Available online at http://scik.org
J. Math. Comput. Sci. 2022, 12:137
https://doi.org/10.28919/jmcs/7310
ISSN: 1927-5307

FOOD CHAIN MODEL WITH LOGISTIC GROWTH AND SELECTIVE OPTIMAL HARVESTING UNDER FUZZY ENVIRONMENT

DIPANKAR SADHUKHAN*

Department of Mathematics, Haldia Government College, Haldia, Debhog, Purba Medinipur-721657, West Bengal, India

Abstract: A multispecies food chain harvesting model is formulated based on Lotka-Voltera model with three species which are affected not only by harvesting but also by the presence of prey, predator and the super predator. In order to understand the dynamics of the system, it is assumed that the all three species follows the logistic law of growth. Further, there is demand for prey predator species in the market and hence selective harvesting of two species is performed. We derive the condition for global stability of the system using a suitable Lyapunov function. The possibility of existence of bioeconomic equilibrium is discussed. The optimal harvest policy is studied and the solution is derived under imprecise inflation in fuzzy environment using Pontryagin's maximal principle. Finally some numerical examples are discussed to illustrate the model.

Keywords: prey-predator-super predator; logistic law of growth; fishery; selective optimal harvesting; fuzzy environment.

2010 AMS Subject Classification: 93A30, 91B76.

1. NOTATIONS

i. x, y, z: size of the prey, predator and super-predator populations respectively at time t.

ii. k_1, k_2, k_3 : environmental carrying capacity for prey, predator and super-predator respectively.

iii. $\lambda_1, \lambda_2, \lambda_3$: are the intrinsic growth rate of prey, predator and super predator.

*Corresponding author

E-mail address: dipankar.sadhukhan2@gmail.com

Received February 27, 2022

iv. q_1, q_2: are the catchability coefficients of prey and predator.

v. E : the common catching effort.

vi. α_{12}: predator response rates towards the prey.

vii. α_{21}: the rate of conversion of prey to predator.

viii. α_{23}: super-predator response rates towards the predator.

ix. α_{32}: the rate of conversion of predator to super-predator.

x. c: constant fishing cost per unit effort.

xi. p_1: constant price per unit biomass of prey species.

xii. p_2: constant price per unit biomass of predator species.

xiii. α_{12}, α_{23} : predation coefficients.

xiv. α_{21}, α_{32} : conversion parameters.

2. INTRODUCTION

In recent days the important part of research on biological modelling is the bioeconomic modelling of exploitation of biological resources such as fisheries and forestry's etc. In the literature, there are some single species [2,3] models in fisheries. But, in reality, marine fisheries consist of multi-species of which one may be prey and others predators and super predators which make a complex ecological food chain. Moreover, both prey and predators are eaten by different sections of people in the society and also used as different medicinal ingredients, so all three species have the demand in the market. Thus, the biological as well as economical study of exploitation of multi-species is now an emerging field of research in society. Also as this field of research includes from fisherman to scientist of all subjects, so now-a-days ecological modelling is very vast area for researchers. Initially, Clark [5,6] introduced this type idea with the technique to approaching for the result.

Normally the main objective of the study of multi-species marine fisheries problems is to investigate the conditions/constraints for bionomic equilibrium of the species and to determine the optimum harvesting policy of the species in order to maximize the present value of the revenues earned from them without disturbing the ecological balance amongst the species.

Initially in this field of study Clark [5] first presented an optimal equilibrium policy for the harvesting of two independent species. Later using this concept, Chaudhuri [1, 2, 3] formulated and solved the optimal control problem for combined harvesting of two competing species in deterministic environment. Later Chaudhuri and Saha Ray [4], Mesterton-Gibbons [7], Kar &

Chaudhuri [12] and some others studied the two species prey-predator fishery models for optimal harvesting of both the species. There are only few fisheries models with three species-prey, predator and super predator with harvesting. Recently Kar and Chaudhury [9] considered a model with two competing prey and one predator. Some more investigations on biological food-chain models [8, 13, 14, 15, 16, 17, 18, 19, 20] have also been reported recently. However, till now none considered food chain model with logistic growth and harvesting for three species-one prey, one predator and one super predator with harvesting fuzzy environment. As mentioned above, in the world, there are some communities who eat even super-predators. Moreover, these may now-a-days be used for some other purposes also i.e. for medicines, etc. They also did not consider the optimal harvesting policy in fuzzy environment taking imprecise inflation and discount rates for food-chain system.

In this paper, an optimal harvesting of three species food chain-the first one is a prey, the second is a predator and third is a super predator which feeds on predator is formulated. The logistic growth of all three species is assumed and selective harvesting of prey and predator species is considered. The local stability, global stability and the bioeconomic equilibrium of the system are studied and the necessary conditions/constraints are derived. Taking the inflation and discount into account and considering these to be imprecise in nature *cf.* Maiti and Maiti [10], the problem is formulated as an optimal control problem for maximum return of revenue and solved for optimum harvesting of the species using Pontryagin's maximal principle. Lastly, some numerical experiments and simulations are depicted to illustrate the model.

3. MODEL FORMULATION

Let us consider three marine fish species for example Scoliodon sorakwa (shark), letes calcaifera (bhetki), sardinella longicepts (sardine) which make a food chain system and prey-predator are subjected to harvesting continuously. In this system we assume that the super predator (shark) feeding on predator species only and there is no competition between the species. Here the predator which lives on prey and super predator which lives on predator both these species grow according to the logistic growth along with the prey species (i.e. the population density of each species is resource limited).

With those above assumption the governing equations describing the system is as follows:

$$\frac{dx}{dt} = \lambda_1 x\left(1 - \frac{x}{k_1}\right) - \alpha_{12}xy - Eq_1x$$
$$\frac{dy}{dt} = \lambda_2 y\left(1 - \frac{y}{k_2}\right) + \alpha_{21}xy - \alpha_{23}yz - Eq_2y \qquad (1)$$
$$\frac{dz}{dt} = \lambda_3 z\left(1 - \frac{z}{k_3}\right) + \alpha_{32}yz$$

where $0 \le x \le k_1, 0 \le y \le k_2, 0 \le z \le k_3$ and $\alpha_{12}, \alpha_{21}, \alpha_{23}, \alpha_{32}$ are positive constants. The catch rate functions Eq_1x and Eq_2y are based on CPUE (CATCH-PER-UNIT EF- FORT).

4. THE STEADY STATES

Solving the above system (1) he steady states are obtained and the possible states i.e. points may be assumed as: $P_0(0,0,0)$, $P_1(0,y_1,z_1)$, $P_2(x_2,0,z_2)$, $P_3(x_3,y_3,0)$, $P_4(0,0,z_4)$, $P_5(0,y_5,0)$, $P_6(x_6,0,0)$ and $P_7(x^*,y^*,z^*)$.

The existence of P_0 , P_4 , P_5 and P_6 are obvious and unstable. Therefore to show the existence of other equilibria we check one by one.

STEADY STATE $\mathbf{P_1(0,y_1,z_1)}$:

$$x_1 = 0$$
$$y_1 = \frac{\frac{\lambda_3}{k_3}(\lambda_2 - Eq_2) - \alpha_{23}\lambda_3}{\alpha_{23}\alpha_{32} + \frac{\lambda_2\lambda_3}{k_2k_3}} \qquad (2)$$
$$z_1 = \frac{\alpha_{23}(\lambda_2 - Eq_2) + \frac{\lambda_2\lambda_3}{k_2}}{\alpha_{23}\alpha_{32} + \frac{\lambda_2\lambda_3}{k_2k_3}}$$

The equilibrium point P_1 exist if $(\lambda_2 - Eq_2) > k_3\alpha_{23}$.

STEADY STATE $\boldsymbol{P_2(x_2,0,z_2)}$:

$$x_2 = \frac{k_1}{\lambda_1}(\lambda_1 - Eq_1)$$
$$y_2 = 0 \qquad (3)$$
$$z_2 = k_3$$

The equilibrium point P_2 exist if $(\lambda_1 - Eq_1) > 0$.

STEADY STATE $\boldsymbol{P_3(x_3,y_3,0)}$:

$$x_3 = \frac{-\alpha_{12}(\lambda_2 - Eq_2) + \frac{\lambda_2}{k_2}(\lambda_1 - Eq_1)}{\alpha_{12}\alpha_{21} + \frac{\lambda_1\lambda_2}{k_1k_2}}$$
$$y_3 = \frac{\frac{\lambda_1}{k_1}(\lambda_2 - Eq_2) + \alpha_{21}(\lambda_1 - Eq_1)}{\alpha_{12}\alpha_{21} + \frac{\lambda_1\lambda_2}{k_1k_2}} \qquad (4)$$
$$z_3 = 0$$

In this case the equilibrium will exist if $(\lambda_1 - Eq_1) > 0$, $(\lambda_2 - Eq_2) > 0$ and $\frac{(\lambda_1 - Eq_1)}{(\lambda_2 - Eq_2)} > \frac{\alpha_{12}k_2}{\lambda_2}$.

Therefore the all three equilibrium points P_1, P_2 and P_3 exist together if $(\lambda_1 - Eq_1) > \frac{\alpha_{12}\alpha_{23}k_2k_3}{\lambda_2}$ and $(\lambda_2 - Eq_2) > k_3\alpha_{23}$ holds together.

Now we assume that the positive interior equilibrium point $P_7(x^*, y^*, z^*)$ exist and the point can be obtained by solving the system of equations (1) in the positive octant. So (x^*, y^*, z^*) is the solution of the system of equations

$$\begin{aligned} &\frac{\lambda_1}{k_1}x + \alpha_{12}y = \lambda_1 - Eq_1 \\ &\alpha_{21}x - \frac{\lambda_2}{k_2}y - \alpha_{23}z = -(\lambda_2 - Eq_2) \\ &\alpha_{32}y - \frac{\lambda_3}{k_3}z = -\lambda_3 \end{aligned} \tag{5}$$

Corresponding solution is:

$$\begin{aligned} x^* &= \frac{\left(\alpha_{23}\alpha_{32} + \frac{\lambda_2\lambda_3}{k_2k_3}\right)(Eq_1 - \lambda_1) + \alpha_{12}\frac{\lambda_3}{k_3}(\lambda_2 - Eq_2) - \alpha_{12}\alpha_{23}\lambda_3}{\Delta} \\ y^* &= \frac{\frac{\lambda_1\lambda_3}{k_1k_3}(Eq_2 - \lambda_2) + \alpha_{21}\frac{\lambda_3}{k_3}(Eq_1 - \lambda_1) + \frac{\lambda_1\lambda_3}{k_1}\alpha_{23}}{\Delta} \\ z^* &= \frac{-\lambda_3\left(\alpha_{12}\alpha_{21} + \frac{\lambda_1\lambda_3}{k_1k_3}\right) + \alpha_{32}\frac{\lambda_1}{k_1}(Eq_2 - \lambda_2) + \alpha_{21}\alpha_{32}\lambda_3(Eq_1 - \lambda_1)}{\Delta} \end{aligned} \tag{6}$$

With the coefficient determinant $\Delta = -\frac{\lambda_1}{k_1}\alpha_{23}\alpha_{32} - \frac{\lambda_3}{k_3}\alpha_{12}\alpha_{21} - \frac{\lambda_1}{k_1}\frac{\lambda_2}{k_2}\frac{\lambda_3}{k_3}\ (\neq 0)$ (7)

5. Local Stability

To discuss the local stability of the system first we need to construct the variational matrix $V(x, y, z)$ corresponding to the system (1).

The variational matrix $V(x, y, z)$ is given by:

$$V(x, y, z) = \begin{bmatrix} V_{11} & V_{12} & V_{13} \\ V_{21} & V_{22} & V_{23} \\ V_{31} & V_{32} & V_{33} \end{bmatrix} \tag{8}$$

Where,

$V_{11} = \lambda_1 - \frac{2x\lambda_1}{k_1} - \alpha_{12}y - q_1E$, $V_{22} = \lambda_2 - \frac{2y\lambda_2}{k_2} + \alpha_{21}x - \alpha_{23}z - q_2E$, $V_{33} = \lambda_3 - \frac{2z\lambda_3}{k_3} + \alpha_{32}y$,

$V_{12} = -\alpha_{12}x$,

$V_{13} = 0$, $V_{23} = -\alpha_{23}y$, $V_{21} = \alpha_{21}y$, $V_{31} = 0$ and $V_{32} = \alpha_{32}z$. (9)

For the equilibrium point P_7 the characteristic equation $V(x^*, y^*, z^*)$ is given by: $\mu^3 + a_1\mu^2 + a_2\mu + a_3 = 0$, where, $a_1 = \frac{\lambda_1}{k_1}x^* + \frac{\lambda_2}{k_2}y^* + \frac{\lambda_3}{k_3}z^*$, $a_2 = \frac{\lambda_1\lambda_2}{k_1k_2}x^*y^* + \frac{\lambda_2\lambda_3}{k_2k_3}y^*z^* + \frac{\lambda_1\lambda_3}{k_1k_3}x^*z^* + \alpha_{12}\alpha_{21}x^*y^* + \alpha_{23}\alpha_{32}y^*z^*$ and $a_3 = \left(\frac{\lambda_1}{k_1}\frac{\lambda_2}{k_2}\frac{\lambda_3}{k_3} + \frac{\lambda_1}{k_1}\alpha_{23}\alpha_{32} + \frac{\lambda_3}{k_3}\alpha_{12}\alpha_{21}\right)x^*y^*z^*$.

As in the positive octant $a_1 > 0$, so by Routh-Hurwitz condition P_7 will be stable if and only if $\begin{vmatrix} a_1 & a_3 \\ 1 & a_2 \end{vmatrix} > 0$.

6. BIONOMIC EQUILIBRIUM

The term bionomic equilibrium of a biological system is the combination of biological equilibrium as well as economic equilibrium. As we already know that the biological equilibrium is obtained by solving $\frac{dx}{dt} = \frac{dy}{dt} = \frac{dz}{dt} = 0$. Also the economic equilibrium is said to be achieved when TR (the total revenue obtained by selling the harvested biomass) equals TC (the total cost for the effort devoted to harvesting).

The economic rent (net revenue) at any time given by $\pi(x, y, z, E) = (p_1q_1x + p_2q_2y - c)E$ (10)

And also for system (1) we have

$$x = 0 \ or \ E = \frac{\lambda_1}{k_1} - \frac{\lambda_1 x}{k_1 q_1} - \frac{\alpha_{12} y}{q_1} \tag{11}$$

$$y = 0 \ or \ E = \frac{\lambda_2}{q_2} - \frac{\lambda_2 y}{k_2 q_2} + \frac{\alpha_{21} x}{q_2} - \frac{\alpha_{23} z}{q_2} \tag{12}$$

$$z = 0 \ or \ \lambda_3 - \frac{\lambda_3}{k_3}z + \alpha_{32}y = 0 \tag{13}$$

Equating (11) and (12) we have

$$\left(\frac{\alpha_{21}}{q_2} + \frac{\lambda_1}{k_1q_1}\right)x + \left(\frac{\alpha_{12}}{q_1} - \frac{\lambda_2}{k_2q_2}\right)y - \frac{\alpha_{23}}{q_2}z = \left(\frac{\lambda_1}{q_1} - \frac{\lambda_2}{q_2}\right) \tag{14}$$

So, the bionomic equilibrium is obtained by solving (10), (13) and (14) that is

$$\begin{aligned} &p_1q_1x + p_2q_2y - c = 0. \\ &\lambda_3 - \frac{\lambda_3}{k_3}z + \alpha_{32}y = 0. \\ &\left(\frac{\alpha_{21}}{q_2} + \frac{\lambda_1}{k_1q_1}\right)x + \left(\frac{\alpha_{12}}{q_1} - \frac{\lambda_2}{k_2q_2}\right)y - \frac{\alpha_{23}}{q_2}z = \left(\frac{\lambda_1}{q_1} - \frac{\lambda_2}{q_2}\right) \end{aligned} \tag{15}$$

7. GLOBAL STABILITY

To check the global stability off the system, in this section we prove a theorem by choosing an appropriate Lyapunov function.

Theorem: The interior equilibrium point $P_7(x^*, y^*, z^*)$ is globally asymptotically stable if $\alpha_{12} = \alpha_{21}$ and $\alpha_{23} = \alpha_{32}$.

Proof: Let us consider Lyapunov function of the form $V(x, y, z) = (x - x^*) - x^* \log\frac{x}{x^*} + (y - y^*) - y^* \log\frac{y}{y^*} + (z - z^*) - z^* \log\frac{z}{z^*}$ (16)

This Lyapunov function V is obviously positive definite and continuous $\forall\ x, y, z > 0$.

$$\frac{dV}{dt} = \frac{\partial V}{\partial x}\frac{dx}{dt} + \frac{\partial V}{\partial y}\frac{dy}{dt} + \frac{\partial V}{\partial z}\frac{dz}{dt} \quad (17)$$

After simplification we get

$$\frac{dV}{dt} = -[(x - x^*) \quad (y - y^*) \quad (z - z^*)]\begin{bmatrix} \frac{\lambda_1}{k_1} & \frac{\alpha_{12}-\alpha_{21}}{2} & 0 \\ \frac{\alpha_{12}-\alpha_{21}}{2} & \frac{\lambda_2}{k_2} & \frac{\alpha_{23}-\alpha_{32}}{2} \\ 0 & \frac{\alpha_{23}-\alpha_{32}}{2} & \frac{\lambda_3}{k_3} \end{bmatrix}\begin{bmatrix} (x - x^*) \\ (y - y^*) \\ (z - z^*) \end{bmatrix} = -X^T A X \quad (18)$$

Where $X = \begin{bmatrix} (x - x^*) \\ (y - y^*) \\ (z - z^*) \end{bmatrix}$ and $A = \begin{bmatrix} \frac{\lambda_1}{k_1} & \frac{\alpha_{12}-\alpha_{21}}{2} & 0 \\ \frac{\alpha_{12}-\alpha_{21}}{2} & \frac{\lambda_2}{k_2} & \frac{\alpha_{23}-\alpha_{32}}{2} \\ 0 & \frac{\alpha_{23}-\alpha_{32}}{2} & \frac{\lambda_3}{k_3} \end{bmatrix}$ (19)

Therefore $\frac{dV}{dt} < 0$ if A is positive definite. The matrix A is positive definite if $\alpha_{12} = \alpha_{21}$ and $\alpha_{23} = \alpha_{32}$. (20)

8. OPTIMAL HARVESTING POLICY

Let J is the present value of revenues of a continuous time-stream then J can be expressed as

$$J = \int_0^\infty e^{-\tilde{R}t}\{(p_1 q_1 x + p_2 q_2 y) - c\}E(t)dt \quad (21)$$

Where $\tilde{R} = \tilde{r} - \tilde{k}$, the discount rate of inflation with $\tilde{r}$ and $\tilde{k}$ be the respectively representing discount rates and inflation for the time value of money and these are fuzzy in nature. The harvesting effort $E(t)$ is a control variable satisfies $0 \le E(t) \le E_{max}$, so we can write $[0, E_{max}] = V_t$ is the control set. The fuzzy number $\tilde{R}$ can be expressed as interval number and following Maiti and Maiti [10] and Grazegorzewsky [11], J of (21) can be expressed as

$$Max\left[J_{L,}J_R\ \right] = \int_0^\infty e^{-[R_L,R_R]t}\{(p_1q_1x + p_2q_2y) - c\}E(t)dt \tag{22}$$

Where,

$$J_L = \int_0^\infty e^{-R_R t}\{(p_1q_1x + p_2q_2y) - c\}E(t)dt \tag{23}$$

$$J_R = \int_0^\infty e^{-R_L t}\{(p_1q_1x + p_2q_2y) - c\}E(t)dt \tag{24}$$

$$R_L = r_L - k_R \text{ and } R_R = r_R - k_L, \tilde{r} = [r_L,\ r_R],\ \tilde{k} = [k_L,\ k_R] \tag{25}$$

Subject to the constraints (1). Now using weights w_1 and w_2 with $w_1+w_2 = 1; w_1, w_2 \geq 0$, we have from (22)

$$MaxJ = Max\left[J_{L,}J_R\ \right] = w_1J_L\ + w_2\,J_R \tag{26}$$

Therefor the corresponding Hamiltonian can be written as

$$H = (w_1e^{-R_Rt} + w_2e^{-R_Lt})(p_1q_1x + p_2q_2y - c)E + \mu_1\left(-\frac{\lambda_1}{k_1}x^2 - \alpha_{12}xy + x(\lambda_1 - q_1E)\right)$$
$$+\mu_2\left(-\frac{\lambda_2}{k_2}y^2 + \alpha_{21}xy - \alpha_{23}yz + y(\lambda_2 - q_2E)\right) + \mu_3\left(-\frac{\lambda_3}{k_3}z^2 + \alpha_{32}yz + \lambda_3z\right) \tag{27}$$

Where, μ_1, μ_2 *and* μ_3 are adjoint variables.

The adjoint equations are: $\frac{d\mu_1}{dt} = -\frac{\partial H}{\partial x}\ ,\ \frac{d\mu_2}{dt} = -\frac{\partial H}{\partial y}\ \ and\ \frac{d\mu_3}{dt} = -\frac{\partial H}{\partial z}$ (28)

So,

$$\begin{aligned}
\frac{d\mu_1}{dt} &= \frac{\lambda_1}{k_1}\mu_1x - \alpha_{21}y\mu_2 - p_1q_1E(w_1e^{-R_Rt} + w_2e^{-R_Lt})\\
\frac{d\mu_2}{dt} &= \alpha_{12}x\mu_1 + \frac{\lambda_2}{k_2}y\mu_2 - \alpha_{32}z\mu_3 - p_2q_2E(w_1e^{-R_Rt} + w_2e^{-R_Lt})\\
\frac{d\mu_3}{dt} &= \alpha_{23}y\mu_2 + \frac{\lambda_3}{k_3}z\mu_3
\end{aligned} \tag{29}$$

Now solving the system of linear differential equations we have

$$\mu_1 = A_1e^{m_1t} + A_2e^{m_2t} + A_3e^{m_3t} + \frac{M_{1L}}{N_L}e^{-R_Lt} + \frac{M_{1R}}{N_R}e^{-R_Rt} \tag{30}$$

Where m_1, m_2 & m_3 are the roots of the equation

$$\nu_0m^3 + \nu_1m^2 + \nu_2m + \nu_3 = 0 \tag{31}$$

Where

$$\begin{aligned}
\nu_0 &= 1\\
\nu_1 &= -\left(\frac{\lambda_1}{k_1}x + \frac{\lambda_2}{k_2}y + \frac{\lambda_3}{k_3}z\right)\\
\nu_2 &= \left(\frac{\lambda_1}{k_1}\frac{\lambda_2}{k_2}xy + \frac{\lambda_2}{k_2}\frac{\lambda_3}{k_3}yz + \frac{\lambda_1}{k_1}\frac{\lambda_3}{k_3}xz + \alpha_{23}\alpha_{32}yz + \alpha_{12}\alpha_{21}xy\right)\\
\nu_3 &= \left(\frac{\lambda_1}{k_1}\frac{\lambda_2}{k_2}\frac{\lambda_3}{k_3} + \frac{\lambda_1}{k_1}\alpha_{23}\alpha_{32} + \frac{\lambda_3}{k_3}\alpha_{12}\alpha_{21}\right)xyz.
\end{aligned} \tag{32}$$

Now μ_1 will be bounded if $m_i < 0, (\ i = 1, 2, 3)$ or $A_i = 0\ for\ i = 1, 2, 3$.

The Hurwitz matrix corresponding to the cubic equation (31) can be written as $\begin{bmatrix} \nu_1 & 1 & 0 \\ \nu_3 & \nu_2 & \nu_1 \\ 0 & 0 & \nu_3 \end{bmatrix}$ and

$\Delta_1 = \nu_1,\ \ \Delta_2 = \nu_1\nu_2 - \nu_3,\ \ \Delta_3 = \nu_3(\nu_1\nu_2 - \nu_3)$.

So the roots of the above equation are real negative or complex conjugate with negative real parts if and only if all of Δ_1, Δ_2 and Δ_3 are positive. But since in positive octant $\Delta_1 < 0$, then it is difficult to check for $m_i < 0, (\ i = 1, 2, 3)$ and hence we consider the case $A_i = 0\ for\ i = 1, 2, 3$.

Therefore

$$\mu_1 = \frac{M_{1L}}{N_L} e^{-R_L t} + \frac{M_{1R}}{N_R} e^{-R_R t} \tag{33}$$

And with similar reason we also have

$$\mu_2 = \frac{M_{2L}}{N_L} e^{-R_L t} + \frac{M_{2R}}{N_R} e^{-R_R t} \tag{34}$$

And

$$\mu_3 = \frac{M_{3L}}{N_L} e^{-R_L t} + \frac{M_{3R}}{N_R} e^{-R_R t} \tag{35}$$

With,

$$M_{1L} = -Ew_1 \left[p_1 q_1 \left\{ R_R^2 + R_R \left(\frac{\lambda_2}{k_2} y + \frac{\lambda_3}{k_3} z \right) + \frac{\lambda_2}{k_2} \frac{\lambda_3}{k_3} yz + \alpha_{23}\alpha_{32} yz \right\} + p_2 q_2 \left\{ \alpha_{21} y R_R + \alpha_{21} yz \frac{\lambda_3}{k_3} \right\} \right] \tag{36}$$

$$M_{1R} = -Ew_2 \left[p_1 q_1 \left\{ R_L^2 + R_R \left(\frac{\lambda_2}{k_2} y + \frac{\lambda_3}{k_3} z \right) + \frac{\lambda_2}{k_2} \frac{\lambda_3}{k_3} yz + \alpha_{23}\alpha_{32} yz \right\} + p_2 q_2 \left\{ \alpha_{21} y R_L + \alpha_{21} yz \frac{\lambda_3}{k_3} \right\} \right] \tag{37}$$

$$M_{2L} = -Ew_1 \left[p_2 q_2 \left\{ R_R^2 + R_R \left(\frac{\lambda_1}{k_1} x + \frac{\lambda_3}{k_3} z \right) + \frac{\lambda_1}{k_1} \frac{\lambda_3}{k_3} xz \right\} + p_1 q_1 \left\{ \alpha_{12} x R_R - \alpha_{12} xz \frac{\lambda_3}{k_3} \right\} \right] \tag{38}$$

$$M_{2R} = -Ew_2 \left[p_2 q_2 \left\{ R_L^2 + R_L \left(\frac{\lambda_1}{k_1} x + \frac{\lambda_3}{k_3} z \right) + \frac{\lambda_1}{k_1} \frac{\lambda_3}{k_3} xz \right\} + p_1 q_1 \left\{ \alpha_{12} x R_L - \alpha_{12} xz \frac{\lambda_3}{k_3} \right\} \right] \tag{39}$$

$$M_{2L} = \ Ew_1 p_1 q_1 \alpha_{12} \alpha_{23} \tag{40}$$

$$M_{3R} = Ew_2 p_1 q_1 \alpha_{12} \alpha_{23} \tag{41}$$

Also,

$$N_R = -(\nu_0 R_R^3 - \nu_1 R_R^2 + \nu_2 R_R - \nu_3) \neq 0 \tag{42}$$

$$N_L = -(\nu_0 R_L^3 - \nu_1 R_L^2 + \nu_2 R_L - \nu_3) \neq 0 \tag{43}$$

For $t \to \infty$, we find that the shadow prices of three species $\mu_i(t) e^{R_L t}\ \forall\ i = 1,2,3$. Remain bounded, so they satisfy transversality condition at ∞. Therefore for $E \in [0, E_{max}]$ the Hamiltonian must be maximized. Assuming that the optimal equilibrium will not occur either at $E = 0$ or $E = E_{max}$, therefore we consider singular control.

Now,

$$\frac{\partial H}{\partial E} = w_1 e^{-R_R t}(p_1 q_1 x + p_2 q_2 y - c) + w_2 e^{-R_L t}(p_1 q_1 x + p_2 q_2 y - c) - \mu_1 q_1 x - \mu_2 q_2 y \quad (44)$$

i.e.,

$$(w_1 e^{-R_R t} + w_2 e^{-R_L t}) \frac{d\pi}{dE} = \mu_1 q_1 x + \mu_2 q_2 y \quad (45)$$

As, from (10) we have,

$$\frac{d\pi}{dE} = (p_1 q_1 x + p_2 q_2 y - c) \quad (46)$$

Therefore,

$$(w_1 e^{-R_R t} + w_2 e^{-R_L t})(p_1 q_1 x + p_2 q_2 y - c) = \mu_1 q_1 x + \mu_2 q_2 y \quad (47)$$

Therefore, substituting the values of μ_1, μ_2, μ_3 we get,

$$M_L e^{-R_R t} + M_R e^{-R_L t} = c(w_1 e^{-R_R t} + w_2 e^{-R_L t}) \quad (48)$$

Where we have,

$$M_L = \left(p_1 w_1 - \frac{M_{1L}}{N_L}\right) q_1 x + \left(p_2 w_1 - \frac{M_{2L}}{N_L}\right) q_2 y$$

$$M_R = \left(p_1 w_2 - \frac{M_{1R}}{N_R}\right) q_1 x + \left(p_2 w_2 - \frac{M_{2R}}{N_R}\right) q_2 y$$

Now (48) can be written as,

$$e^{-R_L t}\left(M_L e^{-[R_R - R_L]t} + M_R\right) = c e^{-R_L t}\left(w_1 e^{-[R_R - R_L]t} + w_2\right) \quad (49)$$

$$\Rightarrow \left(M_L e^{-[R_R - R_L]t} + M_R\right) = c\left(w_1 e^{-[R_R - R_L]t} + w_2\right) \quad (50)$$

Now when $[R_R,\ R_L] \to \infty\ \Rightarrow R_L \to \infty, R_R \to \infty$ and $(R_R - R_L) \to \infty$, which means when $\widetilde{R} \to \infty$, then the equation (50) reduces to

$$M_R = c w_2 \quad (51)$$

At the positive equilibrium the value of E is given by

$$E = \frac{\lambda_1}{q_1} - \frac{\lambda_1}{k_1}\frac{x}{q_1} - \frac{\alpha_{12} y}{q_1} = \frac{\lambda_2}{q_2} - \frac{\lambda_2}{k_2}\frac{y}{q_2} - \frac{\alpha_{23} z}{q_3} \quad (52)$$

As, $\frac{M_{iR}}{N_R} = o(R_L^{-1}), i = 1, 2, 3.$ Therefore, $\frac{M_{iR}}{N_R} \to 0$ as $R_L \to \infty$. Therefore (48) becomes

$$p_1 w_2 q_1 x + p_2 w_2 q_2 y = c w_2 \Rightarrow p_1 q_1 x + p_2 q_2 y = c. \quad (53)$$

Which implies that $\pi(x_\infty, y_\infty, z_\infty, c) = 0.$ (54)

Which indicates that for three species of a food chain also an infinite inflation leads to complete dissipation of economic revenue. This result was also initially investigated by Clark [5] in a combined harvesting of two species and recently by Chaudhuri [1] and also by Kar & Chaudhuri [9].

Now,

$$M_R - cw_2 = \left(p_1 w_1 - \frac{M_{1R}}{N_R}\right) q_1 x + \left(p_2 w_1 - \frac{M_{2R}}{N_R}\right) q_2 y - cw_2 = 0$$

$$\Rightarrow p_1 w_2 q_1 x + p_2 w_2 q_2 y - cw_2 = \left(\frac{M_{1R}}{N_R}\right) q_1 x + \left(\frac{M_{2R}}{N_R}\right) q_2 y$$

$$\Rightarrow \ (p_1 q_1 x + p_2 q_2 y - c)E = \frac{1}{w_2}\left[\left(\frac{M_{1R}}{N_R}\right) q_1 x + \left(\frac{M_{2R}}{N_R}\right) q_2 y\right] E$$

$$\Rightarrow \pi = \frac{1}{w_2}\left[\left(\frac{M_{1R}}{N_R}\right) q_1 x + \left(\frac{M_{2R}}{N_R}\right) q_2 y\right] E \tag{55}$$

Similarly, also we get,

$$\pi = \frac{1}{w_1}\left[\left(\frac{M_{1L}}{N_L}\right) q_1 x + \left(\frac{M_{2L}}{N_L}\right) q_2 y\right] E \tag{56}$$

Therefore from (55) and (56) we have

$$\pi = \frac{1}{N_R}[M_{1R} q_1 x + M_{2R} q_2 y]E + \frac{1}{N_L}[M_{1L} q_1 x + M_{2L} q_2 y]E \tag{57}$$

As, we have $\frac{M_{iR}}{N_R} = o(R_L^{-1}), i = 1, 2, 3.$ So, from (57) π is of $o(R_L^{-1})$ and hence π is a decreasing function of $R_L(\geq 0)$. Therefore, we can conclude that $R_L = 0$ (that is the economic environment when inflation rate r_L it and discount rate k_R are equal) and which gives the maximization of π.

9. Numerical Example and Simulations

Let us consider $\lambda_1 = 6.09$, $\lambda_2 = 5.07$, $\lambda_3 = 0.6$; $k_1 = 500$, $k_2 = 200$, $k_3 = 100$,$\alpha_{12} = 0.05$, $\alpha_{21} = 0.005$, $\alpha_{23} = 0.03$, $\alpha_{32} = 0.003$, $p_1 = 50$, $p_2 = 40$, $q_1 = 0.05$, $q_2 = 0.001$, $c = 45$, $E = 25$, $w_1 = 0.5$ and $w_2 = 0.5$.

Now with this set of parametric values we have

(i)$P_0(0,0,0)$ is unstable.

(ii) $P_1(0,78.95,263.16)$ is unstable.

(iii) $P_2(397.37,0,100)$ is unstable.

(iv) $P_3(259.02, 23.93,0)$is unstable.

(v) $P_4(125.6, 66.92,133.1)$is stable.

The stability diagram and phase portrait corresponding to the interior equilibrium are depicted in figure-1 and figure-2 respectively.

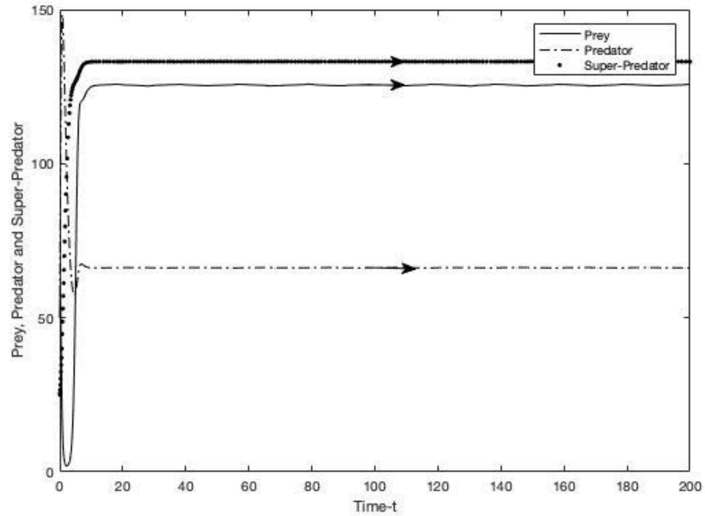

Figure-1: Stability diagram of the system (1) with initial values $x(0) = y(0) = z(0) = 25$.

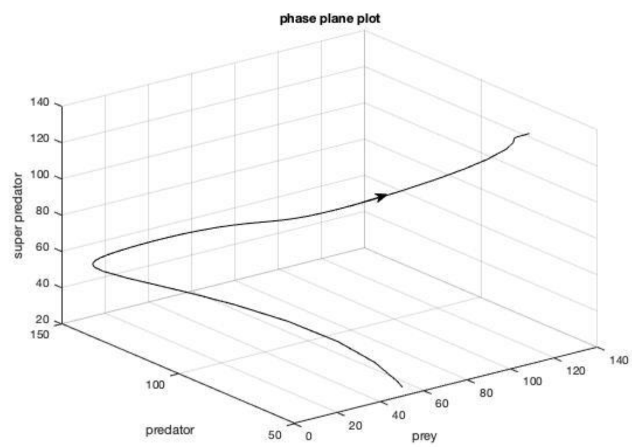

Figure-2: Phase-Space Trajectory.

10. CONCLUSION

In this paper we formulated a food chain model of three species, prey, predator and super predator with logistic law of growth and selective harvesting of prey and predator species is considered. The existence and stability of this system under possible steady states are investigated. The possibility of existence of bioeconomic equilibrium and global stability has been discussed and optimal harvesting policy is investigated with imprecise inflation and discount using Pontryagin's

Maximal principle. Finally, the model is illustrated with the help of a numerical example and MATLAB simulations.

CONFLICT OF INTERESTS

The author declares that there is no conflict of interests.

REFERENCES

[1] K.S. Chaudhuri, A bioeconomic model of harvesting a multispecies fishery, Ecol. Model. (1986), 267-279.

[2] K.S. Chaudhuri, Dynamic Optimization of combined harvesting of a two species fishery, Ecol. Model. 41(1988), 17-25.

[3] K.S. Chaudhuri, T. Johnson, Bioeconomic dynamics of fishery modelled as an S-system, Math. Biosci. 99 (1990), 231-249.

[4] K.S. Chaudhuri, S. Saha Ray, On the combined harvesting of a prey predator system, J. Biol. Syst. 4 (1996), 376-389.

[5] C.W. Clark, Mathematical bioeconomics: the optimal management of renewable resources, Wiley, New York, (1976).

[6] C.W. Clark, Bioeconomic modelling and fisheries management, Wiley, New York, (1985).

[7] M. Mesterton-Gibbons, On the optimal policy for the combined harvesting of predator and prey, Nat. Resour. Model. 3 (1988), 63-90.

[8] H.D. Landhal, B. D. Hansen, A three stage population model with cannibalism, Bull. Math. Biol. 37 (1975), 11-17.

[9] T.K. Kar, K.S. Chaudhuri, Harvesting in a two-prey one-predator fishery: A bioeconomic model, Australian Mathematical Society, (2003).

[10] [10] K. Maiti and M. Maiti, A numerical approach to a multi-objective optimal control problem for deteriorating multi-items under fuzzy inflation and discounting, Computers mathematics with applications, 55 (2008), 1794-1807.

[11] P. Grzegorzewski, Nested interval approximation of a fuzzy number, Fuzzy Sets Syst. 130 (2002), 321-330.

[12] T.K. Kar, K.S. Chaudhuri, Regulation of a prey-predator fishery by taxation, Int. J. Math. Educ. Technol. 34 (2003), 403-416.

[13] A. Maiti, B. Patra, G.P. Samanta, Persistence and stability of a food chain model with mixed selection of function responses, Nonlinear Anal.: Model. Control, 11 (2006), 171-185.

[14] A. Maiti, G.P. Samanta, Complex dynamics of a food chain model with mixed selection of functional response, Bull. Cal. Math. Soc. 97 (2005), 393-412.

[15] S. Gakkar, B. sing, The dynamics of a food web consisting of two preys and a harvesting predator, Chaos Solitons Fractals, 34 (2007), 1346-1356.

[16] S. Gakkar, R.K. Naji, On a food web consisting of a specialist predator, J. Biol. Syst. 11 (2003), 365-376.

[17] S. Kumar, S.K. Srivastava, P. Chingakham, Hopf bifurcation and stability in a harvested one-predator-two prey model, Appl. Math comput., 129 (2002), 107-118.

[18] S. Gakkar, R.K. Naji, Existence of chaos in two prey-one predator system, Chaos Solitons Fractals, 17 (2003), 639-649.

[19] S.B. Hsu, T.W. Hwang, Y. Kuang, A ratio-dependent food chain model and its application to biological control, Math. Biosci. 181 (2003), 55-83.

[20] A. Maiti, G.P. Samanta, Rich dynamics of a time-delayed food chain model, J. Biol. Syst. 14 (2006), 387-412.