Next we will prove that the set $\mathbb{B}(S)$ is uniformly bounded in S, for any $x \in S$, we have,

$$|\mathbb{B}x(t)| = \left| \begin{array}{r} \dfrac{x(t_0)}{f(t_0, x(t_0), x(\delta(t_0)))} - \dfrac{\lambda}{\Gamma(\alpha)} \displaystyle\int_{t_0}^{t} \dfrac{x(s)}{f(s, x(s), x(\delta(s)))} (t-s)^{\alpha-1} ds \\ + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_{t_0}^{t} \dfrac{g(s, x(s), x(\delta(s)))}{(t-s)^{1-\alpha}} ds \end{array} \right|$$

$$\leq \left| \frac{x(t_0)}{f(t_0, x(t_0), x(\delta(t_0)))} \right| + \left| -\frac{\lambda}{\Gamma(\alpha)} \int_{t_0}^{t} \frac{x(s)}{f(s, x(s), x(\delta(s)))} (t-s)^{\alpha-1} ds \right|$$
$$+ \left| \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} \frac{g(s, x(s), x(\delta(s)))}{(t-s)^{1-\alpha}} ds \right|$$
$$\leq \mathbb{P}_0 + \frac{\lambda}{\Gamma(\alpha)} \int_{t_0}^{t} \left| \frac{x(s)}{f(s, x(s), x(\delta(s)))} \right| (t-s)^{\alpha-1} ds + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} \frac{|g(s, x(s), x(\delta(s)))|}{(t-s)^{1-\alpha}} ds$$
$$\leq \mathbb{P}_0 + \frac{\lambda}{\Gamma(\alpha)} \int_{t_0}^{t} \frac{\mathbb{P}(s)}{(t-s)^{1-\alpha}} ds + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} \frac{h(s)}{(t-s)^{1-\alpha}} ds$$

Taking supremum over t, we obtain

$$\|\mathbb{B}x\| \leq \mathbb{P}_0 + \frac{\lambda u(t)}{\Gamma(\alpha)} + \frac{v(t)}{\Gamma(\alpha)} \leq \mathbb{P} + K_1 + K_2 = \mathbb{K} \text{ say.}$$

Therefore $\|\mathbb{B}x\| \leq \mathbb{K}$, which shows that $\mathbb{B}$ is uniformly bounded on S. Now we will show that $\mathbb{B}(\mathrm{S})$ is equicontinuous set in $\mathbb{X}$. Let $t_1, t_2 \in \mathbb{R}_+$ with $t_2 > t_1$ and $x \in S$, then we have $|\mathbb{B}x(t_2) - \mathbb{B}x(t_1)|$

$$= \left| \begin{array}{l} \left\{ \begin{array}{r} \dfrac{x(t_0)}{f(t_0, x(t_0), x(\delta(t_0)))} - \dfrac{\lambda}{\Gamma(\alpha)} \displaystyle\int_{t_0}^{t} \dfrac{x(s)}{f(s, x(s), x(\delta(s)))} (t-s)^{\alpha-1} ds \\ + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_{t_0}^{t} \dfrac{g(s, x(s), x(\delta(s)))}{(t-s)^{1-\alpha}} ds \end{array} \right\} \\ - \left\{ \begin{array}{r} \dfrac{x(t_0)}{f(t_0, x(t_0), x(\delta(t_0)))} - \dfrac{\lambda}{\Gamma(\alpha)} \displaystyle\int_{t_0}^{t} \dfrac{x(s)}{f(s, x(s), x(\delta(s)))} (t-s)^{\alpha-1} ds \\ + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_{t_0}^{t} \dfrac{g(s, x(s), x(\delta(s)))}{(t-s)^{1-\alpha}} ds \end{array} \right\} \end{array} \right|$$

$$\leq \left| \frac{x(t_0)}{f(t_0, x(t_0), x(\delta(t_0)))} - \frac{x(t_0)}{f(t_0, x(t_0), x(\delta(t_0)))} \right| + \left| \begin{array}{l} \dfrac{\lambda}{\Gamma(\alpha)} \displaystyle\int_{t_0}^{t} \dfrac{x(s)}{f(s, x(s), x(\delta(s)))} (t-s)^{\alpha-1} ds \\ - \dfrac{\lambda}{\Gamma(\xi)} \displaystyle\int_{t_0}^{t} \dfrac{x(s)}{f(s, x(s), x(\delta(s)))} (t-s)^{\alpha-1} ds \end{array} \right|$$
$$+ \left| \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} \frac{g(s, x(s), x(\delta(s)))}{(t-s)^{1-\alpha}} ds - \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} \frac{g(s, x(s), x(\delta(s)))}{(t-s)^{1-\alpha}} ds \right|$$

$$\leq \left| \frac{\lambda}{\Gamma(\alpha)} \int_{t_0}^{t_2} \mathbb{P}(t_2 - s)^{\alpha-1} ds - \int_{t_0}^{t_1} \mathbb{P}(t_1 - s)^{\alpha-1} ds \right| +$$
$$\frac{\lambda}{\Gamma(\alpha)} \left| \int_{t_0}^{t_2} h(s)(t_2 - s)^{\alpha-1} ds - \int_{t_0}^{t_1} h(s)(t_1 - s)^{\alpha-1} ds \right|$$

$$\leq \frac{\lambda}{\Gamma(\alpha)} \|\mathbb{P}\| \left| \begin{array}{l} \displaystyle\int_{t_0}^{t_2} (t_2 - s)^{\alpha-1} ds - \int_{t_0}^{t_1} (t_1 - s)^{\alpha-1} ds \\ + \displaystyle\int_{t_0}^{t_2} (t_2 - s)^{\alpha-1} ds - \int_{t_0}^{t_1} (t_1 - s)^{\alpha-1} ds \end{array} \right| + \frac{\|h\|}{\Gamma(\alpha)} \left| \begin{array}{l} \displaystyle\int_{t_0}^{t_2} (t_2 - s)^{\alpha-1} ds - \int_{t_0}^{t_1} (t_1 - s)^{\alpha-1} ds \\ + \displaystyle\int_{t_0}^{t_2} (t_2 - s)^{\alpha-1} ds - \int_{t_0}^{t_1} (t_1 - s)^{\alpha-1} ds \end{array} \right|$$

$$\leq \frac{\lambda \|\mathbb{P}\|}{\Gamma(\alpha)} \left(\begin{array}{r} \left| \displaystyle\int_{t_0}^{t_2} (t_2 - s)^{\alpha-1} ds - \int_{t_0}^{t_1} (t_1 - s)^{\alpha-1} ds \right| \\ + \left| \displaystyle\int_{t_1}^{t_2} (t_1 - s)^{\alpha-1} ds \right| \end{array} \right) + \frac{\|h\|}{\Gamma(\alpha)} \left(\begin{array}{r} \left| \displaystyle\int_{t_0}^{t_2} (t_2 - s)^{\alpha-1} ds - \int_{t_0}^{t_1} (t_1 - s)^{\alpha-1} ds \right| \\ + \left| \displaystyle\int_{t_1}^{t_2} (t_1 - s)^{\alpha-1} ds \right| \end{array} \right)$$

$$\leq \frac{\lambda}{\Gamma(\xi)}\|\mathbb{P}\|\left\{\begin{aligned}&\left|\left[\frac{(t_2-s)^\alpha}{-\alpha}\right]_{t_0}^{t_2}-\left[\frac{(t_1-s)^\alpha}{-\alpha}\right]_{t_0}^{t_2}\right|\\&\qquad+\left|\left[\frac{(t_1-s)^\alpha}{-\alpha}\right]_{t_1}^{t_2}\right|\end{aligned}\right\}+\frac{\|h\|}{\Gamma(\alpha)}\left\{\begin{aligned}&\left|\left[\frac{(t_2-s)^\alpha}{-\alpha}\right]_{t_0}^{t_2}-\left[\frac{(t_1-s)^\alpha}{-\alpha}\right]_{t_0}^{t_2}\right|\\&\qquad+\left|\left[\frac{(t_1-s)^\alpha}{-\alpha}\right]_{t_1}^{t_2}\right|\end{aligned}\right\}$$

$$\leq \frac{\lambda}{\Gamma(\alpha+1)}\|\mathbb{P}\|\left\{\begin{aligned}&\left|\begin{aligned}&-\left[(t_2-t_2)^\alpha-(t_2-t_0)^\alpha\right]\\&-\left[(t_1-t_2)^\alpha-(t_1-t_0)^\alpha\right]\end{aligned}\right|\\&+\left|-\left[(t_1-t_2)^\alpha-(t_2-t_2)^\xi\right]\right|\end{aligned}\right\}+\frac{\|h\|}{\Gamma(\alpha+1)}\left\{\begin{aligned}&\left|\begin{aligned}&-\left[(t_2-t_2)^\alpha-(t_2-t_0)^\alpha\right]\\&-\left[(t_1-t_2)^\alpha-(t_1-t_0)^\alpha\right]\end{aligned}\right|\\&+\left|-\left[(t_1-t_2)^\alpha-(t_2-t_2)^\alpha\right]\right|\end{aligned}\right\}$$

$$\leq\left\{\frac{\lambda}{\Gamma(\alpha+1)}\|\mathbb{P}\|+\frac{\|h\|}{\Gamma(\alpha+1)}\right\}\left\{|(t_2-t_0)^\alpha-(t_1-t_0)^\alpha|\right\}\to 0$$

as $t_1\to t_2, \forall n\in\mathbb{N}$. Implies $\mathbb{B}$ is equicontinuous. Therefore by Arzela Ascoli theorem that $\mathbb{B}$ is completely continuous operator on s.

Step III: To show $x=\mathbb{A}x\mathbb{B}y \implies x\in S, \forall y\in S$ Let $x\in\mathbb{X}$ and $y\in S$ such that $x=\mathbb{A}x\mathbb{B}y$ By assumptions $(\mathscr{H}_1,\mathscr{H}_2,\mathscr{H}_3)$

$$\begin{aligned}|x(t)|&=|\mathbb{A}x(t)\mathbb{B}x(t)|\\&\leq||\mathbb{A}x(t)|||\mathbb{B}x(t)|\end{aligned}$$

$$\leq |f(t,x(t),x(\delta(t)))|\left|\begin{aligned}&\frac{x(t_0)}{f(t_0,x(t_0),x(\delta(t_0)))}-\frac{\lambda}{\Gamma(\xi)}\int_{t_0}^t\frac{x(s)}{f(s,x(s),x(\delta(s)))}(t-s)^{\alpha-1}ds\\&\qquad+\frac{1}{\Gamma(\xi)}\int_{t_0}^t\frac{g(s,x(s),x(\delta(s)))}{(t-s)^{1-\alpha}}ds\end{aligned}\right|$$

$$\leq\left\{\begin{aligned}&\mathbb{F}\left|\frac{x(t_0)}{f(t_0,x(t_0)x(\delta(t_0)))}\right|+\left|-\frac{\lambda}{\Gamma(\xi)}\int_{t_0}^t\frac{x(s)}{f(s,x(s),x(\delta(s)))}(t-s)^{\alpha-1}ds\right|\\&\qquad+\left|\frac{1}{\Gamma(\alpha)}\int_{t_0}^t\frac{g(s,x(s),x(\delta(s)))}{(t-s)^{1-\alpha}}ds\right|\end{aligned}\right\}$$

$$\leq\mathbb{F}\left\{\mathbb{P}_0+\frac{\lambda}{\Gamma(\xi)}\int_{t_0}^t\left|\frac{x(s)}{f(s,x(s),x(\delta(s)))}\right|(t-s)^{\alpha-1}ds+\frac{1}{\Gamma(\alpha)}\int_{t_0}^t\frac{|g(s,x(s),x(\delta(s)))|}{(t-s)^{1-\alpha}}ds\right\}$$

Taking supremum over t, we obtain

$$\leq\mathbb{F}\left\{\mathbb{P}_0+\frac{\lambda}{\Gamma(\alpha)}\int_{t_0}^t\frac{\mathbb{P}(s)}{(t-s)^{1-\alpha}}ds+\frac{1}{\Gamma(\xi)}\int_{t_0}^t\frac{h(s)}{(t-s)^{1-\alpha}}ds\right\}$$

$$\leq\mathbb{F}\left\{\mathbb{P}_0+\frac{\lambda u(t)}{\Gamma(\alpha)}+\frac{v(t)}{\Gamma(\xi)}\right\}\leq\mathbb{F}\left\{\mathbb{P}_0+K_1+K_2\right\}=\mathbb{F}\mathbb{K}\leq r$$

Therefore $\|x\| \leq \mathbb{F}\mathbb{K} \leq r$.

That is we have, $\|x\| = \|\mathbb{A}x\mathbb{B}x\| \leq r, \forall x \in S$.

Hence assumption (c) of theorem (2.9) is proved.

Step IV: Also we have $M = \|\mathbb{B}(s)\| = \sup\{\|\mathbb{B}x\|$

$$= \sup\left\{\sup_{t\in\mathbb{R}_+}\left[\left|\begin{array}{r}\dfrac{x(t_0)}{f(t_0,x(t_0),x(\delta(t_0)))} - \dfrac{\lambda}{\Gamma(\alpha)}\displaystyle\int_{t_0}^{t}\dfrac{x(s)}{f(s,x(s),x(\delta(s)))}(t-s)^{\alpha-1}ds \\ +\dfrac{1}{\Gamma(\alpha)}\displaystyle\int_{t_0}^{t}\dfrac{g(s,x(s),x(\delta(s)))}{(t-s)^{1-\alpha}}ds\end{array}\right|\right]\right\}$$

$$\leq \sup\left\{\sup_{t\in\mathbb{R}_+}\left\{\begin{array}{c}\left|\dfrac{x(t_0)}{f(t_0,x(t_0),x(\delta(t_0)))}\right| + \left|-\dfrac{\lambda}{\Gamma(\xi)}\displaystyle\int_{t_0}^{t}\dfrac{x(s)}{f(s,x(s),x(\delta(s)))}(t-s)^{\alpha-1}ds\right| \\ +\left|\dfrac{1}{\Gamma(\alpha)}\displaystyle\int_{t_0}^{t}\dfrac{g(s,x(s),x(\delta(s)))}{(t-s)^{1-\alpha}}ds\right|\end{array}\right\}\right\}$$

$$\leq \sup_{t\in\mathbb{R}_+}\left\{\mathbb{P}_0 + \frac{\lambda}{\Gamma(\alpha)}\int_{t_0}^{t}\left|\frac{x(s)}{f(s,x(s),x(\delta(s)))}\right|(t-s)^{\alpha-1}ds + \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}\frac{|g(s,x(s),x(\delta(s)))|}{(t-s)^{1-\alpha}}ds\right\}$$

$$\leq \sup_{t\in\mathbb{R}_+}\left\{\mathbb{P}_0 + \frac{\lambda}{\Gamma(\alpha)}\int_{t_0}^{t}\frac{\mathbb{P}(s)}{(t-s)^{1-\alpha}}ds + \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}\frac{h(s)}{(t-s)^{1-\alpha}}ds\right\}$$

taking supremum over t, we obtain

$$\leq \left\{\mathbb{P}_0 + \frac{\lambda u(t)}{\Gamma(\alpha)} + \frac{v(t)}{\Gamma(\alpha)}\right\}$$

$$\leq \{\mathbb{P}_0 + K_1 + K_2\} = \mathbb{K}$$

and therefore $MK = \mathbb{K}K < 1$.

Thus the condition (d) of theorem (2.9) is satisfied.

Hence all the conditions of theorem (2.9) are satisfied and therefore the operator equation $\mathbb{A}x\mathbb{B}x = x$ has a solution in. As a result, the FNQDE (1) has a solution defined on $\mathbb{R}_+$.

Step V: Finally we have to show that the locally attractivity of the solution for FNQDE (1).

Let x and y be two solutions of FNQDE (1) in S defined on $\mathbb{R}_+$. Then we have

$$|x(t)-y(t)| = \left|\begin{array}{l}\left\{[f(t,x(t),x(\delta(t)))]\left[\begin{array}{l}\frac{x(t_0)}{f(t_0,x(t_0),x(\delta(t_0)))} - \frac{\lambda}{\Gamma(\alpha)}\int_{t_0}^{t}\frac{x(s)}{f(s,x(s),x(\delta(s)))}(t-s)^{\alpha-1}ds \\ +\frac{1}{\Gamma(\xi)}\int_{t_0}^{t}\frac{g(s,x(s),x(\delta(s)))}{(t-s)^{1-\alpha}}ds\end{array}\right]\right\} \\ -\left\{[f(t,y(t),y(\delta(t)))]\left[\begin{array}{l}\frac{y(t_0)}{f(t_0,y(t_0),x(\delta(t_0)))} - \frac{\lambda}{\Gamma(\alpha)}\int_{t_0}^{t}\frac{y(s)}{f(s,y(s),y(\delta(s)))}(t-s)^{\alpha-1}ds \\ +\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}\frac{g(s,y(s),x(\delta(s)))}{(t-s)^{1-\alpha}}ds\end{array}\right]\right\}\end{array}\right|$$

$$\leq \left\{ |f(t,x(t),x(\delta(t)))| \left[\begin{array}{l} \left|\frac{x(t_0)}{f(t_0,x(t_0),x(\delta(t_0)))}\right| + \left| -\frac{\lambda}{\Gamma(\xi)} \int_{t_0}^{t} \frac{x(s)}{f(s,x(s),x(\delta(s)))}(t-s)^{\alpha-1} ds \right| \\ + \left| \frac{1}{\Gamma(\xi)} \int_{t_0}^{t} \frac{g(s,x(s),x(\delta(s)))}{(t-s)^{1-\alpha}} ds \right| \end{array} \right] \right\}$$

$$+ \left\{ |f(t,y(t),y(\delta(t_0)))| \left[\begin{array}{l} \left|\frac{y(t_0)}{f(t_0,y(t_0),y(\delta(t_0)))}\right| + \left| -\frac{\lambda}{\Gamma(\xi)} \int_{t_0}^{t} \frac{y(s)}{f(s,y(s),y(\delta(t_0)))}(t-s)^{\alpha-1} ds \right| \\ + \left| \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} \frac{g(s,y(s),y(\delta(t_0)))}{(t-s)^{1-\alpha}} ds \right| \end{array} \right] \right\}$$

$$\leq \mathbb{F}\left\{ \mathbb{P}_0 + \frac{\lambda}{\Gamma(\xi)} \int_{t_0}^{t} \left| \frac{x(s)}{f(s,x(s),x(\delta(s)))} \right| (t-s)^{\alpha-1} ds + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} \frac{|g(s,x(s),x(\delta(s)))|}{(t-s)^{1-\alpha}} ds \right\} +$$

$$\mathbb{F}\left\{ \mathbb{P}_0 + \frac{\lambda}{\Gamma(\alpha)} \int_{t_0}^{t} \left| \frac{y(s)}{f(s,y(s),y(\delta(s)))} \right| (t-s)^{\alpha-1} ds + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} \frac{|g(s,y(s),y(\delta(s)))|}{(t-s)^{1-\alpha}} ds \right\}$$

$$\leq \mathbb{F}\left\{ \mathbb{P}_0 + \frac{\lambda}{\Gamma(\alpha)} \int_{t_0}^{t} \frac{\mathbb{P}(s)}{(t-s)^{1-\alpha}} ds + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} \frac{h(s)}{(t-s)^{1-\alpha}} ds \right\} +$$

$$\mathbb{F}\left\{ \mathbb{P}_0 + \frac{\lambda}{\Gamma(\alpha)} \int_{t_0}^{t} \frac{\mathbb{P}(s)}{(t-s)^{1-\alpha}} ds + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} \frac{h(s)}{(t-s)^{1-\alpha}} ds \right\}$$

$$\leq 2\mathbb{F}\left\{ \mathbb{P}_0 + \frac{\lambda}{\Gamma(\alpha)} \int_{t_0}^{t} \frac{\mathbb{P}(s)}{(t-s)^{1-\alpha}} ds + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} \frac{h(s)}{(t-s)^{1-\alpha}} ds \right\}$$

Taking supremum over t, we obtain

$$\leq \mathbb{F}\left\{ \mathbb{P}_0 + \frac{\lambda u(t)}{\Gamma(\alpha)} + \frac{v(t)}{\Gamma(\alpha)} \right\}$$

Since $\lim_{t\to\infty} v(t) = 0, \lim_{t\to\infty} u(t) = 0, \lim_{t\to\infty} \mathbb{P}(t) = 0$ for $\varepsilon > 0$, there exist a real number $\mathbb{T}' > 0, \mathbb{T}'' > 0$ and $\mathbb{T}''' > 0$ such that $\mathbb{P}_0 \leq \frac{\varepsilon}{6\,\mathrm{F}}, u(t) \leq \frac{\Gamma(\alpha)\varepsilon}{\lambda 6\mathbb{F}}$ and $v(t) \leq \frac{\Gamma(\xi)\varepsilon}{6F}$ for all $t \geq \mathbb{T}^*$, if we choose $\mathbb{T}^* = \max\{\mathbb{T}', \mathbb{T}'', \mathbb{T}''\}$.

Then from above inequality it follows that $|x(t) - y(t)| < \varepsilon$ for all $t \geq \mathbb{T}^*$.

Hence FNQIE (1) has a locally attractive solution on $\mathbb{R}_+$.

4. Existence of Extremal Solutions

A closed and non-empty set $\mathbb{K}$ in a Banach Algebra $\mathbb{X}$ is called a cone if

i. $\mathbb{K} + \mathbb{K} \subseteq \mathbb{K}$

ii. $\lambda\mathbb{K} \subseteq \mathbb{K}$ for $\lambda \in \mathbb{R}, \lambda \geq 0$

iii. $\{-\mathbb{K}\} \cap \mathbb{K} = 0$ where 0 is the zero element of $\mathbb{X}$.

and is called positive cone if

iv. $\mathbb{K} \circ \mathbb{K} \subseteq \mathbb{K}$

And the notation o is a multiplication composition in $\mathbb{X}$

We introduce an order relation $\leq$ in $\mathbb{X}$ as follows.

Let $x,y \in \mathbb{X}$ then $x \leq y$ if and only if $y-x \in \mathbb{K}$. A cone $\mathbb{K}$ is called normal if the norm $\|\cdot\|$ is monotone increasing on $\mathbb{K}$. It is known that if the cone $\mathbb{K}$ is normal in $\mathbb{X}$ then every order-bounded set in $\mathbb{X}$ is normbounded set in $\mathbb{X}$. We equip the space $\mathscr{C}(\mathbb{R}_+,\mathbb{R})$ of continuous real valued function on $\mathbb{R}_+$ with the order relation $\leq$ with the help of cone defined by,

$\mathbb{K} = \{x \in \mathscr{C}(\mathbb{R}_+,\mathbb{R}) : x(t) \geq 0 \;\forall\, t \in \mathbb{R}_+\}$

We well known that the cone $\mathbb{K}$ is normal and positive in $\mathscr{C}(\mathbb{R}_+,\mathbb{R})$. As a result of positivity of the cone $\mathbb{K}$ we have:

Lemma 4.1. [7] Let $p_1,p_2,q_1,q_2 \in \mathbb{K}$ be such that $p_1 \leq q_1$ and $p_2 \leq q_2$ then $p_1p_2 \leq q_1q_2$.

For any $p,q \in \mathbb{X} = \mathscr{C}(\mathbb{R}_+,\mathbb{R}), p \leq q$ the order interval $[p,q]$ is a set in $\mathbb{X}$ given by,

$[p,q] = \{x \in \mathbb{X} : p \leq x \leq q\}$

Definition 4.2. [7] A mapping $G : [p,q] \to \mathbb{X}$ is said to be nondecreasing or monotone increasing if $x \leq y$ implies $Gx \leq Gy$ for all $x,y \in [p,q]$

For proving the existence of extremal solutions of the equations (1) under certain monotonicity conditions by using following fixed pint theorem of Dhage [7].

Theorem 4.3. [7] Let $\mathbb{K}$ be a cone in Banach Algebra $\mathbb{X}$ and let $[p,q] \in \mathbb{X}$. Suppose that $\mathbb{A},\mathbb{B} : [p,q] \to \mathbb{K}$ are two operators such that

a. $\mathbb{A}$ is a Lipschitz with Lipschitz constant α,

b. $\mathbb{B}$ is completely continuous,

c. $\mathbb{A}x\mathbb{B}x \in [p,q]$ for each $x \in [p,q]$ and

d. $\mathbb{A}$ and $\mathbb{B}$ are nondecreasing.

Further if the cone $\mathbb{K}$ is normal and positive then the operator equation $\mathbb{A}x\mathbb{B}x = x$ has the least and greatest positive solution in $[p,q]$ whenever $\alpha M < 1$, where $M = \|\mathbb{B}([p,q])\| = \sup\{\|\mathbb{B}x\| : x \in [p,q]\}$ We need following definitions and hypothesis for existence the extremal solution of FNQDE (1).

Definition 4.4. A function $p \in \mathscr{AC}(\mathbb{R}_+,\mathbb{R})$ is called a lower solution of the FNQDE (1) on $\mathbb{R}_+$ if the function $t \to \frac{p(t)}{f(t,p(t),p(\delta(t)))}$ is continuous and

$$D^{\alpha}\left[\frac{p(t)}{f(t,p(t),p(\delta(t)))}\right] \le g(t,p(t)) - \lambda\left[\frac{p(t)}{f(t,p(t),p(\delta(t)))}\right], a.e., t \in \mathbb{R}_+ \tag{10}$$

$$p(t_0) = p_0 p(\delta(t_0)) = c_0 \in \mathbb{R} \tag{11}$$

$$f(t_0,p(t_0),p(\delta(t_0))) = f(t_0,p_0,c_0). \tag{12}$$

Again a function $q \in \mathscr{AC}(\mathbb{R}_+,\mathbb{R})$ is called an upper solution of the FNQDE (1) on $\mathbb{R}_+$ if function the $t \to \frac{q(t)}{f(t,q(t),q(\delta(t)))}$ is continuous and

$$D^{\alpha}\left[\frac{q(t)}{f(t,q(t),q(\delta(t)))}\right] \le g(t,q(t),q(\delta(t))) - \lambda\left[\frac{q(t)}{f(t,q(t),q(\delta(t)))}\right], a.e., t \in \mathbb{R}_+ \tag{13}$$

$$q(t_0) = q_0, \quad q(\delta(t_0)) = d_0 \tag{14}$$

$$f(t_0,q(t_0), \quad p(\delta(t_0))) = f(t_0,q_0,d_0) \tag{15}$$

Definition 4.5. A solution x_M of the FNQDE (1) is said to be maximal if for any other solution x to FNQDE (1) one has $x(t) \le x_M(t)$ for all t $\in \mathbb{R}_{+.}$ Again a solution x_M of the FNQDE (1) is said to be minimal if $x_M(t) \le x(t)$ for all t $\in \mathbb{R}_+$ where x is any solution of the FNQDE (1) on $\mathbb{R}_+$.

Definition 4.6. (Caratheodory Case): We consider the following set of assumptions: $\mathfrak{B}1$

- $\mathfrak{B}1$ The functions $g(t,x(t),x(\delta(t)))$ and $\frac{x(t)}{f(t,x(t),x(\delta(t)))}$ are Caratheodory.
- $\mathfrak{B}2$ The functions $(t,x(t),x(\delta(t))), g(t,x(t),x(\delta(t)))$ and $\frac{x(t)}{f(t,x(t),x(\delta(t)))}$ are non-decreasing in x almost everywhere for all $t \in \mathbb{R}_+$.
- $\mathfrak{B}3$ The FNQDE (1) has a lower solution p and an upper solution q on $\mathbb{R}_+$ with $p \le q$.
- $\mathfrak{B}4$ The function $l : \mathbb{R}_+,\mathbb{R}$ defined by, $l(t) = |g(t,p(t),p(\delta(t)))| + |g(t,q(t),q(\delta(t)))|$ is Lebesgue measurable.

$\mathfrak{B}5$ The function q: $\mathbb{R}_+, \mathbb{R}$ defined by, $\shortparallel(t) = \left|\frac{p(t)}{f(t,p(t),p(\delta(t)))}\right| + \left|\frac{q(t)}{f(t,q(t),q(\delta(t)))}\right|$ is Lebesgue measurable.

Remark 4.7: Assume that $(\mathfrak{B}1 - \mathfrak{B}5)$ hold. Then $|g(t,x(t),x(\delta(t)))| \leq l(t)$, a.e. $t \in \mathbb{R}_+$, for all $x \in [p,q]$ and $\left|\frac{x(t)}{f(t,x(t),x(\delta(t)))}\right| \leq \shortparallel(t)$

Theorem 4.7. Suppose that the assumptions $(\mathscr{H}_1) - (\mathscr{H}_4)$ and $(\mathfrak{B}1) - (\mathfrak{B}5)$ holds and $l, \shortparallel$ are given in remark (4.7) and $\alpha\left\{\mathbb{P}_0 + \frac{T^\xi}{\Gamma(\xi+1)}\left[\lambda\|\shortparallel\|_{\mathscr{L}^1} + \|l\|_{\mathscr{L}^1}\right] < 1\right.$ hold then FNQDE (1) has a minimal and maximal positive solution on $\mathbb{R}_+$.

Proof: Now FNQDE (1) is equivalent to FNQIE (3.1)on $\mathbb{R}_+$. Let $\mathbb{X} = \mathscr{C}(\mathbb{R}_+, \mathbb{R})$ and define an order relation "\leq" by the cone $\mathbb{K}$ given by (4.7). Clearly $\mathbb{K}$ is a normal cone in $\mathbb{X}$. Define two operators $\mathbb{A}$ and $\mathbb{B}$ on $\mathbb{X}$ by (7) and (8) respectively. Then FNQIE (5) is transformed into an operator equation $\mathbb{A}x\mathbb{B}x = x$ in BanachAlgebraX. Notice that $(\mathfrak{B}1)$ implies $\mathbb{A}, \mathbb{B} : [p,q] \to \mathbb{K}$ Since the cone $\mathbb{K}$ in $\mathbb{X}$ is normal, $[p,q]$ is a norm bounded set in $\mathbb{X}$. Now it is shown, as in the proof of Theorem (3.1), that $\mathbb{A}$ is a Lipschitz with a Lipschitz constant $\|\alpha\|$ and $\mathbb{B}$ is completely continuous operator on $[p,q]$.

Step I: Again the hypothesis ($\mathfrak{B}2$) implies that $\mathbb{A}$ and $\mathbb{B}$ are non-decreasing on $[p,q]$. To see this, let $x, y \in [p,q]$ be such that $x \leq y$. Then by ($\mathfrak{B}2$) we have, $\mathbb{A}x(t) = f(t,x(t),x(\delta(t))) \leq f(t,y(t),y(\delta(t))) \leq \mathbb{A}y(t), \forall\, t \in \mathbb{R}_+$

Similarly,

$$\begin{aligned}
\mathbb{B}x(t) = &\frac{x(t_0)}{f(t_0,x(t_0),x(\delta(t_0)))} - \frac{\lambda}{\Gamma(\alpha)}\int_{t_0}^{t}\frac{x(s)}{f(s,x(s),x(\delta(s)))}(t-s)^{\alpha-1}ds \\
&+\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}\frac{g(s,x(s),x(\delta(s)))}{(t-s)^{1-\alpha}}ds \\
\leq &\frac{y(t_0)}{f(t_0,y(t_0),y(\delta(t_0)))} - \frac{\lambda}{\Gamma(\alpha)}\int_{t_0}^{t}\frac{y(s)}{f(s,y(s),y(\delta(s)))}(t-s)^{\alpha-1}ds \\
&+\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}\frac{g(s,y(s),y(\delta(s)))}{(t-s)^{1-\alpha}}ds \\
\leq &\mathbb{B}y(t), \forall\, t \in \mathbb{R}_+
\end{aligned}$$

Implies that $\mathbb{A}$ and $\mathbb{B}$ are non-decreasing operators on $[p,q]$

Step II: Again definition (4.4) and hypothesis (B3) implies that,

$$p(t) \leq f(t,p(t),p(\delta(t_0))) \left\{ \begin{array}{r} \dfrac{p(t_0)}{f(t_0,p(t_0),p(\delta(t_0)))} - \dfrac{\lambda}{\Gamma(\alpha)} \displaystyle\int_{t_0}^{t} \dfrac{p(s)}{f(s,p(s),p(\delta(s)))} (t-s)^{\alpha-1} ds \\ + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_{t_0}^{t} \dfrac{g(s,p(s),p(\delta(s)))}{(t-s)^{1-\alpha}} ds \end{array} \right\}$$

$$\leq f(t,x(t),x(\delta(t_0))) \left\{ \begin{array}{r} \dfrac{x(t_0)}{f(t_0,x(t_0),x(\delta(t_0)))} - \dfrac{\lambda}{\Gamma(\alpha)} \displaystyle\int_{t_0}^{t} \dfrac{x(s)}{f(s,x(s),x(\delta(s)))} (t-s)^{\alpha-1} ds \\ + \dfrac{1}{\Gamma(\xi)} \displaystyle\int_{t_0}^{t} \dfrac{g(s,x(s),x(\delta(s)))}{(t-s)^{1-\alpha}} ds \end{array} \right\}$$

$$\leq f(t,q(t),q(\delta(s))) \left\{ \begin{array}{r} \dfrac{q(t_0)}{f(t_0,q(t_0),q(\delta(t_0)))} - \dfrac{\lambda}{\Gamma(\alpha)} \displaystyle\int_{t_0}^{t} \dfrac{q(s)}{f(s,q(s),q(\delta(s)))} (t-s)^{\alpha-1} ds \\ + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_{t_0}^{t} \dfrac{g(s,q(s),q(\delta(s)))}{(t-s)^{1-\alpha}} ds \end{array} \right\}$$

$\leq q(t), \forall t \in \mathbb{R}_+$and $x \in [p,q]$

As a result $p(t) \leq \mathbb{A}x(t)\mathbb{B}x(t)$

$\leq q(t), \forall t \in \mathbb{R}_+$and $x \in [p,q]$

Hence $\mathbb{A}x\mathbb{B}x \in [p,q], \forall x \in [p,q]$

StepIII :$M = \|\mathbb{B}([p,q])\| = \sup\{\|\mathbb{B}x\| : x \in [p,q]\}$

$$\leq \sup \left\{ \sup_{t\in\mathbb{R}_+} \left\{ \left| \begin{array}{r} \dfrac{x(t_0)}{f(t_0,x(t_0),x(\delta(t_0)))} - \dfrac{\lambda}{\Gamma(\alpha)} \displaystyle\int_{t_0}^{t} \dfrac{x(s)}{f(s,x(s),x(\delta(s)))} (t-s)^{\alpha-1} ds \\ + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_{t_0}^{t} \dfrac{g(s,x(s),x(\delta(s)))}{(t-s)^{1-\alpha}} ds \end{array} \right| \right\} \right\}$$

$$\leq \sup \left\{ \sup_{t\in\mathbb{R}_+} \left\{ \begin{array}{r} \left| \dfrac{x(t_0)}{f(t_0,x(t_0),x(\delta(t_0)))} \right| + \left| - \dfrac{\lambda}{\Gamma(\alpha)} \displaystyle\int_{t_0}^{t} \dfrac{x(s)}{f(s,x(s),x(\delta(s)))} (t-s)^{\alpha-1} ds \right| \\ + \left| \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_{t_0}^{t} \dfrac{g(s,x(s),x(\delta(s)))}{(t-s)^{1-\alpha}} ds \right| \end{array} \right\} \right\}$$

$$\leq \sup_{t\in\mathbb{R}_+} \left\{ \left\{ \begin{array}{r} \mathbb{P}_0 + \dfrac{\lambda}{\Gamma(\alpha)} \displaystyle\int_{t_0}^{t} \left| \int_{t_0}^{t} \dfrac{x(s)}{f(s,x(s),x(\delta(s)))} \right| (t-s)^{\alpha-1} ds \\ + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_{t_0}^{t} \dfrac{|g(s,x(s),x(\delta(s)))|}{(t-s)^{1-\alpha}} ds \end{array} \right\} \right\}$$

$$\leq \sup_{t\in\mathbb{R}_+}\left\{\left\{\mathbb{P}_0+\frac{\lambda}{\Gamma(\alpha)}\int_{t_0}^{t}\frac{\shortparallel}{(t-s)^{1-\alpha}}ds+\frac{1}{\Gamma(\xi)}\int_{t_0}^{t}\frac{l(s)}{(t-s)^{1-\alpha}}ds\right\}\right\}$$

$$\leq \mathbb{P}_0+\frac{\lambda\|q\|}{\Gamma(\alpha+1)}\left|\left[\frac{(t-s)^{\alpha}}{-\alpha}\right]\right|+\frac{\|l\|}{\Gamma(\alpha)}\left|\left[\frac{(t-s)^{\alpha}}{-\alpha}\right]\right\|$$

$$\leq \mathbb{P}_0+\frac{\lambda\|q\|}{\Gamma(\alpha+1)}\{(t-t)^{\alpha}-(t-t_0)^{\alpha}\}+\frac{1}{\Gamma(\alpha)}\frac{l(s)}{(t-s)^{1-\alpha}}\{(t-t)^{\alpha}-(t-t_0)^{\alpha}\}$$

$$\leq \mathbb{P}_0+\frac{\lambda\|q\|}{\Gamma(\alpha+1)}(t-t_0)^{\alpha}+\frac{\|l\|}{\Gamma(\alpha+1)}(t-t_0)^{\alpha}$$

$$\leq \mathbb{P}_0+\frac{\lambda\|q\|}{\Gamma(\alpha+1)}T^{\alpha}+\frac{\|l\|}{\Gamma(\alpha+1)}T^{\alpha}$$

$$\leq \mathbb{P}_0+\frac{T^{\alpha}}{\Gamma(\alpha+1)}\left[\lambda\|\shortparallel\|_{\mathscr{L}^1}+\|l\|_{\mathscr{L}^1}\right]$$

5. APPLICATION

Example 5.1. Consider the following FNQDE of type (1)

$$\mathfrak{D}^{\frac{1}{2}}\left[\frac{x(t)}{f(t,x(t),x(\delta(t)))}\right]+2\left[\frac{x(t)}{f(t,x(t),x(\delta(t)))}\right]=g(t,x(t),x(\delta(t))),t\in\mathbb{R}_+$$

$x(0)=0$

$f(0,0,0)=(0,0)$ Where the functions $f(t,x(t),x(\delta(t)))=\cos t\left[\frac{x(t)}{1-x(t)}+e^{-t}\right]$, $g(t,x(t),x(\delta(t)))=\frac{1}{t^5(1+x(4t))}, h(t)=\frac{1}{t^5}$ and $\mathbb{P}(t)=\frac{1}{\cos t}$ and $\alpha=\frac{1}{2},\lambda=2$

$(\mathscr{H}_1)$ Now $|f(t,x(t),x(\delta(t)))-f(t,y(t),y(\delta(t)))|$

$$=\left|\left\{\cos t\left[\frac{x(t)}{1-x(t)}+e^{-t}\right]\right\}-\left\{\cos t\left[\frac{x(t)}{1-x(t)}+e^{-t}\right]\right\}\right|$$

$$=\left|\cos t\left[\frac{x(t)}{1-x(t)}-\frac{y(t)}{1-y(t)}\right]\right|.$$

$$\leq|\cos t|\left|\frac{x(t)y(t)+x(t)-y(t)-x(t)y(t)}{x(t)y(t)-x(t)-y(t)+1}\right|$$

$$\leq|\cos t||x(t)-y(t)|$$

$$\leq\kappa(t)|x(t)-y(t)|$$

$$\leq\|\kappa\||x(t)-y(t)|$$

Since $\kappa(t)=\cos t$ say which is continuous and bounded on $\mathbb{R}_+$ has bound $\|\kappa\|$.

$(\mathscr{H}_2)$ Take $h(t)=\frac{1}{t^5}$, it is continuous on $\mathbb{R}_+$.

Implies $g(t,x(t),x(\delta(t))\leq h(t)$ that is $\frac{1}{t^5(1+x(4t))}\leq\frac{1}{t^5}$

Implies g is caratheodory satisfy above condition.

$(\mathscr{H}_3)$ The function $\frac{x(t)}{f(t,x(t),x(\delta(t))}$ is again caratheodory function with continuous function $\mathbb{P}(t)$: $\mathbb{R}_+ \to \mathbb{R}$ such that

$\mathbb{P}(t) = \frac{1}{\cos t}$ and satisfying $\frac{x(t)}{f(t,x(t),x(\delta(t))} \leq \mathbb{P}(t)$

It follows that all the conditions $(\mathscr{H}_1) - (\mathscr{H}_3)$ satisfied. Thus by theorem (2.9) above problem has a solution on $\mathbb{R}_+$.

Example 5.2. Consider the following FNQDE of type (1)

$$\mathfrak{D}^{\frac{1}{2}}\left[\frac{x(t)}{f(t,x(t),x(\delta(t)))}\right] + 2\left[\frac{x(t)}{f(t,x(t),x(\delta(t)))}\right] = g(t,x(t),x(\delta(t))), t \in \mathbb{R}_+$$

$$x(0) = 0$$

$$f(0,0,0) = (0,0)$$

Where the functions $f(t,x(t),x(\delta(t))) = \sin t\left[\frac{x(t)}{1-x(t)} + e^{-t}\right]$,

$g(t,x(t),x(\delta(t))) = \frac{1}{t^2(1+x(2t))}, h(t) = \frac{1}{t^2}$ and $\mathbb{P}(t) = \frac{1}{\cos t}$ and $\alpha = \frac{1}{2}, \lambda = 2$

$(\mathscr{H}_1)$ Now $|f(t,x(t),x(\delta(t))) - f(t,y(t),y(\delta(t)))|$

$$\begin{aligned}
&= \left|\left\{\sin t\left[\frac{x(t)}{1-x(t)} + e^{-t}\right]\right\} - \left\{\sin t\left[\frac{x(t)}{1-x(t)} + e^{-t}\right]\right\}\right| \\
&= \left|\sin t\left[\frac{x(t)}{1-x(t)} - \frac{y(t)}{1-y(t)}\right]\right|. \\
&\leq |\sin t|\left|\frac{x(t)y(t) + x(t) - y(t) - x(t)y(t)}{x(t)y(t) - x(t) - y(t) + 1}\right| \\
&\leq |\sin t||x(t) - y(t)| \\
&\leq \kappa(t)|x(t) - y(t)| \\
&\leq \|\kappa\||x(t) - y(t)|
\end{aligned}$$

Since $\kappa(t) = \sin t$ say which is continuous and bounded on $\mathbb{R}_+$ has bound $\|\kappa\|$.

$(\mathscr{H}_2)$ Take $h(t) = \frac{1}{t^2}$, it is continuous on $\mathbb{R}_+$.

Implies $g(t,x(t),x(\delta(t)) \leq h(t)$ that is $\frac{1}{t^2(1+x(2t))} \leq \frac{1}{t^2}$

Implies g is caratheodory satisfy above condition.

$(\mathscr{H}_3)$ The function $\frac{x(t)}{f(t,x(t),x(\delta(t))}$ is again caratheodory function with continuous function $\mathbb{P}(t)$: $\mathbb{R}_+ \to \mathbb{R}$ such that

$\mathbb{P}(t) = \frac{1}{\sin t}$ and satisfying $\frac{x(t)}{f(t,x(t),x(\delta(t))} \leq \mathbb{P}(t)$

It follows that all the conditions $(\mathscr{H}_1) - (\mathscr{H}_3)$ satisfied. Thus by theorem (2.9) above problem has a solution on $\mathbb{R}_+$.

6. Conclusion

We investigated the existence of a solution to a fractional order nonlinear quadratic differential equation in this study. The result was derived by using Dhage's hybrid fixed point theorem to three operators in Banach space. With the assistance of an example, the primary outcome is vividly shown.

Conflict of Interests

The authors declare that there is no conflict of interests.

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